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Generalized Daugavet equations, affine operators and unique best approximation

by

Paweł Wójcik (Kraków)

Abstract. We introduce and investigate the notion of generalized Daugavet equation $\|A_1 + \cdots + A_n\| = \|A_1\| + \cdots + \|A_n\|$ for affine operators $A_1, \ldots, A_n$ on a reflexive Banach space into another Banach space. This extends the well-known Daugavet equation $\|T + I\| = \|T\| + 1$, where I denotes the identity operator. A new characterization of the Daugavet equation in terms of extreme points is given. We also present a result concerning uniqueness of best approximation.

1. Introduction. For Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from X into Y, and $\mathcal{L}(X) := \mathcal{L}(X,X)$. The set of compact linear operators from X into Y is denoted by $\mathcal{K}(X,Y)$, and $\mathcal{K}(X) := \mathcal{K}(X,X)$. The Banach space of all continuous affine mappings from X into Y is denoted by $\mathcal{A}(X,Y)$. This means that $\mathcal{A}(X,Y) = \{a+T : a \in Y, T \in \mathcal{L}(X,Y)\}$, and the norm is defined by $\|A\| := \sup\{\|Ax\| : x \in B(X)\}$. It is worth mentioning that $\|A\| \neq \inf\{c > 0 : \forall_{x\in X}\ \|Ax\| \le c\|x\|\}$ (unless A is linear).

In 1963, Daugavet [Dau] proved that each compact operator T on the Banach space $\mathcal{C}[0,1]$ satisfies the equation

$$\|T + I\| = \|T\| + 1.$$

It turns out that various classes of operators on many other Banach spaces satisfy this equation, which is known as the *Daugavet equation* (DE). This result was then generalized by Foiaş and Singer to weakly compact operators acting on arbitrary atomless $\mathcal{C}(\Omega)$ (see [FS]). Most of the results on the (DE) concern Banach spaces with the *Daugavet property*, i.e. spaces X for which every weakly compact operator (equivalently, every rank one operator) in

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$\mathcal{L}(X)$ satisfies the (DE). It is worth mentioning that every Banach space with the Daugavet property is nonreflexive. However, there are interesting problems concerning the (DE) for some operators on a reflexive Banach space (see e.g. [AAB], [Lin1], [Lin2]). In particular, Lin [Lin1, Theorem 1] proved the following result.

THEOREM 1.1 ([Lin1, Theorem 1]). *Let X be a uniformly convex Banach space. For $S, T \in \mathcal{L}(X)$, the following are equivalent:*

(i) $\|I + S + T\| = 1 + \|S\| + \|T\|$;
(ii) *there exists a sequence $(x_n)_{n=1}^{\infty}$ of unit vectors in X such that*

$$(S + T)x_n - (\|S\| + \|T\|)x_n \to 0 \quad \text{as } n \to \infty;$$

(iii) *there exists a sequence $(x_n)_{n=1}^{\infty}$ of unit vectors in X such that*

$$\|(I + S + T)x_n\| - (1 + \|Sx_n\| + \|Tx_n\|) \to 0 \quad \text{as } n \to \infty,$$
$$\|Sx_n\| + \|Tx_n\| \to \|S\| + \|T\| \quad \text{as } n \to \infty.$$

For the equation $\|T_1 + \cdots + T_m\| = \|T_1\| + \cdots + \|T_m\|$ a result similar to Theorem 1.1 was obtained in [Lin2].

The above results motivate the following definition. We say that a family $\mathcal{F} := \{A_t \in \mathcal{A}(X, Y) : t \in \mathcal{T}\} \subseteq \mathcal{A}(X, Y)$ has *property* (D) if for any finite collection $A_{t_1}, \ldots, A_{t_n}$ of elements of $\mathcal{F}$,

$$\|A_{t_1} + \cdots + A_{t_n}\| = \|A_{t_1}\| + \cdots + \|A_{t_n}\|.$$

In particular, the equation $\|I+T\| = 1+\|T\|$ (or $\|I+S+T\| = 1+\|S\|+\|T\|$) corresponds to $\mathcal{F} = \{I, T\}$ (or $\mathcal{F} = \{I, S, T\}$). Indeed, if $\|I + S + T\| = 1 + \|S\| + \|T\|$, then $\|I + S\| = 1 + \|S\|$ and $\|I + T\| = 1 + \|T\|$.

Our results generalize and complement Theorem 1.1 to some extent. Our method of proof is different from that of [AAB], [Lin1], [Lin2]. In [AAB] it is proved that a compact operator on a uniformly convex Banach space satisfies the Daugavet equation if and only if its norm is an eigenvalue. Section 4 of the present paper shows that assuming that the operator is compact is not necessary (under certain circumstances). Sometimes it suffices to assume that the space is strictly convex (instead of uniformly strictly convex). In the last section, we investigate best approximation.

2. Preliminaries. Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The closed unit ball of X is denoted by $B(X)$. The unit sphere of X is denoted by $S(X)$. Fix $x \in X \setminus \{0\}$. We set

$$J(x) := \{x^* \in X^* : \|x^*\| = 1,\ x^*(x) = \|x\|\}.$$

It is easy to check that the set $J(x) \subseteq S(X^*)$ is convex and closed. By the Hahn–Banach theorem, $J(x) \neq \emptyset$ for all $x \in X \setminus \{0\}$.

