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## PROJECTIONS IN UNIFORM POLYNOMIAL APPROXIMATIONS

**1. Introduction.** Let  $C_{\langle a, b \rangle}$  denote the space of real and continuous functions in the interval  $\langle a, b \rangle$  and let  $W_n$  be the class of algebraic polynomials of degree at most  $n$  ( $n > 0$ ). For the function  $f \in C_{\langle a, b \rangle}$  we define, in the ordinary way, the norm

$$\|f\| = \max_{a \leq x \leq b} |f(x)|$$

and the  $n$ -th error of optimal approximation as

$$\varepsilon_n(f) = \inf_{w_n \in W_n} \|f - w_n\|.$$

The polynomial  $w_n^* \in W_n$  satisfying the equality  $\varepsilon_n(f) = \|f - w_n^*\|$  will be called the  $n$ -th *optimal polynomial* for the function  $f$ . With regard to the complicated dependence of the optimal polynomial upon the function  $f$  it was possible to find such polynomials only for a few functions. In this connection, in practice we often resigned of the optimal polynomial and are seeking a polynomial well approximating the function  $f \in C_{\langle a, b \rangle}$ .

The present paper concerns operators defined on the space  $C_{\langle a, b \rangle}$  the values of which are polynomials well approximating functions in a certain precisely defined sense.

In section 2, these operators and some their properties are defined. Sections 3, 4 and 5 are devoted to interpolating operators. In sections 6 and 7, the construction of some discrete non-interpolating operators is described. In section 8, examples of these operators are given. The last section contains some unsolved problems.

**2. Definition and properties of a projection.** Let  $L_n$  be a linear operator such that

$$(2.1) \quad L_n f = w_n \quad (f \in C_{\langle a, b \rangle}, n > 0, w_n \in W_n)$$

and

$$(2.2) \quad L_n w_n \equiv w_n$$

for every  $w_n \in W_n$ .

The linear operator satisfying conditions (2.1) and (2.2) will be called a *projection*.

Remark. The operator, whose value is the  $n$ -th optimal polynomial, satisfies condition (2.2) but is not linear. One uses in the approximation theory the so-called *Bernstein operators*, denoted by  $B_n$ , and

$$B_n f(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} \quad (x \in \langle 0, 1 \rangle).$$

They are obviously linear and such that, for every  $x \in \langle 0, 1 \rangle$ , there is

$$\lim_{n \rightarrow \infty} B_n f(x) = f(x)$$

but, for example, they do not fulfil (2.2) for  $w_2(x) = x^2$ .

We quote without a proof a known theorem (see [1]) which gives the quality information about the approximation of the function  $f \in C_{\langle a, b \rangle}$  with help of the projection  $L_n$ , satisfying conditions (2.1) and (2.2). This theorem follows also from F. Riesz's theorem about the general form of the linear functional defined on the space  $C_{\langle a, b \rangle}$ .

**THEOREM 2.1.** *Let  $L_n$  ( $n > 0$ ) be a projection such that  $L_n: C_{\langle a, b \rangle} \rightarrow W_n$ . For the function  $f \in C_{\langle a, b \rangle}$ , the operator  $L_n$  is expressed by the formula*

$$(2.3) \quad L_n f(x) = \sum_{k=0}^n x^k \int_a^b f(t) dg_k(t) = \int_a^b f(t) d \sum_{k=0}^n x^k g_k(t),$$

where  $g_k$  ( $k = 0, 1, \dots, n$ ) are functions with bounded variation in the interval  $\langle a, b \rangle$  such that

$$(2.4) \quad \int_a^b t^j dg_k(t) = \delta_{jk} \quad (j, k = 0, 1, \dots, n),$$

where  $\delta_{jk}$  is Kronecker's delta. The equality

$$(2.5) \quad \|L_n\| = \|A_n\|,$$

where

$$(2.6) \quad A_n(x) = \text{Var}_{\langle a, b \rangle} \sum_{k=0}^n x^k g_k(t)$$

and  $\|L_n\| \geq 1$ , holds. For every function  $f \in C_{\langle a, b \rangle}$ , the inequality

$$(2.7) \quad \|f - L_n f\| \leq (1 + \|L_n\|) \varepsilon_n(f)$$

holds.

The function  $A_n$  (the symbol  $\text{Var}_{\langle a, b \rangle} \varphi(t)$  denotes the total variation of the function  $\varphi$  in the interval  $\langle a, b \rangle$ ) is called the *Lebesgue function* for the operator  $L_n$ .

From Daugavet's theorem (see [3]) it follows that the quantity  $1 + \|L_n\|$  occurring in expression (2.7) is the smallest. In connection with this, the following definition of a minimal projection is justified:

**Definition.** The projection with the minimal norm for a fixed  $n$  is called *minimal*.

Up to now the minimal operator has not been found and the problem seems to be difficult to solve. The problem simplifies somewhat if we restrict ourselves to a certain subclass of projections which will be dealt with later. At present, we prove a certain invariability property of the projection satisfying conditions (2.1) and (2.2). Before, we introduce a certain additional notation.

Let  $\bar{L}_n$  be a projection such that  $\bar{L}_n: C_{\langle c, d \rangle} \rightarrow W_n$  ( $n > 0$ ,  $\langle c, d \rangle \neq \langle a, b \rangle$ ). In virtue of (2.3),

$$(2.8) \quad \bar{L}_n h(y) = \int_c^d h(u) d \sum_{k=0}^n y^k \bar{g}_k(u) \quad (y \in \langle c, d \rangle, h \in C_{\langle c, d \rangle}),$$

where  $\bar{g}_k(u)$  ( $k = 0, 1, \dots, n$ ) are functions with bounded variation in the interval  $\langle c, d \rangle$ .

**THEOREM 2.2.** *If the functions  $g_k(t)$  and  $\bar{g}_k(u)$  ( $t \in \langle a, b \rangle$ ,  $u \in \langle c, d \rangle$ ,  $k = 0, 1, \dots, n$ ) are such that*

$$g_k(t) = g_k \frac{u - \beta}{a} = \bar{g}_k(u),$$

where  $a$  and  $\beta$  are constants such that the transformation  $u = at + \beta$  converts the interval  $\langle a, b \rangle$  into the interval  $\langle c, d \rangle$ , then operators (2.3) and (2.8) have identical norms.

