

## On Möbius orthogonality for subshifts of finite type with positive topological entropy

by

D. KARAGULYAN (Stockholm)

**Abstract.** We prove that Möbius orthogonality does not hold for subshifts of finite type with positive topological entropy. This, in particular, shows that all  $C^{1+\alpha}$  surface diffeomorphisms with positive entropy correlate with the Möbius function.

**1. Introduction.** Let  $\mu$  denote the Möbius function, i.e.

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 \cdots p_k \text{ for distinct primes } p_i, \\ 0 & \text{otherwise.} \end{cases}$$

In [11], [12] Sarnak introduced the following related conjecture. Recall that a *topological dynamical system*  $(Y, T)$  is a compact metric space  $Y$  with a homeomorphism  $T : Y \rightarrow Y$ , and the *topological entropy*  $h(Y, T)$  of such a system is defined as

$$h(Y, T) = \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log N(\epsilon, n),$$

where  $N(\epsilon, n)$  is the largest number of  $\epsilon$ -separated points in  $Y$  using the metric  $d_n : Y \times Y \rightarrow (0, \infty)$  defined by

$$d_n(x, y) = \max_{0 \leq i \leq n} d(T^i x, T^i y).$$

A sequence  $f : \mathbb{Z} \rightarrow \mathbb{C}$  is said to be *deterministic* if it is of the form

$$f(n) = F(T^n x),$$

for all  $n$  and some topological dynamical system  $(Y, T)$  with zero topological entropy, a base point  $x \in Y$ , and a continuous function  $F : Y \rightarrow \mathbb{C}$ .

---

2010 *Mathematics Subject Classification*: Primary 37B10; Secondary 37A35, 11Y35.

*Key words and phrases*: subshifts of finite type, entropy, Möbius orthogonality.

Received 21 July 2016; revised 24 October 2016.

Published online 3 March 2017.

CONJECTURE 1 (P. Sarnak). *Let  $f : \mathbb{N} \rightarrow \mathbb{C}$  be a deterministic sequence. Then*

$$(1.1) \quad S_n(T(x), f) = \frac{1}{n} \sum_{k=1}^n \mu(k)f(k) = o(1).$$

In this case we also say that the Möbius function *does not correlate with* or *is orthogonal to* the sequence  $f(k)$ .

The conjecture is known to be true for several dynamical systems. For a Kronecker flow (that is, a translation on a compact abelian group) it is proved in [13] and [5], while when  $(X, f)$  is a translation on a compact nilmanifold it is proved in [7]. In [3] it is established also for horocycle flows. For orientation preserving circle homeomorphisms and continuous interval maps of zero entropy it is proved in [8]. For other references see [1], [2], [9].

In this paper we study the opposite direction of the conjecture. That is, we are interested in systems with positive entropy and their correlation properties with the Möbius function. Peter Sarnak, in his famous exposition [12], mentions that for a given sequence  $\epsilon(n)$  with properties similar to those of  $\mu(n)$ , one can construct a positive entropy flow orthogonal to  $\epsilon(n)$  (he attributes this to Bourgain—private communication). But this example has never been published. Assuming Bourgain’s claim, it becomes of interest to construct a positive entropy system which does not correlate with the Möbius function. In this context the subshifts of finite type are a natural class to examine. We mention that the conjecture in the opposite direction has previously been considered in [2] and [6]. The authors construct examples of non-regular Toeplitz sequences for which the orthogonality to the Möbius function does not hold. We point out that for any measure preserving dynamical system  $(X, \mathcal{B}, \nu, T)$  Sarnak’s conjecture holds almost surely with respect to  $\nu$  (see [11]). Hence the orthogonality to the Möbius function may fail only on a pathological set. Another motivation for studying subshifts of finite type comes from Katok’s famous horseshoe theorem which states that for any  $C^{1+\epsilon}$  smooth surface diffeomorphism with positive entropy, there is a compact invariant set  $A$  such that the restriction of the map to  $A$  is topologically conjugate to a subshift of finite type with positive topological entropy. So this shows that all sufficiently smooth surface diffeomorphisms with positive entropy correlate with the Möbius function. Horseshoes also emerge in many other systems with positive entropy, such as unimodal maps, Hénon maps etc.