2.1. Extreme points in duals of operator spaces. The main tool in this paper is a theorem which characterizes the extremal points of the unit ball in $\mathcal{K}(X,Y)^*$ in terms of the extremal points of the closed unit balls in Y^* and X^{**}. We denote by $\operatorname{Ext} W$ the set of all extremal points of a given set W. By the Krein–Milman Theorem, the closed unit ball $B(Y^*)$ has many extreme points. In particular, $\operatorname{Ext} B(\mathcal{K}(X,Y)^*) \neq \emptyset$.

THEOREM 2.1 ([F], [LO], [RS]). *If X and Y are Banach spaces, then*

$$\begin{aligned}&\operatorname{Ext} B(\mathcal{K}(X,Y)^*)\\ &\qquad = \{x^{**}\otimes y^* \in \mathcal{K}(X,Y)^* : x^{**} \in \operatorname{Ext} B(X^{**}),\ y^* \in \operatorname{Ext} B(Y^*)\},\end{aligned}$$

where $x^{**}\otimes y^*\colon \mathcal{K}(X,Y) \to \mathbb{K}$, $(x^{**}\otimes y^*)(T) := x^{**}(T^*y^*)$ for every $T \in \mathcal{K}(X,Y)$.

Fakhoury [F] proved $\subseteq$; Ruess and Stegall [RS] proved $\supseteq$ for the real case; and Lima and Olsen [LO] proved $\supseteq$ for the complex case.

In particular, if X is a reflexive Banach space, then $\operatorname{Ext} B(X) \neq \emptyset$. From Theorem 2.1 we obtain the following result.

COROLLARY 2.2. *If X is a reflexive Banach space, then*

$$\operatorname{Ext} B(\mathcal{K}(X,Y)^*) = \{y^*\otimes x \in \mathcal{K}(X,Y)^* : x \in \operatorname{Ext} B(X),\ y^* \in \operatorname{Ext} B(Y^*)\},$$

where $y^*\otimes x\colon \mathcal{K}(X,Y) \to \mathbb{K}$, $(y^*\otimes x)(T) := y^*(Tx)$ for every $T \in \mathcal{K}(X,Y)$.

2.2. Decomposition for $\mathcal{L}(X,Y)^*$. Let X be a Banach space and V a closed subspace of X. The subspace V is said to be an *M-ideal in X* if $X^* = V^* \oplus_1 V^\perp$ where $V^\perp := \{x^* \in X^* : V \subseteq \ker x^*\}$, and if $x^* = x_1^* + x_2^*$ is the unique decomposition of x^* in X^*, then $\|x^*\| = \|x_1^*\| + \|x_2^*\|$.

We recall several situations when $\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$. Hennefeld [H] and Saatkamp [Sa] have proved that $\mathcal{K}(l^p,l^q)$ is an M-ideal when $1 < p \le q < \infty$. Note that if $1 \le q < p < \infty$, then $\mathcal{K}(l^p,l^q) = \mathcal{L}(l^p,l^q)$ and $\mathcal{K}(c_0,l^q) = \mathcal{L}(c_0,l^q)$ [LT, p. 76, Proposition 2.c.3]. Several authors have observed that $\mathcal{K}(X,c_0)$ is an M-ideal for all Banach spaces X [Lim, Sa, SW]. It is known that $\mathcal{K}(l^1,l^1)$ and $\mathcal{K}(l^\infty,l^\infty)$ are not M-ideals [SW].

3. Main results. The Banach space of all compact affine mappings from X into Y is denoted by $\mathcal{AK}(X,Y)$. Thus $\mathcal{AK}(X,Y) = \{a+T : a \in Y,\ T \in \mathcal{K}(X,Y)\}$. If Ω is a compact topological space, we let $\mathcal{C}(\Omega,Y)$ denote the space of all continuous functions f from Ω to Y with $\|f\|_\infty := \sup\{\|f(t)\|_Y : t \in \Omega\}$.

Brosowski and Deutsch [BD] proved that

$$\operatorname{Ext} B(C(\Omega,Y)^*) = \{y^*\circ\psi_t \in \mathcal{C}(\Omega;Y)^* : t \in \Omega,\ y^* \in \operatorname{Ext} B(Y^*)\}, \tag{3.1}$$

where $\psi_t\colon \Omega \to Y$ is the evaluation functional at t, i.e., $\psi_t(f) := f(t)$ for all f in $\mathcal{C}(\Omega,Y)$. It is well known [Si] that if W is a subspace of a normed linear space Z, then

$$\operatorname{Ext} B(W^*) \subseteq \{z^*|_W : z^* \in \operatorname{Ext} B(Z^*)\}. \tag{3.2}$$

Now suppose that X is reflexive. Then $B(X)$ with the w-topology is compact, and $\mathcal{AK}(X,Y)$ can be identified in a natural way with a subspace of $\mathcal{C}(B(X),Y)$. Namely, an isometric embedding $\varphi\colon \mathcal{AK}(X,Y) \to \mathcal{C}(B(X),Y)$ is defined by $\varphi(A) := A|_{B(X)}$. The next result is a slight generalization of Corollary 2.2.

Theorem 3.1. *Let X be a reflexive Banach space. Then*

$$\operatorname{Ext} B(\mathcal{AK}(X,Y)^*) = \{y^* \otimes x : x \in \operatorname{Ext} B(X),\ y^* \in \operatorname{Ext} B(Y^*)\}, \tag{3.3}$$

where $y^* \otimes x\colon \mathcal{AK}(X,Y) \to \mathbb{K}$, $(y^* \otimes x)(A) := y^*(Ax)$ for every $A \in \mathcal{AK}(X,Y)$.