**Proof.** For simplicity, we may assume that the transformation  $u = at + \beta$  converts the point  $a$  in the point  $c$  and the point  $b$  in the point  $d$ , i. e. that  $aa + \beta = c$  and  $ab + \beta = d$  hold.  $y = ax + \beta$  and  $u = at + \beta$  are affine transformations such that if  $x, t \in \langle a, b \rangle$ , then  $y, u \in \langle c, d \rangle$ . Because  $u = at + \beta$ ,

$$g_k(t) = g_k \left( \frac{u - \beta}{a} \right) = \bar{g}_k(u)$$

(where the functions  $\bar{g}_k$  are the same as in (2.8)) and

$$\text{Var}_{\langle a, b \rangle} g_k(t) = \text{Var}_{\langle c, d \rangle} \bar{g}_k(u) \quad (k = 0, 1, \dots, n).$$

Hence

$$\|\bar{L}_n\| = \max_{c \leq y \leq d} \text{Var}_{\langle c, d \rangle} \sum_{k=0}^n y^k \bar{g}_k(u) = \max_{a \leq x \leq b} \text{Var}_{\langle a, b \rangle} \sum_{k=0}^n x^k g_k(t) = \|L_n\|.$$

As an example of a projection, take the operator  $L_n$  which will be called *discrete*:

$$(2.9) \quad L_n f(x) = \sum_{l=0}^p f(x_l) \lambda_l(x) \quad (\lambda_l \in W_n, x_l \in \langle a, b \rangle, p \geq n).$$

Conditions (2.2) for operator (2.9) can now be written in the form

$$\sum_{l=0}^p x_l^j \lambda_l(x) \equiv x^j \quad (j = 0, 1, \dots, n)$$

and the respective Lebesgue function (denoted by  $A_n$ , as above) will be expressed by the formula

$$(2.10) \quad A_n(x) = \sum_{l=0}^p |\lambda_l(x)| \quad (x \in \langle a, b \rangle).$$

The simplest example of the discrete operator, for  $p = n$ , is the so-called *interpolating operator* whose value for the function  $f$  is the interpolation polynomial with nodes  $x_l$ ,

$$(2.11) \quad L_n f(x) = \sum_{l=0}^n f(x_l) \lambda_l(x),$$

where

$$(2.12) \quad \lambda_l(x) = \prod_{\substack{i=0 \\ i \neq l}}^n \frac{x - x_i}{x_l - x_i} \quad (x_l \in \langle a, b \rangle, l = 0, 1, \dots, n).$$

Because of theorem 2.2 we assume in what follows that  $\langle a, b \rangle = \langle -1, 1 \rangle$ .

Considering interpolating operators we assume that the nodes of the interpolation  $x_0, x_1, \dots, x_n$  are such that  $-1 \leq x_0 < x_1 < \dots < x_n \leq 1$  and, in addition, that  $x_{-1} = -1$ ,  $x_{n+1} = 1$ . It is known that, for  $x \in \langle x_k, x_{k+1} \rangle$  ( $k = -1, 0, \dots, n$ ),

$$(2.13) \quad \text{sign } \lambda_l(x) = \begin{cases} (-1)^{l+k+1} & (k < l), \\ (-1)^{l+k} & (k \geq l). \end{cases}$$

Remark. The particular version of theorem 2.2 for the interpolating operators was proved by Luttmann and Rivlin [4].

**3. Properties of the Lebesgue function for the interpolating operator.** The need of investigation of the function  $A_n$  follows from expression (2.7) and equality (2.5).

Notice that if  $n = 1$  and  $-x_0 = x_1 = 1$ , then from (2.12) and (2.10) (for  $p = n$ ) we obtain  $A_1(x) \equiv 1$  ( $x \in \langle -1, 1 \rangle$ ); therefore, in this case

the interpolating operator with the above-mentioned nodes is minimal.

We assume in what follows that  $n > 1$ .

We give certain elementary properties of the Lebesgue function.

PROPERTY 1.  $A_n(x_k) = 1$  ( $k = 0, 1, \dots, n$ ).

PROPERTY 2. For  $x \in \langle -1, x_0 \rangle$  the function  $A_n$  is convex and strictly decreasing, and for  $x \in \langle x_n, 1 \rangle$  the function  $A_n$  is convex and strictly increasing. In every of the intervals  $\langle x_k, x_{k+1} \rangle$  ( $k = 0, 1, \dots, n-1$ ), the function  $A_n$  is strictly concave. Hence, in every of the intervals  $\langle x_k, x_{k+1} \rangle$  ( $k = -1, 0, \dots, n$ ), the Lebesgue function has exactly one local maximum.

PROPERTY 3. The equality  $A_n(x) = A_n(-x)$  holds if and only if  $-x_k = x_{n-k}$  ( $x \in \langle -1, 1 \rangle$ ;  $k = 0, 1, \dots, n$ ).

From (2.5) it follows that the norm of the interpolating operator depends upon the local maxima of the Lebesgue function connected with this operator, and these, in turn, depend upon the nodes  $x_k$  ( $k = 0, 1, \dots, n$ ). In this connection, it is worth to investigate how the change of nodes influences the change of the maxima of the function  $A_n$ . In the next section we prove theorems describing the change of local maxima of the Lebesgue function caused by the shift of one node. Subsequently, it will be given a theorem about the increments of the maxima of the function  $A_n$  in the case where all nodes are locally shifted.

#### 4. Theorems about the Lebesgue function. Let

$$X = \{x_0, x_1, \dots, x_n\}, \quad \text{where } -1 \leq x_0 < x_1 < \dots < x_n \leq 1,$$

$$\bar{X} = \{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_n\}, \quad \text{where } -1 \leq \bar{x}_0 < \bar{x}_1 < \dots < \bar{x}_n \leq 1,$$

and, for fixed  $k = 0, 1, \dots, n$ ,

$$x_j = \bar{x}_j \quad \text{for } j = 0, 1, \dots, k-1, k+1, \dots, n \text{ and } x_k \neq \bar{x}_k.$$

Let  $\lambda_l$  and  $\bar{\lambda}_l$  ( $l = 0, 1, \dots, n$ ) denote polynomials (2.12) formed for the sets  $X$  and  $\bar{X}$ , respectively, and let  $A_n$  and  $\bar{A}_n$  denote suitable Lebesgue functions.

We take the intervals  $X_p = \langle x_p, x_{p+1} \rangle$  and  $\bar{X}_p = \langle \bar{x}_p, \bar{x}_{p+1} \rangle$  and define numbers  $e_p$  and  $\bar{e}_p$  ( $p = -1, 0, \dots, n$ ):

$$e_p = \max_{x \in X_p} A_n(x) \quad \text{and} \quad \bar{e}_p = \max_{x \in \bar{X}_p} \bar{A}_n(x).$$

We also define the sets of indices  $P_p$  and  $Q_p$  ( $p = -1, 0, \dots, n$ ):

$$P_p = \{l \mid \lambda_l(x) < 0, \quad l = 0, 1, \dots, p; \quad x \in X_p\},$$

$$Q_p = \{l \mid \lambda_l(x) < 0, \quad l = p+1, p+2, \dots, n; \quad x \in X_p\}.$$

In virtue of equality (2.13), we obtain

$$P_p = \begin{cases} \{1, 3, \dots, p-1\} & \text{for } p \text{ even,} \\ \{0, 2, \dots, p-1\} & \text{for } p \text{ odd,} \end{cases} \quad (4.1)$$

$$Q_p = \begin{cases} \{p+2, p+4, \dots, n\} & \text{for } p \text{ and } n \text{ both even or both odd,} \\ \{p+2, p+4, \dots, n-1\} & \text{in the opposite case.} \end{cases}$$

Let

$$N_p = P_p \cup Q_p. \quad (4.2)$$

We now define the auxiliary function

$$\Phi_n = \frac{1}{2}(\Lambda_n - \bar{\Lambda}_n). \quad (4.3)$$

We will use this function in the proof of theorems; for example, having verified that in a certain interval  $\Phi_n(x) > 0$  holds, we conclude that in this interval the maximum of the function  $\Lambda_n$  is greater than the maximum of the function  $\bar{\Lambda}_n$ .