**2. Statement and proof of the main theorem.** We are going to prove the following theorem.

THEOREM 1. For any subshift  $(T, \Sigma_A^+)$  of finite type with positive topological entropy there exist a sequence  $z \in \Sigma_A^+$  and a continuous test function  $\phi$  for which

$$\lim_{n \rightarrow \infty} S_n(T(z), \phi) \neq 0.$$

First we recall the definition of a subshift of finite type. Let  $V$  be a finite set of  $n$  symbols and  $A$  be an  $n \times n$  adjacency matrix with entries in  $\{0, 1\}$ . Define

$$\Sigma_A^+ = \{(x_0, x_1, \dots) : x_j \in V, A_{x_j x_{j+1}} = 1, j \in \mathbb{N}\}.$$

The shift operator  $T$  maps a sequence in the one-sided shift to another by shifting all symbols to the left, i.e.

$$(T(x))_j = x_{j+1}.$$

The topological entropy of a subshift of finite type can be computed by computing the number of different admissible words of length up to  $n$ , i.e. if

$$B_n = \{(v_0, \dots, v_{n-1}) : v_j = x_j \text{ for } 0 \leq j < n \text{ for some } x \in \Sigma_A^+\},$$

then the topological entropy of  $T$  equals

$$(2.1) \quad h(T) = \limsup_{n \rightarrow \infty} \frac{\log \#B_n}{n}.$$

We say that the sequence  $X = \{x_k\}_{k=0}^n$  of symbols from  $V$  is a *word* in  $\Sigma_A^+$  if  $A_{x_i, x_{i+1}} = 1$  for all  $i = 0, \dots, n - 1$ . For the word  $X$ , the number of its elements will be denoted by  $|X|$ . We say that we have an *admissible loop* at  $v \in V$  if there exists a finite sequence  $\{x_k\}_{k=0}^n, n \geq 1$ , of elements from  $V$  such, that  $x_0 = v, x_n = v$  and  $x_k \neq v$  for  $0 < k < n$  and  $A_{x_i, x_{i+1}} = 1$  for all  $i = 0, \dots, n - 1$ .

The main property of the Möbius function which will be used in the proof is the following well known fact (see e.g. [10], [4]):

$$(2.2) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \mu^2(k) = \frac{6}{\pi^2}.$$

The following lemma is an easy consequence of (2.2). It can also be obtained from Mirsky’s [10] theorem on the patterns of arithmetic progressions in square free numbers.

LEMMA 1. For any  $M \in \mathbb{N}$  there exists an integer  $0 \leq s < M$  for which

$$(2.3) \quad \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{1 \leq k \leq N \\ k \equiv s \pmod{M}}} \mu^2(k) > 0.$$

*Proof.* For this it is enough to note that

$$\frac{1}{N} \sum_{k=1}^N \mu^2(k) = \sum_{l=0}^{M-1} \frac{1}{N} \sum_{\substack{1 \leq k \leq N \\ k \equiv l \pmod{M}}} \mu^2(k),$$

and since the sum on the left hand side does not converge to 0, the same must hold for at least one of the  $M$  sums on the right hand side. ■

Let us return to subshifts of finite type with positive entropy. Note that for some  $v \in V$  there are at least two different admissible loops at  $v$ . Otherwise the space  $\Sigma_A^+$  will consist of only periodic orbits, and from (2.1) it will follow that the topological entropy of  $T$  is zero. Let  $\gamma_1 = \{x_1, \dots, x_n\}$  and  $\gamma_2 = \{y_1, \dots, y_m\}$  be two loops. Define also  $\gamma'_1 = \{x_1, \dots, x_{n-1}\}$  and  $\gamma'_2 = \{y_1, \dots, y_{m-1}\}$ .