Proof. Combining (3.1) and (3.2), we immediately get

$$\begin{aligned} &\operatorname{Ext} B(\varphi(\mathcal{AK}(X,Y))^*) \\ &\qquad \subseteq \{(y^* \otimes x)|_{\mathcal{AK}(X;Y)} : x \in B(X),\ y^* \in \operatorname{Ext} B(Y^*)\}. \end{aligned} \tag{3.4}$$

To prove (3.3) fix $f \in \operatorname{Ext} B(\varphi(\mathcal{AK}(X,Y))^*)$. By (3.4), $f = y^* \otimes x$ for some $x \in B(X)$ and $y^* \in \operatorname{Ext} B(Y^*)$. It is easy to see that $\mathcal{K}(X,Y) \subseteq \mathcal{AK}(X,Y)$ and $(y^*\otimes x)|_{\mathcal{K}(X,Y)} = f|_{\varphi(\mathcal{K}(X,Y))} \in \operatorname{Ext} B(\varphi(\mathcal{K}(X,Y))^*)$, by (3.2). Now it follows from Corollary 2.2 that $x \in \operatorname{Ext} B(X)$. Thus $f \in \{y^* \otimes x : x \in \operatorname{Ext} B(X),\ y^* \in \operatorname{Ext} B(Y^*)\}$.

We now prove the converse inclusion. Since $\mathcal{K}(X,Y) \subseteq \mathcal{AK}(X,Y)$, by (3.2) we get

$$\operatorname{Ext} B(\mathcal{K}(X,Y)^*) \subseteq \{f|_{\mathcal{K}(X,Y)} : f \in \operatorname{Ext} B(\mathcal{AK}(X,Y)^*)\}. \tag{3.5}$$

Combining (3.5) and Corollary 2.2, we immediately get

$$\{y^* \otimes x : x \in \operatorname{Ext} B(X),\ y^* \in \operatorname{Ext} B(Y^*)\} \subseteq \operatorname{Ext} B(\mathcal{AK}(X,Y)^*). \ \blacksquare$$

The main goal of this paper is to prove the following two results.

Theorem 3.2. *Let X,Y be Banach spaces with X reflexive. Then for any subset $\mathcal{F} \subseteq \mathcal{AK}(X,Y)$ the following conditions are equivalent:*

(a) *$\mathcal{F}$ has property (D);*
(b) *there exists $x_o \in \operatorname{Ext} B(X)$ such that $\|Ax_o\| = \|A\|$ for every $A \in \mathcal{F}$ and $\|A_1x_o + \cdots + A_nx_o\| = \|A_1x_o\| + \cdots + \|A_nx_o\|$ for any finite collection $A_1,\ldots,A_n \in \mathcal{F}$.*

Proof. (b)$\Rightarrow$(a). If (b) holds, then

$$\begin{aligned} \|A_1\| + \cdots + \|A_n\| &= \|A_1x_o\| + \cdots + \|A_nx_o\| \\ &= \|A_1x_o + \cdots + A_nx_o\| \le \|A_1 + \cdots + A_n\| \\ &\le \|A_1\| + \cdots + \|A_n\|. \end{aligned}$$

It follows that $\|A_1 + \cdots + A_n\| = \|A_1\| + \cdots + \|A_n\|$.

(a)⇒(b). First we show that $E := \bigcap_{A\in\mathcal{F}} J(A) \neq \emptyset$. Since every $J(A)$ is a nonempty weak*-closed subset of the weak*-compact unit ball of $\mathcal{AK}(X,Y)$, it is enough to show that $\{J(A) : A \in \mathcal{F}\}$ has the finite intersection property, that is, $\bigcap_{k=1}^{n} J(A_k) \neq \emptyset$ for every n and all $A_1,\dots,A_n \in \mathcal{F}$. Fix $A_1,\dots,A_n \in \mathcal{F}$. Then $J(\sum_{k=1}^{n} A_k) \neq \emptyset$ (see Preliminaries). We show that

$$J\Big(\sum_{k=1}^{n} A_k\Big) \subseteq \bigcap_{k=1}^{n} J(A_k),$$

which is enough by the above. Fix any $f \in J(\sum_{k=1}^{n} A_k)$. Then $f \in \mathcal{AK}(X,Y)^*$, $\|f\| = 1$ and

$$\Big\|\sum_{k=1}^{n} A_k\Big\| = f\Big(\sum_{k=1}^{n} A_k\Big) = \sum_{k=1}^{n} f(A_k) \leq \sum_{k=1}^{n} \|A_k\| \overset{\text{(a)}}{=} \Big\|\sum_{k=1}^{n} A_k\Big\|,$$

which implies $\sum_{k=1}^{n} f(A_k) = \sum_{k=1}^{n} \|A_k\|$. Taking into account that $|f(A_k)| \leq \|A_k\|$ for all k, we deduce that $f(A_k) = \|A_k\|$ for all k, which means that $f \in \bigcap_{k=1}^{n} J(A_k)$. So, $E \neq \emptyset$ is proved.

Now E is nonempty, convex, and weak*-closed, and so it has an extreme point h by the Krein–Milman theorem. In particular, $h \in E$ and $h \in \operatorname{Ext} E$.

We claim that h is also an extreme point of $B(\mathcal{AK}(X,Y)^*)$. Suppose $h = \frac{1}{2}g + \frac{1}{2}p$, where $g, p \in B(\mathcal{AK}(X,Y)^*)$.