Because of

$$\sum_{l=0}^n \lambda_l(x) = \sum_{l=0}^n \bar{\lambda}_l(x) = 1,$$

we have

$$\Phi_n(x) = \frac{1}{2} \sum_{l=0}^n |\lambda_l(x)| - \frac{1}{2} \sum_{l=0}^n |\bar{\lambda}_l(x)| - \frac{1}{2} \sum_{l=0}^n [\lambda_l(x) - \bar{\lambda}_l(x)].$$

For  $x \in X_p \cap \bar{X}_p$  we obtain

$$\Phi_n(x) = \sum_{l \in N_p} [\bar{\lambda}_l(x) - \lambda_l(x)]. \quad (4.4)$$

**THEOREM 4.1.** *If, for fixed  $k$  ( $k = 0, 1, \dots, n$ ), the inequality  $\bar{x}_k < x_k$  ( $x_k < \bar{x}_k$ , respectively) holds, then  $\bar{e}_{k-1} < e_{k-1}$  and  $e_k < \bar{e}_k$  ( $e_{k-1} < \bar{e}_{k-1}$  and  $\bar{e}_k < e_k$ , respectively).*

**Proof.** We verify the thesis of the theorem in the case where  $\bar{x}_k < x_k$  (by changing the notation we obtain the case where  $x_k < \bar{x}_k$ ). At first, we prove that the inequality  $e_k < \bar{e}_k$  holds.

Let

$$a_{lk}(x) = \prod_{\substack{i=0 \\ i \neq l, k}}^n \frac{x - x_i}{x_l - x_i} \quad (l = 0, 1, \dots, n; l \neq k).$$

Then

$$\lambda_l(x) = a_{lk}(x) \frac{x - x_k}{x_l - x_k}, \quad \bar{\lambda}_l(x) = a_{lk}(x) \frac{x - \bar{x}_k}{x_l - \bar{x}_k}.$$

From (4.4) we obtain, for  $x \in X_k$ ,

$$\Phi_n(x) = \sum_{l \in N_k} a_{lk}(x) \left( \frac{x - \bar{x}_k}{x_l - \bar{x}_k} - \frac{x - x_k}{x_l - x_k} \right) = \sum_{l \in N_k} a_{lk}(x) b_{lk}(x),$$

where

$$b_{lk}(x) = \frac{(x - x_l)(\bar{x}_k - x_k)}{(x_l - x_k)(x_l - \bar{x}_k)}.$$

From (4.1) we obtain, for  $x \in X_k$ ,

$$(4.5) \quad \text{sign } b_{lk}(x) = \begin{cases} -1 & \text{for } x \in P_k, \\ 1 & \text{for } x \in Q_k. \end{cases}$$

Because

$$a_{lk}(x) = \frac{x_l - x_k}{x - x_k} \lambda_l(x)$$

and, on the other hand, for  $x \in X_k$  and  $l \in N_k$ , we have  $\lambda_l(x) < 0$  and

$$\text{sign}(x_l - x_k) = \begin{cases} -1 & \text{for } l \in P_k, \\ 1 & \text{for } l \in Q_k, \end{cases}$$

thus

$$\text{sign } a_{lk}(x) = \begin{cases} 1 & \text{for } l \in P_k, \\ -1 & \text{for } l \in Q_k. \end{cases}$$

Hence and from (4.5) it follows that  $a_{lk}(x)b_{lk}(x) \leq 0$ , whence  $\Phi_n(x) \leq 0$  (the equality holds for  $x = x_{k+1}$  only), and also (4.3) implies

$$\max_{x \in X_k} \Lambda_n(x) < \max_{x \in X_k} \bar{\Lambda}_n(x).$$

Because  $X_k \subset \bar{X}_k$ , we have

$$\max_{x \in X_k} \Lambda_n(x) < \max_{x \in \bar{X}_k} \bar{\Lambda}_n(x).$$

Hence, in view of our notation, we have

$$(4.6) \quad e_k < \bar{e}_k.$$

The proof of the inequality  $\bar{e}_{k-1} < e_{k-1}$  can be brought to the previous one. In fact, it suffices to take the set  $\{-x_n, -x_{n-1}, \dots, -x_{k+1}, -\bar{x}_k$ ,

$-x_{k-1}, \dots, -x_0\}$  instead of the set  $X$  and the set  $\{-x_n, -x_{n-1}, \dots, -x_0\}$  instead of  $\bar{X}$ . Such a choice of sets  $X$  and  $\bar{X}$  and inequality (4.6) imply the inequality  $\bar{e}_{n-k-1} < e_{n-k-1}$ . If we put  $k$  in the place of  $n-k$  and take into account theorem 2.2 (for the transformation  $u = -t$ ), we obtain  $\bar{e}_{k-1} < e_{k-1}$ .

An immediate consequence of theorem 4.1 is the following

**CONCLUSION 4.1.** *If all local maxima of the Lebesgue function for the interpolating operator are identical, then the change of one of them increases the norm of the operator.*

Let us call a *locally minimal operator* the operator with nodes  $x_k^*$  which has the smallest norm among all operators with nodes  $x_k$  belonging to sufficiently small neighbourhoods of corresponding nodes  $x_k^*$ .

From conclusion 4.1 it follows at once

**CONCLUSION 4.2.** *If all local maxima of the Lebesgue function for the interpolating operator are identical, then this operator is locally minimal in the class of the interpolating operators.*

**Proof.** The change of one of the nodes causes an increase of the norm of the operator. Further, local changes of all nodes cause that the norm of the operator does not return to the initial value, because all local maxima of the Lebesgue function depend in a continuous way upon the nodes  $x_k$  ( $k = 0, 1, \dots, n$ ).

In the case where the shifted node is  $x_0$  ( $x_n$ , respectively), it is possible to obtain additional information about the change of the local maxima of the function  $A_n$  in the intervals  $\langle X_{n-1}, X_n \rangle$  ( $\langle X_{-1}, X_0 \rangle$ , respectively). This follows from the following theorem concerning the node  $x_0$ :

**THEOREM 4.2.** *If  $\bar{x}_0 < x_0$  ( $x_0 < \bar{x}_0$ , respectively), then  $\bar{e}_n < e_n$  and  $\bar{e}_{n-1} < e_{n-1}$  ( $e_n < \bar{e}_n$  and  $e_{n-1} < \bar{e}_{n-1}$ , respectively).*

**Proof.** Similarly as in theorem 4.1, we verify the thesis of the theorem in the case of shifting the node  $x_0$  to the left, i. e. when  $\bar{x}_0 < x_0$ . At first, we prove the inequality  $\bar{e}_n < e_n$ . Putting  $p = n$  from (4.1) and (4.2), we obtain

$$N_n = \begin{cases} \{1, 3, \dots, n-1\} & \text{for } n \text{ even,} \\ \{0, 2, \dots, n-1\} & \text{for } n \text{ odd.} \end{cases}$$

For  $l = 0$ , we have

$$\begin{aligned} \bar{\lambda}_0(x) - \lambda_0(x) &= \prod_{i=1}^n (x - x_i) \left[ \prod_{i=1}^n \frac{1}{\bar{x}_0 - x_i} - \prod_{i=1}^n \frac{1}{x_0 - x_i} \right] \\ &= (-1)^{n+1} \prod_{i=1}^n \frac{x - x_i}{(x_i - x_0)(x_i - \bar{x}_0)} \left[ \prod_{i=1}^n (x_i - \bar{x}_0) - \prod_{i=1}^n (x_i - x_0) \right]. \end{aligned}$$