LEMMA 2. *For any subshift  $(T, \Sigma_A^+)$  of finite type with positive topological entropy there exists a positive integer  $l \in \mathbb{N}$  such that for any integer  $s$  with  $0 \leq s < l$ , there exist  $z_s \in \Sigma_A^+$  and  $\phi \in C(\Sigma_A^+)$  with the property that for any  $n \in \mathbb{N}$ ,*

$$(2.4) \quad \phi(T^n(z_s)) = \begin{cases} \mu(n) & \text{if } n \equiv s \pmod{l} \text{ and } \mu(n) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We say that two finite words  $a$  and  $b$  of equal length have the *recognizability property* if in any concatenation of these two words ( $aa$ ,  $ab$ ,  $bb$  or  $ba$ ) neither  $a$  nor  $b$  has any additional occurrences (not shown in the writing of the concatenated blocks). For example, it is very easy to check that  $a = 00110$  and  $b = 01010$  have the recognizability property. Following the patterns of 0's and 1's in  $a$  and  $b$ , we construct the words

$$x = \gamma'_1 \gamma'_1 \gamma'_2 \gamma'_2 \gamma'_1 \quad \text{and} \quad y = \gamma'_1 \gamma'_2 \gamma'_1 \gamma'_2 \gamma'_1.$$

Notice that

$$|x| = |y| = 3|\gamma'_1| + 2|\gamma'_2| =: l.$$

Since  $\gamma'_1$  and  $\gamma'_2$  both start with the symbol  $v$ , and  $v$  does not occur inside these blocks, and since the blocks  $x$ ,  $y$  are built of  $\gamma'_1$  and  $\gamma'_2$  following the  $a$ ,  $b$  patterns, and have the same length, it is obvious that  $x$ ,  $y$  also have the recognizability property. One can also check that any word obtained through concatenation of  $x$  and  $y$  will be admissible in  $\Sigma_A^+$ . We now define a sequence  $z_s = \tilde{z}z_0z_1 \dots z_k \dots$ , where  $\tilde{z}$  is the word consisting of the last  $s$  symbols of either  $x$  or  $y$ , and  $\{z_k\}_{k=0}^\infty$  is chosen as follows:

$$(2.5) \quad z_k = \begin{cases} x & \text{if } \mu(kl + s) = -1, \\ y & \text{otherwise.} \end{cases}$$

Clearly  $z_s$  is admissible in  $\Sigma_A^+$ . We also define a test function  $\phi$  by

$$(2.6) \quad \phi(u) = \begin{cases} 1 & \text{if } u \text{ begins with the word } x, \\ -1 & \text{if } u \text{ begins with the word } y, \\ 0 & \text{otherwise,} \end{cases}$$

or alternatively

$$\phi(u) = \chi_x(u) - \chi_y(u),$$

where  $\chi_\gamma(u)$  is the characteristic function of the word  $\gamma$ . It is easy to check that  $\chi_\gamma(u)$  is a continuous function, and hence so is  $\phi(u)$ .

Since the words  $x$  and  $y$  have the recognizability property, it is clear that in the sequence  $z_s$  they may appear only at positions  $n \equiv s \pmod{l}$ . But from the construction of  $\phi$  and the choice of the subwords  $\{z_k\}$  in  $z_s$  it follows that  $\phi(T^n(z_s)) = \mu(n)$  if  $\mu(n) \neq 0$  and  $n \equiv s \pmod{l}$ . ■

*Proof of Theorem 1.* Using Lemma 2 we can find a positive integer  $l$  such that for any integer  $s$  with  $0 \leq s < l$ , there exist  $z_s \in \Sigma_A^+$  and  $\phi \in C(\Sigma_A^+)$  such that (2.4) holds. Now we use Lemma 1 with  $M = l$  to find an integer  $s$  for which (2.3) holds. According to (2.4),

$$\sum_{n=1}^N \mu(n)\phi(T^n(z_s)) = \sum_{\substack{1 \leq k \leq N \\ k \equiv s \pmod{l}}} \mu^2(n).$$

But from the choice of  $s$ ,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{1 \leq k \leq N \\ k \equiv s \pmod{l}}} \mu^2(n) > 0.$$