Fix $A \in \mathcal{F}$. Since $h \in \bigcap_{A\in\mathcal{F}} J(A)$, we have

$$\|A\| = h(A) = \tfrac{1}{2}g(A) + \tfrac{1}{2}p(A) \leq \tfrac{1}{2}\|A\| + \tfrac{1}{2}\|L\| = \|A\|.$$

Therefore

$$\|A\| = |g(A)| = |p(A)|.$$

Thus $h(A), g(A), p(A)$ are in $\{\alpha \in \mathbb{K} : |\alpha| = \|A\|\}$, and since one of them is a convex combination of the others, they must all be the same scalar. Therefore $\|A\| = h(A) = g(A) = p(A)$. Hence, g and p are both in $J(A)$. Since A was arbitrary, we get $g, p \in \bigcap_{A\in\mathcal{F}} J(A) = E$. Since $h \in \operatorname{Ext} E$, we must have $h = g = p$. To summarize, it has been shown that $h \in \operatorname{Ext} B(\mathcal{AK}(X,Y)^*)$.

By Theorem 3.1, the functional h has the form $h = y^* \otimes x_o$ for some $x_o \in \operatorname{Ext} B(X)$, $y^* \in \operatorname{Ext} B(Y^*)$.

In order to prove (b), take $A_1,\dots,A_n$ in $\mathcal{F}$. Since $E \subseteq J(A_k)$, we have $h \in J(A_k)$ for all k. Then

$$\begin{aligned}\|A_1\| + \dots + \|A_n\| &= h(A_1) + \dots + h(A_n) = y^*(A_1x_o) + \dots + y^*(A_nx_o)\\ &\leq y^*(A_1x_o + \dots + A_nx_o) \leq \|A_1x_o + \dots + A_nx_o\|\\ &\leq \|A_1x_o\| + \dots + \|A_nx_o\| \leq \|A_1\| + \dots + \|A_n\|,\end{aligned}$$

and therefore $\|A_1x_o + \dots + A_nx_o\| = \|A_1x_o\| + \dots + \|A_nx_o\|$.

It remains to show that $\|Ax_o\| = \|A\|$ for all $A \in \mathcal{F}$. Since $y^* \otimes x_o = h \in J(A)$ for every $A \in \mathcal{F}$, we conclude that $\|A\| = h(A) = y^*(Ax_o) \leq \|Ax_o\| \leq \|A\|$. Thus $\|Ax_o\| = \|A\|$ for every $A \in \mathcal{F}$. ∎

The proof of the next theorem is similar to that of Theorem 3.2.

THEOREM 3.3. *Let X, Y be Banach spaces with X reflexive. Assume that $\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$. Let $L_c \in \mathcal{L}(X,Y)$ be an operator with* $\operatorname{dist}(L_c, \mathcal{K}(X,Y)) < \|L_c\|$. *Then for any subset $\mathcal{F} \subseteq \mathcal{L}(X,Y)$ with $L_c \in \mathcal{F}$ the following conditions are equivalent:*

(a) *$\mathcal{F}$ has property (D);*
(b) *there exists $x_o \in \operatorname{Ext} B(X)$ such that $\|Lx_o\| = \|L\|$ for every $L \in \mathcal{F}$ and $\|L_1x_o + \cdots + L_nx_o\| = \|L_1x_o\| + \cdots + \|L_nx_o\|$ for any finite collection $L_1, \ldots, L_n \in \mathcal{F}$.*

Proof. In a similar way to the proof of Theorem 3.2, we obtain (b)⇒(a).

We now prove (a)⇒(b). Much as in the proof of Theorem 3.2 we can show that

$$E := \bigcap_{L \in \mathcal{F}} J(L) \neq \emptyset.$$

Now E is nonempty, convex, and weak*-closed, and so it has an extreme point h by the Krein–Milman theorem. As in the proof of Theorem 3.2, it can be shown that $h \in \operatorname{Ext} B(\mathcal{L}(X,Y)^*)$.

The only difficult point is to show that $h \in \operatorname{Ext} B(\mathcal{K}(X,Y)^*)$. In particular, $h \in J(L_c)$ and hence

$$h(L_c) = \|L_c\|. \tag{3.6}$$

By assumption we have

$$\mathcal{L}(X,Y)^* = \mathcal{K}(X,Y)^* \oplus_1 \mathcal{K}(X,Y)^{\perp}.$$

Let $h = h_1 + h_2$ with $h_1 \in \mathcal{K}(X,Y)^*$ and $h_2 \in \mathcal{K}(X,Y)^{\perp}$. Then $\|h\| = \|h_1\| + \|h_2\|$. We show that $h_2 = 0$. Assume that $h_2 \neq 0$. Since $\operatorname{dist}(L_c, \mathcal{K}(X,Y)) < \|L_c\|$, there exists $T \in \mathcal{K}(X,Y)$ such that $\|L_c - T\| < \|L_c\|$. So, we have

$$\begin{aligned}
\|L_c\| &\overset{(3.6)}{=} h(L_c) = h_1(L_c) + h_2(L_c) = h_1(L_c) + h_2(L_c) - 0 \\
&= h_1(L_c) + h_2(L_c) - h_2(T) = h_1(L_c) + h_2(L_c - T) \\
&\leq \|h_1\| \cdot \|L_c\| + \|h_2\| \cdot \|L_c - T\| < \|h_1\| \cdot \|L_c\| + \|h_2\| \cdot \|L_c\| \\
&= (\|h_1\| + \|h_2\|) \cdot \|L_c\| = \|h\| \cdot \|L_c\| = \|L_c\|,
\end{aligned}$$

a contradiction. So, $h_2 = 0$ and $h = h_1 \in \mathcal{K}(X,Y)^*$. The rest of the proof is similar to that of Theorem 3.2. ∎

4. Applications. Now, we present some applications of our main results, which generalize some results of [AAB].

THEOREM 4.1 ([AAB, Corollary 2.4]). *A compact operator $T \in \mathcal{L}(X)$ on a uniformly convex Banach space satisfies the Daugavet equation if and only if its norm $\|T\|$ is an eigenvalue of T.*

The next lemmas will be useful later in this section.