Hence

$$(4.7) \quad \bar{\lambda}_0(x) - \lambda_0(x) = -\lambda_0(x) \prod_{i=1}^n \frac{1}{x_i - \bar{x}_0} \left[ \prod_{i=1}^n (x_i - \bar{x}_0) - \prod_{i=1}^n (x_i - x_0) \right].$$

For  $l = 1, 2, \dots, n$ , we have

$$(4.8) \quad \begin{aligned} \bar{\lambda}_l(x) - \lambda_l(x) &= \frac{(\bar{x}_0 - x_0)(x - x_l)}{(x_l - \bar{x}_0)(x_l - x_0)} \prod_{\substack{i=1 \\ i \neq l}}^n \frac{x - x_i}{x_l - x_i} \\ &= \frac{(\bar{x}_0 - x_0)(x - x_l)}{(x - x_0)(x_l - x_0)} \lambda_l(x). \end{aligned}$$

Using (2.13), and (4.7) and (4.8) for  $x \in X_n$  and  $l \in n$ , we obtain

$$\text{sign}[\bar{\lambda}_l(x) - \lambda_l(x)] = \begin{cases} 0 & \text{for } x = x_n, \\ 1 & \text{for } x \in (x_n, 1). \end{cases}$$

Hence, in view of (4.4), we obtain the inequality  $\bar{e}_n < e_n$ .

For the proof of the inequality  $\bar{e}_{n-1} < e_{n-1}$ , we put  $p = n - 1$  in (4.1). Hence and from (4.2) we infer that

$$N_{n-1} = \begin{cases} \{0, 2, \dots, n-2\} & \text{for } n \text{ even,} \\ \{1, 3, \dots, n-2\} & \text{for } n \text{ odd.} \end{cases}$$

Using (2.13), and (4.7) and (4.8) for  $x \in X_{n-1}$  and  $l \in N_{n-1}$ , we obtain

$$\text{sign}[\bar{\lambda}_l(x) - \lambda_l(x)] = \begin{cases} 0 & \text{for } x = x_{n-1} \text{ and } x = x_n, \\ 1 & \text{for } x \in (x_{n-1}, x_n). \end{cases}$$

Finally, we have  $\bar{e}_{n-1} < e_{n-1}$ .

Theorems 4.1 and 4.2 describe, in particular, the change of local maxima of the Lebesgue function in the intervals  $\langle -1, x_0 \rangle$ ,  $\langle x_0, x_1 \rangle$ ,  $\langle x_{n-1}, x_n \rangle$ ,  $\langle x_n, 1 \rangle$  while shifting the node  $x_0$  to the left (right).

Now, we prove that without additional assumptions about the distribution of nodes  $x_i$  ( $i = 0, 1, \dots, n$ ) it is not possible to obtain information about the change of local maxima in the remaining intervals  $X_p$  ( $p = 1, 2, \dots, n-2$ ).

**THEOREM 4.3.** For  $\bar{x}_0 < x_0$  ( $x_0 < \bar{x}_0$ , respectively),  $n \geq 3$ , arbitrary nodes  $x_0, x_1, \dots, x_{p-2}, x_p, x_{p+1}, \dots, x_n$ , sufficiently small number  $\varepsilon > 0$  and for every node  $x_{p-1}$  such that  $x_p - x_{p-1} \leq \varepsilon$ , we have  $\bar{e}_p < e_p$  ( $e_p < \bar{e}_p$ , respectively). For  $\bar{x}_0 < x_0$  ( $x_0 < \bar{x}_0$ , respectively),  $n \geq 3$ , arbitrary nodes  $x_0, x_1, \dots, x_p, x_{p+1}, x_{p+3}, \dots, x_n$ , sufficiently small number  $\varepsilon > 0$  and for every node  $x_{p+2}$  such that  $x_{p+2} - x_{p+1} \leq \varepsilon$ , we have  $e_p < \bar{e}_p$  ( $e_p < \bar{e}_p$ , respectively).

This theorem states that a shift of the node  $x_0$  to the left causes a decrease of the local maxima  $e_p$  ( $p = 1, 2, \dots, n-2$ ) if the nodes  $x_{p-1}$  and  $x_p$  are lying near each other (in relation to the relative situation of the other nodes) or an increase of those maxima if the nodes  $x_{p+1}$  and  $x_{p+2}$  are lying near each other (in relation to the relative situation of the other nodes).

**Proof of theorem 4.3.** We prove the thesis of the theorem in the case where  $\bar{x}_0 < x_0$ . In the presence of (4.7) and (4.8), for  $x \in X_p$  ( $p = 1, 2, \dots, n-2$ ), there holds

$$(4.9) \quad \text{sign}[\bar{\lambda}_l(x) - \lambda_l(x)] = \begin{cases} 1 & \text{for } l \in P_p, \\ -1 & \text{for } l \in Q_p. \end{cases}$$

From (4.1) it is seen that  $p-1 \in P_p$ . For the proof of the first part of the theorem, it suffices to verify that, for  $x \in X_p$  and  $l \in Q_p$  ( $p = 1, 2, \dots, n-2$ ), the differences  $\bar{\lambda}_l(x) - \lambda_l(x)$  are bounded absolutely, the difference  $\bar{\lambda}_{p-1}(x) - \lambda_{p-1}(x)$ , however, is in this interval sufficiently large. From this and from (4.9), we infer, for  $x \in X_p$ , that  $\Phi_n(x) \geq 0$  and, consequently, that  $\bar{e}_p < e_p$ .

From (4.8), for  $l \in Q_p$ , we obtain

$$|\bar{\lambda}_l(x) - \lambda_l(x)| = \frac{x_0 - \bar{x}_0}{x_l - \bar{x}_0} \prod_{i=1}^n |x - x_i| \prod_{\substack{i=0 \\ i \neq l}}^n \frac{1}{|x_l - x_i|}.$$

Let

$$\alpha = \min_{\substack{0 \leq i \leq n \\ l \in Q_p, i \neq l}} |x_l - x_i|.$$

Hence, for  $l \in Q_p$ ,  $|x_l - x_i| \geq \alpha$  ( $i = 0, 1, \dots, n$ ;  $i \neq l$ ) holds, and thus  $x_0 - \bar{x}_0 < x_l - \bar{x}_0$ . For  $x \in X_p$ , we have  $|x - x_i| < 2$  ( $i = 1, 2, \dots, n$ ), thus, for  $l \in Q_p$ ,

$$(4.10) \quad |\bar{\lambda}_l(x) - \lambda_l(x)| < \left(\frac{2}{\alpha}\right)^n$$

holds.