Hence

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mu(n)\phi(T^n(z_s)) = \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{1 \leq k \leq N \\ k \equiv s \pmod{l}}} \mu^2(n) > 0,$$

which finishes the proof of the theorem. ■

One can also show that the entropy of  $T$  is positive. By definition  $|x| = |y| = l$ . As all concatenations of the words  $x$  and  $y$  are admissible, if we consider all admissible words of length  $\{nl : n = 1, 2, \dots\}$ , then for the number of different words of length  $nl$  we will have

$$(2.7) \quad \frac{\log \#B_{nl}}{nl} \geq \frac{\log 2^n}{nl} = \frac{\log 2}{l} = \log 2^{1/l}.$$

Hence  $h(T) \geq \log 2^{1/l}$ . Now, if we consider the subshift of finite type which consists of exactly two loops with  $|\gamma'_1| = |\gamma'_2| = l$ , then in (2.7) we will have equality. Hence the entropy of  $T$  can be made arbitrarily small.

**Acknowledgements.** The author would like to thank Mariusz Lemańczyk for proposing the problem in the opposite direction to Sarnak’s conjecture, to Michael Benedicks for his remark about Katok’s horseshoe theorem and his guidance and many valuable suggestions, and Ana Rodrigues and El Houcein El Abdalaoui for useful comments about the manuscript. He would also like to thank Joanna Kułaga-Przymus for her remark concerning Lemma 1 and also the anonymous referee for carefully reading the manuscript and for constructive comments, which substantially helped improving the quality of the paper.

### References

- [1] H. El Abdalaoui, M. Lemańczyk and T. de la Rue, *On spectral disjointness of powers for rank-one transformations and Möbius orthogonality*, J. Funct. Anal. 266 (2014), 284–317.
- [2] H. El Abdalaoui, S. Kasjan and M. Lemańczyk, *0-1 sequences of the Thue–Morse type and Sarnak’s conjecture*, Proc. Amer. Math. Soc. 144 (2016), 161–176.
- [3] J. Bourgain, P. Sarnak and T. Ziegler, *Disjointness of Moebius from horocycle flows*, in: From Fourier Analysis and Number Theory to Radon Transforms and Geometry, Dev. Math. 28, Springer, New York, 2013, 67–83.
- [4] F. Cellarosi and Ya. G. Sinai, *Ergodic properties of square-free numbers*, J. Eur. Math. Soc. 15 (2013), 1343–1374.
- [5] H. Davenport, *On some infinite series involving arithmetical functions. II*, Quart. J. Math. Oxford Ser. 8 (1937), 313–320.
- [6] T. Downarowicz and S. Kasjan, *Odometers and Toeplitz subshifts revisited in the context of Sarnak’s conjecture*, Studia Math. 229 (2015), 45–72.
- [7] B. Green and T. Tao, *The Möbius function is strongly orthogonal to nilsequences*, Ann. of Math. (7) 175 (2012), 541–566.
- [8] D. Karagulyan, *On Möbius orthogonality for interval maps of zero entropy and orientation-preserving circle homeomorphisms*, Ark. Mat. 53 (2015), 317–327.
- [9] J. Kułaga-Przymus and M. Lemańczyk, *The Möbius function and continuous extensions of rotations*, Monatsh. Math. 178 (2015), 553–582.
- [10] L. Mirsky, *Arithmetical pattern problems relating to divisibility by  $r$ th powers*, Proc. London Math. Soc. (2) 50 (1949), 497–508.
- [11] P. Sarnak, *Three lectures on the Möbius function randomness and dynamics*, <http://www.math.ias.edu/files/wam/2011/PSMobius.pdf>.
- [12] P. Sarnak, *Möbius randomness and dynamics*, Not. S. Afr. Math. Soc. 43 (2012), 89–97.
- [13] I. M. Vinogradov, *Some theorems concerning the theory of primes*, Mat. Sb. 2 (1937), 179–195.

D. Karagulyan  
 Department of Mathematics  
 Royal Institute of Technology  
 S-100 44 Stockholm, Sweden  
 E-mail: davitk@kth.se