LEMMA 4.2 ([Lin2, Theorem 2.1]). *Let X be a normed space. Then $\|u_1 + \cdots + u_m\| = \|u_1\| + \cdots + \|u_m\|$ if and only if $\|\alpha_1 u_1 + \cdots + \alpha_m u_m\| = \|\alpha_1 u_1\| + \cdots + \|\alpha_m u_m\|$ for all $\alpha_1, \ldots, \alpha_m > 0$.*

LEMMA 4.3. *If $\|a_1 + \cdots + a_n\| = \|a_1\| + \cdots + \|a_n\|$ and $k \le n$, then $\|a_1 + \cdots + a_k\| = \|a_1\| + \cdots + \|a_k\|$.*

Proof. Observe that

$$\begin{aligned}\|a_1\| + \cdots + \|a_k\| + \cdots + \|a_n\| &= \|a_1 + \cdots + a_k + a_{k+1} + \cdots + a_n\| \\ &\le \|a_1 + \cdots + a_k\| + \|a_{k+1} + \cdots + a_n\| \\ &\le \|a_1 + \cdots + a_k\| + \|a_{k+1}\| + \cdots + \|a_n\| \\ &\le \|a_1\| + \cdots + \|a_k\| + \|a_{k+1}\| + \cdots + \|a_n\|,\end{aligned}$$

so $\|a_1+\cdots+a_k\|+\|a_{k+1}\|+\cdots+\|a_n\| = \|a_1\|+\cdots+\|a_k\|+\|a_{k+1}\|+\cdots+\|a_n\|$. It follows that $\|a_1 + \cdots + a_k\| = \|a_1\| + \cdots + \|a_k\|$. ∎

In particular, continuous operators $I, T_1, \ldots, T_n \colon X \to X$ on a Banach space satisfy the generalized Daugavet equation if and only if the family $\{I, T_1, \ldots, T_n\}$ has property (D). Now we apply Theorem 3.3 to generalize Theorem 4.1. In particular, we show that it is not necessary to assume that T is a compact operator. Moreover, we may assume that the space is strictly convex (and not necessarily uniformly convex).

THEOREM 4.4. *Let X, Y be Banach spaces with X reflexive and Y strictly convex. Let $T_1, \ldots, T_n \in \mathcal{L}(X, Y)$. Assume that $\mathcal{K}(X, Y)$ is an M-ideal in $\mathcal{L}(X, Y)$. Suppose that $\operatorname{dist}(T_1, \mathcal{K}(X, Y)) < \|T_1\|$. The following conditions are equivalent:*

(a) $\|T_1 + \cdots + T_n\| = \|T_1\| + \cdots + \|T_n\|$;
(b) *there is an $x_o \in \operatorname{Ext} B(X)$ such that $\frac{T_1}{\|T_1\|}x_o = \cdots = \frac{T_n}{\|T_n\|}x_o$ and $\|T_1 x_o\| = \|T_1\|, \ldots, \|T_n x_o\| = \|T_n\|$.*

Proof. To prove that (a) implies (b), suppose

$$\|T_1 + \cdots + T_n\| = \|T_1\| + \cdots + \|T_n\|,$$

and consider the family $\mathcal{F} := \{T_1, \ldots, T_n\}$. As in the proof of Lemma 4.3, we can show that $\mathcal{F}$ has property (D). By Theorem 3.3 this implies that

$$\|T_1 x_o + \ldots + T_n x_o\| = \|T_1 x_o\| + \ldots + \|T_n x_o\| \tag{4.1}$$

and

$$\|T_1 x_o\| = \|T_1\|, \ldots, \|T_n x_o\| = \|T_n\| \tag{4.2}$$

for some $x_o \in \operatorname{Ext} B(X)$. Fix k in $\{2, \ldots, n\}$. By (4.1) and Lemma 4.3, we have $\|Tx_1 + T_k x_o\| = \|T_1 x_o\| + \|T_k x_o\|$. Then by Lemma 4.2,

$$\left\| \frac{1}{2} \frac{T_1}{\|T_1\|} x_o + \frac{1}{2} \frac{T_k}{\|T_k\|} x_o \right\| = \left\| \frac{1}{2} \frac{T_1}{\|T_1\|} x_o \right\| + \left\| \frac{1}{2} \frac{T_k}{\|T_k\|} x_o \right\| = 1.$$

Thus $\frac{1}{2} \frac{T_1}{\|T_1\|} x_o + \frac{1}{2} \frac{T_k}{\|T_k\|} x_o \in S(Y)$. Since Y is strictly convex, this implies that

$$\frac{1}{2} \frac{T_1}{\|T_1\|} x_o = \frac{1}{2} \frac{T_k}{\|T_k\|} x_o. \tag{4.3}$$

Combining (4.2) and (4.3), we immediately get $\frac{T_1}{\|T_1\|} x_o = \cdots = \frac{T_n}{\|T_n\|} x_o$.