Now we prove that, for  $x \in X_p$ , the difference  $\bar{\lambda}_{p-1}(x) - \lambda_{p-1}(x)$  is sufficiently large. For  $p = 1$ , in virtue of (4.7), we obtain

$$\bar{\lambda}_0(x) - \lambda_0(x) = (-1)^{n+1} \prod_{i=1}^n (x - x_i) \left( \prod_{i=1}^n \frac{1}{x_i - x_0} - \prod_{i=1}^n \frac{1}{x_i - \bar{x}_0} \right).$$

Because of

$$(-1)^{n+1} \prod_{i=1}^n (x - x_i) \geq 0 \quad \text{for } x \in X_1$$

and from the assumption  $x_1 - x_0 \leq \varepsilon$ , we have

$$\bar{\lambda}_0(x) - \lambda_0(x) \geq (-1)^{n+1} \prod_{i=1}^n (x - x_i) \left( \frac{1}{\varepsilon} \prod_{i=2}^n \frac{1}{x_i - x_0} - \prod_{i=1}^n \frac{1}{x_i - \bar{x}_0} \right) \geq 0.$$

For  $p > 1$ , from (4.8) we obtain (putting  $l = p - 1$ )

$$\bar{\lambda}_{p-1}(x) - \lambda_{p-1}(x) = \frac{(\bar{x}_0 - x_0)(x - x_{p-1})}{(x - x_0)(x_{p-1} - x_0)} \prod_{\substack{i=0 \\ i \neq p-1}}^n \frac{x - x_i}{x_{p-1} - x_i}.$$

Hence, in view of the inequality  $x_p - x_{p-1} \leq \varepsilon$ , we have

$$\begin{aligned} & \bar{\lambda}_{p-1}(x) - \lambda_{p-1}(x) \\ & \geq \frac{(x - \bar{x}_0)(x - x_{p-1})(x - x_p)}{(x - x_0)(x_{p-1} - x_0)\varepsilon} \prod_{i=0}^{p-2} \frac{x - x_i}{x_{p-1} - x_i} \prod_{i=p+1}^n \frac{x_i - x}{x_i - x_{p-1}} \geq 0. \end{aligned}$$

The proof of the second part of the theorem is similar to the first part.

From (4.1) we obtain  $p + 2 \in Q_p$ . If we verify that the differences  $\bar{\lambda}_l(x) - \lambda_l(x)$  are bounded absolutely for  $x \in X_p$  and  $l \in P_p$  ( $p = 1, 2, \dots, n-2$ ) and that the difference  $\bar{\lambda}_{p+2}(x) - \lambda_{p+2}(x)$  is sufficiently small in this interval, then from this and from (4.9) we infer that  $\Phi_n(x) \leq 0$  or  $e_p < \bar{e}_p$ .

First, we prove that the difference  $\bar{\lambda}_0(x) - \lambda_0(x)$  is bounded absolutely. From (4.7) we obtain

$$|\bar{\lambda}_0(x) - \lambda_0(x)| = \left| -\lambda_0(x) \left( 1 - \prod_{i=1}^n \frac{x_i - x_0}{x_i - \bar{x}_0} \right) \right| < \prod_{i=1}^n \left| \frac{x - x_i}{x_0 - x_i} \right| < \left( \frac{2}{\beta} \right)^n,$$

where  $\beta = x_1 - x_0$ .

Proceeding similarly as in the first part of the proof of the theorem, for the differences  $\bar{\lambda}_l(x) - \lambda_l(x)$  we obtain an identical expression as (4.10), where now

$$\alpha = \min_{\substack{0 \leq i \leq n \\ l \in P_p, i \neq l \neq 0}} |x_l - x_i|.$$

For  $l \in Q_p$  and in the presence of the inequality  $x_{p+2} - x_{p+1} \leq \varepsilon$ , we obtain from (4.8)

$$\begin{aligned} & \frac{(\bar{x}_0 - x_0)(x - x_{p+1})(x - x_{p+2})}{(x - x_0)(x_{p+2} - x_0)\varepsilon} \prod_{i=0}^p \frac{x - x_i}{x_{p+2} - x_i} \prod_{i=p+3}^n \frac{x_i - x}{x_i - x_{p+2}} \\ & \leq \bar{\lambda}_{p+2}(x) - \lambda_{p+2}(x) \leq 0. \end{aligned}$$

We now show that theorem 4.2 concerning the shift of the node  $x_0$  can be completed with the additional assumption about a symmetric distribution of the nodes.

**THEOREM 4.4.** *If  $-x_i = x_{n-i}$  ( $i = 0, 1, \dots, n$ ) and  $\bar{x}_0 < x_0$  ( $x_0 < \bar{x}_0$ , respectively) and, moreover, if, for  $n$  odd,  $\bar{x}_0$  is the node obtained in the result of the local shift of the node  $x_0$ , then  $\bar{e}_{n-2} < e_{n-2}$  ( $e_{n-2} < \bar{e}_{n-2}$ , respectively).*

**Proof.** We prove the theorem in the case  $\bar{x}_0 < x_0$ . Assume, at first, that  $n$  is even. For  $p = n-2$ , from (4.1) we obtain  $N_{n-2} = \{1, 3, \dots, n-3, n\}$ . Hence and from (4.8) it follows that if  $x \in X_{n-2}$ , then

$$(4.11) \quad \text{sign}[\bar{\lambda}_l(x) - \lambda_l(x)] = \begin{cases} 1 & \text{for } l = 1, 3, \dots, n-3, \\ -1 & \text{for } l = n. \end{cases}$$

Now we prove that, for  $x \in X_{n-2}$ , the function  $\gamma(x) = \bar{\lambda}_1(x) - \lambda_1(x) + \bar{\lambda}_n(x) - \lambda_n(x)$  is non-negative. Hence and from (4.11) we obtain  $\Phi_n(x) \geq 0$  and also  $\bar{e}_{n-2} < e_{n-2}$ . Using (4.8) we obtain

$$\begin{aligned} \gamma(x) &= \frac{\bar{x}_0 - x_0}{x - x_0} \left[ \frac{x - x_1}{x_1 - x_0} \lambda_1(x) + \frac{x - x_n}{x_n - x_0} \lambda_n(x) \right] \\ &= (\bar{x}_0 - x_0) \prod_{i=1}^n (x - x_i) \left[ \frac{1}{(x_n - x_0)^2 (x_n - x_1) (x_n - x_{n-1}) \prod_{i=2}^{n-2} (x_n - x_i)} - \right. \\ &\quad \left. - \frac{1}{(x_1 - x_0)^2 (x_{n-1} - x_1) (x_n - x_1) \prod_{i=2}^{n-2} (x_i - x_1)} \right]. \end{aligned}$$

Obviously,  $x_n - x_0 > x_1 - x_0$  and  $x_n - x_0 > x_{n-1} - x_1$ . From the assumption of the symmetrical distribution of the nodes  $x_i$  ( $i = 0, 1, \dots, n$ ) we obtain  $x_n - x_{n-1} = x_1 - x_0$  and  $x_n - x_{n-i} = x_i - x_0 > x_i - x_1$  ( $i = 2, 3, \dots, n-2$ ); thus

$$\prod_{i=2}^{n-2} (x_n - x_i) > \prod_{i=2}^{n-2} (x_i - x_1).$$

From this, the expression in the last square bracket is negative, and thus  $\gamma(x) \geq 0$  for  $x \in X_{n-2}$ .

Now we assume that  $n$  is odd. From (4.1) we obtain  $N_{n-2} = \{0, 2, \dots, n-3, n\}$ . Using (4.7) and (4.8), for  $x \in X_{n-2}$  we obtain

$$(4.12) \quad \text{sign}[\bar{\lambda}_l(x) - \lambda_l(x)] = \begin{cases} 1 & \text{for } l = 0, 2, \dots, n-3, \\ -1 & \text{for } l = n. \end{cases}$$

Let  $\gamma(x) = \bar{\lambda}_0(x) - \lambda_0(x) + \bar{\lambda}_n(x) - \lambda_n(x)$ . Using (4.7) and (4.8), we obtain

$$\begin{aligned}\gamma(x) &= \prod_{i=1}^n \frac{x-x_i}{x_i-x_0} - \prod_{i=1}^n \frac{x-x_i}{x_i-\bar{x}_0} + \frac{(\bar{x}_0-x_0)(x-x_n)}{(x_n-\bar{x}_0)(x_n-x_0)} \prod_{i=1}^{n-1} \frac{x-x_i}{x_n-x_i} \\ &= \prod_{i=1}^n (x-x_i) \left( \prod_{i=1}^n \frac{1}{x_i-x_0} + \frac{\bar{x}_0-x_0}{x_n-\bar{x}_0} \prod_{i=0}^{n-1} \frac{1}{x_n-x_i} - \prod_{i=1}^n \frac{1}{x_i-\bar{x}_0} \right).\end{aligned}$$

Because  $-x_i = x_{n-i}$  ( $i = 0, 1, \dots, n$ ), we have

$$\prod_{i=1}^n \frac{1}{x_i-x_0} = \prod_{i=0}^{n-1} \frac{1}{x_n-x_i}.$$

Hence

$$\gamma(x) = \prod_{i=1}^n (x-x_i) \left[ \left( 1 + \frac{\bar{x}_0-x_0}{x_n-\bar{x}_0} \right) \prod_{i=1}^n \frac{1}{x_i-x_0} - \prod_{i=1}^n \frac{1}{x_i-\bar{x}_0} \right].$$

It is easy to see that the sign of the expression in the square bracket is identical with the sign of the expression which further on will be denoted by  $\beta$ :

$$\beta = \left( 1 + \frac{\bar{x}_0-x_0}{x_n-\bar{x}_0} \right) \prod_{i=1}^n (x_i-\bar{x}_0) - \prod_{i=1}^n (x_i-x_0).$$

By the assumption,  $\bar{x}_0$  is the node obtained as the result of a local shift of the node  $x_0$ , i. e.  $\bar{x}_0 = x_0 + dx_0$ , where  $dx_0$  denotes the differential at the point  $x_0$ . From the assumption,  $\bar{x}_0 < x_0$  holds; thus  $dx_0 < 0$ . Hence

$$\begin{aligned}\beta &= \left( 1 + \frac{dx_0}{x_n-x_0-dx_0} \right) \prod_{i=1}^n (x_i-x_0-dx_0) - \prod_{i=1}^n (x_i-x_0) \\ &= \left( 1 + \frac{dx_0}{x_n-x_0} \right) \left[ \prod_{i=1}^n (x_i-x_0) - dx_0 \sum_{j=1}^n \prod_{\substack{i=1 \\ i \neq j}}^n (x_i-x_0) \right] - \prod_{i=1}^n (x_i-x_0) \\ &= dx_0 \prod_{i=1}^n (x_i-x_0) \left( \frac{1}{x_n-x_0} - \sum_{i=1}^n \frac{1}{x_i-x_0} \right) \\ &= -dx_0 \prod_{i=1}^n (x_i-x_0) \sum_{i=1}^{n-1} \frac{1}{x_i-x_0}.\end{aligned}$$

Because  $dx_0 < 0$ , thus  $\beta > 0$ . Hence, for  $x \in X_{n-2}$ , we obtain  $\gamma(x) \geq 0$  and, in virtue of (4.12), we have  $\bar{e}_{n-2} < e_{n-2}$ .

Now, let us study how all local maxima of the Lebesgue function change at simultaneous local changes of all nodes  $x_i$  ( $i = 0, 1, \dots, n$ ).

Let numbers  $e_k$  ( $k = 0, 1, \dots, n+1$ ) be defined in the following way:

$$e_0 = A_n(-1), \quad e_k = \max_{x \in \langle x_{k-1}, x_k \rangle} A_n(x) \quad (k = 1, 2, \dots, n), \quad e_{n+1} = A_n(1).$$

Because in each of the intervals  $\langle x_{k-1}, x_k \rangle$  ( $k = 0, 1, \dots, n+1$ ;  $n > 1$ ) the function  $A_n$  has exactly one local maximum, thus numbers  $y_0, y_1, \dots, y_{n+1}$  are simultaneously defined such that  $A_n(y_k) = e_k$  ( $k = 0, 1, \dots, n+1$ ),  $-1 = y_0 < y_1 < \dots < y_{n+1} = 1$  and  $A'_n(y_k) = 0$  ( $k = 1, 2, \dots, n$ ). By  $dx_k$  ( $k = 0, 1, \dots, n$ ),  $dy_k$ ,  $de_k$  ( $k = 0, 1, \dots, n+1$ ) we denote the differentials of the nodes, the extremal points and the local maxima of the function  $A_n$ , respectively.

Let

$$(4.13) \quad a_{lk} = |\lambda_l(y_k)| \quad (l = 0, 1, \dots, n; k = 0, 1, \dots, n+1),$$

where  $\lambda_l$  denotes polynomial (2.12).

**THEOREM 4.5.** *Using the above-mentioned notation we have the equalities*

$$(4.14) \quad de_k = - \sum_{l=0}^n dx_l \prod_{\substack{i=0 \\ i \neq l}}^n \left( \frac{a_{lk}}{y_k - x_l} + \frac{a_{lk} + a_{ik}}{x_l - x_i} \right) \quad (k = 0, 1, \dots, n+1).$$

**Proof.** Let  $k = 0$ . Because  $y_0 = -1$ , thus  $dy_0 = 0$ . Hence

$$\begin{aligned} e_0 + de_0 &= A_n(y_0) = \sum_{l=0}^n \left| \prod_{\substack{i=0 \\ i \neq l}}^n \frac{y_0 - x_i - dx_i}{x_l + dx_l - x_i - dx_i} \right| \\ &= \sum_{l=0}^n \prod_{\substack{i=0 \\ i \neq l}}^n \left| \frac{y_0 - x_i}{x_l - x_i} \right| \left| \frac{1 - dx_i/(y_0 - x_i)}{1 + (dx_l - dx_i)/(x_l - x_i)} \right| \\ &= \sum_{l=0}^n |\lambda_l(y_0)| \left[ 1 - \sum_{\substack{i=0 \\ i \neq l}}^n \left( \frac{dx_i}{y_0 - x_i} + \frac{dx_l - dx_i}{x_l - x_i} \right) \right]. \end{aligned}$$

Because

$$e_0 = A_n(y_0) = \sum_{l=0}^n |\lambda_l(y_0)|,$$

thus

$$(4.15) \quad de_0 = - \sum_{l=0}^n |\lambda_l(y_0)| \sum_{\substack{i=0 \\ i \neq l}}^n \left( \frac{dx_i}{y_0 - x_i} + \frac{dx_l - dx_i}{x_l - x_i} \right).$$

Similarly, it can be verified that

$$(4.16) \quad de_{n+1} = - \sum_{l=0}^n |\lambda_l(y_{n+1})| \sum_{\substack{i=0 \\ i \neq l}}^n \left( \frac{dx_i}{y_{n+1} - x_i} + \frac{dx_l - dx_i}{x_l - x_i} \right).$$