Conversely, assume that for some x_o in $\operatorname{Ext} B(X)$ we have $\frac{T_1}{\|T_1\|} x_o = \cdots = \frac{T_n}{\|T_n\|} x_o$ and $\|T_1 x_o\| = \|T_1\|, \ldots, \|T_n x_o\| = \|T_n\|$. Then

$$\begin{aligned} \left\| \frac{T_1}{\|T_1\|} x_o \right\| + \cdots + \left\| \frac{T_n}{\|T_n\|} x_o \right\| &= n \left\| \frac{T_1}{\|T_1\|} x_o \right\| = \left\| n \frac{T_1}{\|T_1\|} x_o \right\| \\ &= \left\| \frac{T_1}{\|T_1\|} x_o + \cdots + \frac{T_n}{\|T_n\|} x_o \right\|. \end{aligned}$$

So the result follows from Theorem 3.3 and Lemma 4.2. ■

Now, we are ready to present a generalization of Theorem 4.1. As an immediate consequence of Theorem 4.4, we have the following.

Theorem 4.5. *Let X be a reflexive and strictly convex* (*not necessarily uniformly*) *Banach space. Let $S_1, \ldots, S_n, T \in \mathcal{L}(X)$. Assume that $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$. Suppose that* $\operatorname{dist}(T, \mathcal{K}(X)) < \|T\|$. *The following conditions are equivalent:*

(a) $\|I + S_1 + \cdots + S_n + T\| = 1 + \|S_1\| + \cdots + \|S_n\| + \|T\|$;
(b) *there is an x_o in $S(X)$ such that $S_1 x_o = \|S_1\| \cdot x_o$, …, $S_n x_o = \|S_n\| \cdot x_o$ and $T x_o = \|T\| \cdot x_o$.*

Theorem 4.6 ([AAB, Corollary 2.5]). *A compact operator $T\colon l^p \to l^p$ $(1 < p < \infty)$ satisfies the Daugavet equation if and only if its norm $\|T\|$ is an eigenvalue of T.*

Again, it is not necessary to assume that T is a compact operator. Theorem 4.6 can be strengthened as follows.

Theorem 4.7. *Assume that $1 < p < \infty$. Let $S_1, \ldots, S_n, T \in \mathcal{L}(l^p)$. Suppose that* $\operatorname{dist}(T, \mathcal{K}(l^p)) < \|T\|$. *The following conditions are equivalent:*

(a) $\|I + S_1 + \cdots + S_n + T\| = 1 + \|S_1\| + \cdots + \|S_n\| + \|T\|$;
(b) *there is an x_o in $S(l^p)$ such that $S_1 x_o = \|S_1\| \cdot x_o$, …, $S_n x_o = \|S_n\| \cdot x_o$ and $T x_o = \|T\| \cdot x_o$.*

Proof. Since $\mathcal{K}(l^p)$ is an M-ideal in $\mathcal{L}(l^p)$, the result follows from Theorem 4.5. ■

5. Best approximation in spaces of continuous affine operators. As an application of Theorems 3.2 and 3.3 we consider the problem of best approximation. Assume that X, Y are Banach spaces. Let $\mathcal{M}$ be a linear subspace (not necessarily closed) of $\mathcal{AK}(X,Y)$. For $U \in \mathcal{AK}(X,Y)$ set

$$\mathcal{P}_{\mathcal{M}}(U) := \{V \in \mathcal{M} : \|U - V\| = \operatorname{dist}(V, \mathcal{M})\}.$$

Let $V_1 \in \mathcal{P}_{\mathcal{M}}(U)$. The operator V_1 (which need not be unique) is called an *element of best approximation* (briefly a *best approximation*) from $\mathcal{M}$ to U. In general, the problem of finding an element of best approximation effectively is complicated. For this reason in approximation theory the following two principal problems are posed:

(e) existence of best approximation ($\mathcal{P}_{\mathcal{M}}(U) \neq \emptyset$);
(u) uniqueness of best approximation ($\operatorname{card} \mathcal{P}_{\mathcal{M}}(U) = 1$).

The aim of this section is to present some results concerning problems (e) and (u) in the case of the space of all linear (or affine) continuous mappings from a Banach space X into a Banach space Y. It is clear that if $\mathcal{M}$ is a finite-dimensional subspace of $\mathcal{A}(X,Y)$, then each $U \in \mathcal{A}(X,Y)$ has a best approximation in $\mathcal{M}$.

Lewicki [Le] obtained characterization theorems of a best approximation operator from a finite-dimensional subspace of $\mathcal{K}(X,Y)$. We present a necessary condition for a subspace of $\mathcal{K}(X,Y)$ to be a non-Chebyshev subspace, which extends some results of [Le].

5.1. Examples and lemmas. We say that an affine operator $A\colon X \to Y$ has *property* (*INJ*) if

$$\ker A = \{0\}, \quad \text{where} \quad \ker A := \{x \in X : A(x) = 0\}.$$

For X, Y set

$$\mathcal{INJ}(X,Y) := \big\{A \in \mathcal{A}(X,Y) : \ker A = \{0\}\big\}.$$

Example 5.1. Let $T \in \mathcal{L}(X,Y)$. It is easy to see that $T \in \mathcal{INJ}(X;Y)$ $\Leftrightarrow T$ is injective.

Example 5.2. Let $T \in \mathcal{L}(X,Y)$ be injective. Let $S \in \mathcal{A}(X,Y)$ be such that $T(X) \cap S(X) = \{0\}$. It is straightforward to verify that if $\alpha \neq 0$, then $\alpha T + \beta S \in \mathcal{INJ}(X,Y)$.

First, we give some preliminary lemmas.