For  $k = 1, 2, \dots, n$ , we obtain

$$\begin{aligned} e_k + de_k &= A_n(y_k + dy_k) = \sum_{l=0}^n \left| \prod_{\substack{i=0 \\ i \neq l}}^n \frac{y_k + dy_k - x_i - dx_i}{x_l + dx_l - x_i - dx_i} \right| \\ &= \sum_{l=0}^n \prod_{\substack{i=0 \\ i \neq l}}^n \left| \frac{y_k - x_i}{x_l - x_i} \right| \left| \frac{1 + (dy_k - dx_i)/(y_k - x_i)}{1 + (dx_l - dx_i)/(x_l - x_i)} \right| \\ &= \sum_{l=0}^n |\lambda_l(y_k)| \left[ 1 + \sum_{\substack{i=0 \\ i \neq l}}^n \left( \frac{dy_k - dx_i}{y_k - x_i} - \frac{dx_l - dx_i}{x_l - x_i} \right) \right]. \end{aligned}$$

Hence, in virtue of the equalities  $e_k = A_n(y_k)$  ( $k = 1, 2, \dots, n$ ),

$$de_k = \sum_{l=0}^n |\lambda_l(y_k)| \sum_{\substack{i=0 \\ i \neq l}}^n \left( \frac{dy_k - dx_i}{y_k - x_i} - \frac{dx_l - dx_i}{x_l - x_i} \right) \quad (k = 1, 2, \dots, n)$$

holds.

Because

$$\frac{\lambda'_l(x)}{\lambda_l(x)} = \sum_{\substack{i=0 \\ i \neq l}}^n \frac{1}{x - x_i} \quad (l = 0, 1, \dots, n),$$

thus

$$\begin{aligned} \lambda'_l(y_k) &= \lambda_l(y_k) \sum_{\substack{i=0 \\ i \neq l}}^n \frac{1}{y_k - x_i}, \\ A'_n(y_k) &= \left( \sum_{l=0}^n |\lambda_l(x)| \right)'_{x=y_k} = \sum_{l=0}^n |\lambda_l(y_k)| \sum_{\substack{i=0 \\ i \neq l}}^n \frac{1}{y_k - x_i}. \end{aligned}$$

It follows from the definition of the point  $y_k$  that  $A'_n(y_k) = 0$ ; hence

$$de_k = - \sum_{l=0}^n |\lambda_l(y_k)| \sum_{\substack{i=0 \\ i \neq l}}^n \left( \frac{dx_i}{y_k - x_i} + \frac{dx_l - dx_i}{x_l - x_i} \right).$$

In virtue of (4.15) and (4.16), the last equality holds for  $k = 0, 1, \dots, n+1$ . Considering (4.13), after a certain transformation, we can write the above-mentioned equalities in form (4.14).

**5. Characterization of minimal interpolating projections.** Cheney and Price [1] raised the following question (problem 14):

Is the operator

$$L_n f(x) = \sum_{l=0}^n f(x_l) \lambda_l(x) \quad (n > 1)$$

minimal in the class of the interpolating projections if all local maxima of the Lebesgue function are identical?

For  $n = 2, 3$  the answer to the question is given by theorem 5.1. The author has failed to generalize this theorem for the case  $n > 3$  because of calculation difficulties stemming from the method used. In the sequel, we use the notation introduced in section 4.

**THEOREM 5.1.** *Let  $n = 2, 3$  and, in addition, let  $-x_i = x_{n-i}$  ( $i = 0, 1, 2, 3$ ). If all local maxima of the Lebesgue function are identical, then the interpolating operator is minimal in the class of interpolating operators. For  $-x_0 = x_n = 1$  ( $n = 2, 3$ ), this operator is unique.*

**Proof.** In virtue of theorem 2.2, the norm of the projection does not change if the interpolation nodes are transformed in an affine way so that  $-x_0 = x_n = 1$ . We still assume that the last equality holds.

Let  $n = 2$ . Denote by  $e_1$  and  $e_2$  the local maxima of the function  $A_2$ . By the assumption, the equality  $e_1 = e_2$  holds. In virtue of theorem 4.1, a shift of the node  $x_1$  to the left or to the right gives rise to an increase of one of the maxima  $e_i$  ( $i = 1, 2$ ), which denotes an increase of the norm of the interpolating operator. From this it follows also that, for  $-x_0 = x_2 = 1$ , the minimal interpolating operator is unique.

In the case  $n = 3$  we use theorem 4.5. We prove that arbitrary nodes, different from those which assured the equality  $e_1 = e_2 = e_3$ , define an operator with a greater norm. Before proving this fact we give some details which will be used later on. For numbers defined by (4.13) and by the assumption about the symmetry of nodes  $x_i$  ( $i = 0, 1, \dots, n$ ) it follows the easy to verify equality  $a_{lk} = a_{n-l, n+1-k}$  ( $l = 0, 1, \dots, n$ ;  $k = 0, 1, \dots, n+1$ ). By the definition, the point  $y_k$  satisfies the equality



$\Lambda'_3(y_3) = 0$ . For  $x \in \langle x_2, 1 \rangle$ , the equality

$$\Lambda_3(x) = \frac{x^2 - x_2^2}{1 - x_2^2} - \frac{x(x^2 - 1)}{x_2(1 - x_2^2)}$$

holds. It follows that  $\Lambda'_3(x) = 0$  for  $x = y_3 = (x_2 + \sqrt{x_2^2 + 3})/3$  and  $x_2 = (3y_3^2 - 1)/2y_3$ . Because  $x_2 > 0$ , we have  $1/\sqrt{3} < y_3$ . Using (4.14), for  $dx_1 = -dx_2$ , we obtain

$$de_1 = \frac{1}{1 - x_2^2} \left[ \frac{2x_2(y_3^2 - 1)}{y_3^2 - x_3^2} (\alpha_{01} + \alpha_{31}) + \frac{3x_2^2 y_3 + 2x_2^3 - y_3}{x_2(y_3 + x_2)} \alpha_{11} + \right. \\ \left. + \frac{3x_2^2 y_3 - 2x_2^3 - y_3}{x_2(y_3 - x_2)} \alpha_{21} \right] dx_2,$$

$$de_2 = \frac{4}{x_2(1 - x_2^2)} (\alpha_{02} + x_2^2 \alpha_{12}) dx_2,$$

$$de_3 = de_1.$$

Using (4.13), we obtain

$$de_1 = \frac{1 - y_3^2}{x_2^2(1 - x_2^2)^2} (3x_2^2 y_3 - 2x_2^3 - y_3) dx_2, \\ de_2 = \frac{4x_2}{(1 - x_2^2)^2} dx_2, \\ de_3 = de_1.$$

Because  $x_2 = (3y_3^2 - 1)/2y_3$ , the expression for  $de_1$  can be written as

$$de_1 = \frac{1 - y_3^2}{4x_2^2 y_3^2 (1 - x_2^2)^2} (5y_3^2 - 1)(y_3^2 - 1) dx_2.$$

For  $1/\sqrt{3} < y_3 < 1$ , the expression standing by  $dx_2$  in the last equality is negative, however, the expression by  $dx_2$  in the equality for  $de_2$  is positive. Hence, for  $dx_2 > 0$  ( $dx_2 < 0$ , respectively), we obtain  $de_1 < 0$  and  $de_2 > 0$  ( $de_1 > 0$  and  $de_2 < 0$ , respectively). Hence, for  $dx_2 \neq 0$ , the norm of the operator is always greater than the norm of the operator for which all local maxima  $e_i$  ( $i = 1, 2, 3$ ) are identical. Hence, also the uniqueness of the minimal interpolating projection follows if  $-x_0 = x_3 = 1$ .