Lemma 5.3. *Suppose that* $(Y, \|\cdot\|)$ *is a strictly convex Banach space. If* $\|a+b\| = \|a\| + \|b\|$ *and* $\|a\| = \|b\|$, *then* $a = b$.

The proof is rather simple, so we omit it.

Lemma 5.4. *Let* X, Y *be Banach spaces with* X *reflexive and* Y *strictly convex. Let* $U, A \in \mathcal{AK}(X,Y)$. *Assume that*

(5.1) $$\|U - \alpha A\| = \|U\|$$

for some $\alpha > 1$. *If* A *has property* (*INJ*), *then* $\|U - A\| < \|U\|$.

Proof. First we will show that

(5.2) $$\|(\alpha - 1)U + U - \alpha A\| < \|(\alpha - 1)U\| + \|U - \alpha A\|.$$

Assume, contrary to our claim, that

$$\|(\alpha - 1)U + U - \alpha A\| = \|(\alpha - 1)U\| + \|U - \alpha A\|.$$

Then by Lemma 4.2 we also have

$$\|U + U - \alpha A\| = \|U\| + \|U - \alpha A\|.$$

We set $\mathcal{F} := \{U, U - \alpha A\}$. By Theorem 3.2 we have

(5.3) $$\|Ux_o\| = \|U\| \overset{(5.1)}{=} \|U - \alpha A\| = \|Ux_o - \alpha Ax_o\|$$

and

(5.4) $$\|Ux_o + Ux_o - \alpha Ax_o\| = \|Ux_o\| + \|Ux_o - \alpha Ax_o\|$$

for some $x_o \in \operatorname{Ext} B(X)$. Since Y is a strictly convex space, this implies that $Ux_o = Ux_o - \alpha Ax_o$ (see (5.3), (5.4) and Lemma 5.3). Thus $Ax_o = 0$. Then by property (*INJ*), we get $x_o = 0$, which is a contradiction. So, (5.2) is proved.

It follows that

$$\begin{aligned}\|U - A\| &= \frac{1}{\alpha}\|\alpha U - \alpha A\| = \frac{1}{\alpha}\|(\alpha - 1)U + U - \alpha A\| \\ &\overset{(5.2)}{<} \frac{1}{\alpha}((\alpha - 1)\|U\| + \|U - \alpha A\|) \\ &\overset{(5.1)}{=} \frac{1}{\alpha}((\alpha - 1)\|U\| + \|U\|) = \|U\|. \ \blacksquare\end{aligned}$$

By a reasoning similar to that for Lemma 5.4, using Theorem 3.3 (instead of 3.2), we can prove

LEMMA 5.5. *Let* X, Y *be Banach spaces with* X *reflexive and* Y *strictly convex. Assume that* $\mathcal{K}(X, Y)$ *is an* M*-ideal. Let* $U, L \in \mathcal{L}(X, Y)$. *Assume that* $\|U - \alpha L\| = \|U\|$ *for some* $\alpha > 1$. *Suppose that* $\operatorname{dist}(U, \mathcal{K}(X, Y)) < \|U\|$ *or* $\operatorname{dist}(U - \alpha L, \mathcal{K}(X, Y)) < \|U - \alpha L\|$. *If* L *is injective, then* $\|U - L\| < \|U\|$.

5.2. Uniqueness of best approximation. Now we formulate the main result of this section.

THEOREM 5.6. *Let* X, Y *be Banach spaces with* X *reflexive and* Y *strictly convex. Let* $\mathcal{M}$ *be a linear subspace of* $\mathcal{A}(X, Y)$ *with* $\mathcal{M} \subseteq \mathcal{INJ}(X, Y) \cup \{0\}$. *Assume that* $U, V \in \mathcal{AK}(X, Y)$ *and* $V \in \mathcal{P}_{\mathcal{M}}(U)$. *Then* $\mathcal{P}_{\mathcal{M}}(U) = \{V\}$.

Proof. Assume, contrary to our claim, that there exists $V_1 \in \mathcal{P}_{\mathcal{M}}(U)$ such that $V \neq V_1$. So, we have

(5.5) $$\|U - V\| = \operatorname{dist}(U, \mathcal{M}) = \|U - V_1\|.$$

Fix $\alpha \in (1,\infty)$. In view of (5.5), we have

$$\|U - V\| = \left\|U - V - \alpha \cdot \frac{1}{\alpha}(V_1 - V)\right\|. \tag{5.6}$$

Define $\widehat{U}, \widehat{A}\colon X \to Y$ by $\widehat{U} := U - V$, $\widehat{A} := \frac{1}{\alpha}(V_1 - V)$. Now (5.6) becomes

$$\|\widehat{U}\| = \|\widehat{U} - \alpha\widehat{A}\|.$$

It is a straightforward verification that $\widehat{A} \neq 0$ and $\widehat{A} \in \mathcal{INJ}(X,Y)$. By Lemma 5.4, $\|\widehat{U} - \widehat{A}\| < \|\widehat{U}\|$. It follows that

$$\left\|U - V - \frac{1}{\alpha}(V_1 - V)\right\| < \|U - V\|. \tag{5.7}$$

In particular, $V + \frac{1}{\alpha}(V_1 - V) \in \mathcal{M}$, which yields

$$\operatorname{dist}(U,\mathcal{M}) \le \left\|U - \left(V + \frac{1}{\alpha}(V_1 - V)\right)\right\| \overset{(5.7)}{<} \|U - V\|,$$

and so $V \notin \mathcal{P}_{\mathcal{M}}(U)$, a contradiction. ∎

By a reasoning similar to that for Theorem 5.6, using Lemma 5.5 (instead of 5.4), we can prove

Theorem 5.7. *Let X, Y be Banach spaces with X reflexive and Y strictly convex. Assume that $\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$. Let $\mathcal{M}$ be a linear subspace of $\mathcal{L}(X,Y)$ with $\mathcal{M} \subseteq \mathcal{INJ}(X,Y) \cup \{0\}$. Assume that $U \in \mathcal{L}(X,Y)$, $V \in \mathcal{P}_{\mathcal{M}}(U)$ and $\operatorname{dist}(U - V, \mathcal{K}(X,Y)) < \|U - V\|$. Then $\mathcal{P}_{\mathcal{M}}(U) = \{V\}$.*

5.3. Chebyshev subspaces in the space of compact operators. Applying Theorem 5.7 we can prove a sufficient condition for $\mathcal{M}$ to be a Chebyshev subspace. Let $\mathcal{B}$ be a Banach space. Recall that a subspace $\mathcal{M} \subset \mathcal{B}$ is called a *Chebyshev subspace* if $\operatorname{card}\mathcal{P}_{\mathcal{M}}(U) = 1$ for every $U \in \mathcal{B}$. In this subsection we will be concerned with the case of $\mathcal{B} = \mathcal{K}(X,Y)$.

Theorem 5.8. *Let X, Y be Banach spaces with X reflexive and Y strictly convex. Assume that $\mathcal{K}(X,Y)$ is an M-ideal. Suppose that $\mathcal{M} \subset \mathcal{K}(X,Y)$ is a finite-dimensional subspace. Assume $\ker T = \{0\}$ for all $T \in \mathcal{M} \setminus \{0\}$. Then $\mathcal{M}$ is a Chebyshev subspace.*

Proof. Fix $U \in \mathcal{K}(X,Y)$. Since $\dim \mathcal{M} < \infty$, we have $\mathcal{P}_{\mathcal{M}}(U) \neq \emptyset$. Then, using Theorem 5.6 we conclude that $\operatorname{card}\mathcal{P}_{\mathcal{M}}(U) = 1$. ∎

Applying Theorem 5.8 we can prove a necessary condition for $\mathcal{M}$ to be a non-Chebyshev subspace.

Corollary 5.9. *Let X, Y be as in Theorem 5.8. Assume $\mathcal{M} \subset \mathcal{K}(X,Y)$ is a non-Chebyshev finite-dimensional subspace. Then there exists $T \in \mathcal{M} \setminus \{0\}$ such that T is not injective.*

To end this paper we present a characterization of one-dimensional Chebyshev subspaces in $\mathcal{K}(H_1, H_2)$.

Proposition 5.10. *Suppose that* H_1, H_2 *are Hilbert spaces such that* $\dim H_1 < \infty$ *and* $\dim H_1 \le \dim H_2$. *If* $T \in \mathcal{K}(H_1, H_2) \setminus \{0\}$ *then* $\operatorname{span}\{T\}$ *is a Chebyshev subspace if and only if* $\ker T = \{0\}$.

Proof. To prove "$\Rightarrow$", it may be assumed that $\|T\| = 1$. Assume, contrary to our claim, that $\ker T \neq \{0\}$, so there exists $z \in \ker T$ such that $\|z\| = 1$. A straightforward computation gives $H_1 = \{z\}^\perp \oplus \operatorname{span}\{z\}$. Fix c in $T(\{z\}^\perp)^\perp$ such that $\|c\| = 1$. Define $U \in \mathcal{K}(H_1, H_2)$ by

$$U(k + \alpha z) := T(k) + \alpha c$$

for $k + \alpha z$ in $H_1 = \{z\}^\perp \oplus \operatorname{span}\{z\}$.

Now if $k + \alpha z \in H_1 = \{z\}^\perp \oplus \operatorname{span}\{z\}$ (i.e., $k \perp \alpha z$), then

$$\begin{aligned}\|(U - T)(k + \alpha z)\| &= \|\alpha c\| = |\alpha| = \|\alpha z\| \\ &= \sqrt{\langle \alpha z \,|\, \alpha z\rangle} \le \sqrt{\langle k + \alpha z \,|\, k + \alpha z\rangle} = \|k + \alpha z\|.\end{aligned}$$

Thus $\|U - T\| \le 1$. In fact, $\|(U - T)z\| = \|c\| = 1$, so that $\|U - T\| = 1$.

Now if $k + \alpha z \in H_1 = \{z\}^\perp \oplus \operatorname{span}\{z\}$, then

$$\begin{aligned}\|U(k + \alpha z)\|^2 &= \|Tk + \alpha c\|^2 = \|Tk\|^2 + \|\alpha c\|^2 \\ &\le \|k\|^2 + \|\alpha z\|^2 = \|k + \alpha z\|^2.\end{aligned}$$

Thus $\|U\| \le 1$. In fact, $\|Uz\| = \|c\| = 1$, so that $\|U\| = 1$. Fix $\lambda \in \mathbb{K}$; then

$$\|U - T\| = \|U\| = \|U - 0\| = 1 = \|c - 0\| = \|Uz - \lambda Tz\| \le \|U - \lambda T\|.$$

Thus $T, 0 \in \mathcal{P}_{\operatorname{span}\{T\}}(U)$, which contradicts the fact that $\operatorname{span}\{T\}$ is a Chebyshev subspace.

The converse statement is immediate from Theorem 5.8. ∎

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Paweł Wójcik
Institute of Mathematics
Pedagogical University of Cracow
Podchorążych 2
30-084 Kraków, Poland
E-mail: pwojcik@up.krakow.pl