Cheney and Price [2] proved a theorem in which they gave the necessary conditions for a minimal interpolating projection.

If  $L_n$  ( $n > 0$ ) is the minimal interpolating projection defined on the space  $C_{\langle -1, 1 \rangle}$  whose values belong to the Haar subspace of the space  $C_{\langle -1, 1 \rangle}$  (the dimension of this subspace is equal  $n + 1$ ), then either  $\|L_n\| = 1$  or at least  $n + 2$  local maxima of the Lebesgue function for the operator  $L_n$  are identical.

**Remark.** The thesis of the above-mentioned theorem is not precise for interpolating operators whose values are algebraic polynomials of degree  $n$ . If the minimal interpolating operator, whose Lebesgue function has  $n+2$  identical local maxima, corresponds to the nodes  $x_i$  ( $i = 0, 1, \dots, n$ ;  $-1 < x_0, x_n < 1$ ), then using theorem 2.2 we can transform the nodes  $x_i$  in an affine way so that the norm of the interpolating operator does not change; but the Lebesgue function will now have  $n$  identical local maxima in the interval  $\langle -1, 1 \rangle$  (where now the ends of the interval are nodes).

**6. Non-interpolating discrete projections.** Here we shall use the notation introduced in section 2. As previously, we assume that the interval of the approximation is  $\langle a, b \rangle = \langle -1, 1 \rangle$  and that  $n > 1$ . From the general form (2.3) of the projection it follows that it depends upon the functions  $g_k$  ( $k = 0, 1, \dots, n$ ) with bounded variation in the interval  $\langle -1, 1 \rangle$ . As  $g_k$  we assume *step functions*, i. e. functions which have discontinuities of the first kind (steps) in points which will be denoted by  $t_l$  ( $l = 0, 1, \dots, p$ ). We assume that the numbers  $t_l$  are ordered increasingly, i. e.  $-1 \leq t_0 < t_1 < \dots < t_p \leq 1$ . Additionally, we assume that  $p > n$ . Of course, every function  $g_k$  has a bounded variation in the interval  $\langle -1, 1 \rangle$ . Let  $s_{kl}$  ( $k = 0, 1, \dots, n$ ) denote the step of the function  $g_k$  in the point  $t_l$ , i. e.

$$s_{kl} = \begin{cases} g_k(t_0 + 0) - g_k(t_0) & (l = 0), \\ g_k(t_l + 0) - g_k(t_l - 0) & (l = 1, 2, \dots, p-1), \\ g_k(t_p) - g_k(t_p - 0) & (l = p). \end{cases}$$

The steps  $s_{kl}$  will be called *parameters*. Because of the above-mentioned choice of the functions  $g_k$ , condition (2.4) receives the form

$$(6.1) \quad \sum_{l=0}^p t_l^j s_{kl} = \delta_{jk} \quad (j, k = 0, 1, \dots, n),$$

and the value of operator (2.3), which will be denoted by  $L_{np}$ , is given by the formula

$$(6.2) \quad L_{np}f(x) = \sum_{l=0}^p f(t_l) \sum_{k=0}^n s_{kl} x^k \quad (p > n).$$

Because of (2.6) the Lebesgue function of operator (6.2) is now expressed by the formula

$$A_{np}(x) = \sum_{l=0}^p \left| \sum_{k=0}^n s_{kl} x^k \right|.$$

The norm of operator (6.2) is equal to

$$(6.3) \quad \|L_{np}\| = \|A_{np}\|.$$

Let us suppose that

$$(6.4) \quad -t_l = t_{p-l} \quad (l = 0, 1, \dots, p).$$

The symmetrical distribution of the step points  $t_l$  does not imply the equality

$$(6.5) \quad \Lambda_{np}(x) = \Lambda_{np}(-x) \quad (x \in \langle -1, 1 \rangle, n > 1, p > n).$$

Let us suppose also that the parameters  $s_{kl}$  are such that

$$(6.6) \quad s_{kl} = \begin{cases} s_{k,p-l} & \text{for } k \text{ even,} \\ -s_{k,p-l} & \text{for } k \text{ odd} \end{cases} \quad (k = 0, 1, \dots, n; l = 0, 1, \dots, p).$$

We require that operator (6.2) satisfy conditions (6.4) and (6.6) and, therefore, satisfy (6.5).

The fact that it suffices to consider such operators is proved by the following theorem (see [6]):

*Let  $P_n$  ( $n > 0$ ) be an arbitrary projection defined on the space  $C_{\langle -1, 1 \rangle}$ . Its value (a polynomial of degree not greater than  $n$  of the variable  $x$ ) for the function  $h(t)$  is denoted by the symbol  $P_n h(t)(x)$ . Let  $R_n$  ( $n > 0$ ) be a projection such that*

$$(6.7) \quad R_n h(t)(x) = \frac{1}{2} [P_n h(t)(x) + P_n h(-t)(-x)].$$

*Then*

$$\|R_n\| \leq \|P_n\|.$$

In particular, it easily follows from this theorem that if  $P_n$  is a discrete operator, then the discrete operator  $R_n$  satisfies the symmetry conditions (6.4) and (6.6).

In virtue of (6.6), the number of independent parameters  $s_{kl}$  determining the operator  $L_{np}$  is equal to  $\frac{1}{2}(n+1)(p+1)$  for  $n$  or  $p$  odd, and is equal to  $\frac{1}{2}(n+1)(p+1) + \frac{1}{2}$  for  $n$  and  $p$  even.

If the step points  $t_l$  ( $l = 0, 1, \dots, p$ ) satisfy (6.4), then, in view of (6.6), the number of conditions (6.1) is equal to  $\frac{1}{2}(n+1)^2$  for  $n$  odd, and is equal to  $\frac{1}{2}(n+1)^2 + \frac{1}{2}$  for  $n$  even. In such a case conditions (6.1) form an underdetermined system of linear equations with unknowns  $s_{kl}$ , in which there are  $[\frac{1}{2}(p-n)(n+1)] = q > 0$  free parameters ( $[a]$  denotes here the integer part of the number  $a$ ). The parameters  $s_{kl}$  which are free will be denoted, for simplicity, by  $y_j$  ( $j = 1, 2, \dots, q$ ).

**7. The calculation of parameters defining the operator (6.2).** From section 6 we see that the Lebesgue function  $\Lambda_{np}$  depends upon free parameters  $y_j$  ( $j = 1, 2, \dots, q$ ) which will be denoted by  $\Lambda_{np}(x) \equiv \Lambda_{np}(x; y_1, y_2, \dots, y_q)$ . In view of (6.3) and of inequality (2.7), the free parameters

$y_j$  ( $j = 1, 2, \dots, q$ ) are to be selected so that the norm of the operator  $L_{np}$  will be smallest. In this connection, the problem of finding the best operator among operators  $L_{np}$  for fixed  $n$  and  $p$  and the step points  $t_l$  reduces to a certain problem which will be called *continuous*.

THE CONTINUOUS PROBLEM. Find the vector  $y^* = (y_1^*, y_2^*, \dots, y_q^*)$  such that

$$(7.1) \quad \min_{y_1, y_2, \dots, y_q} \max_{-1 \leq x \leq 1} \Lambda_{np}(x; y_1, y_2, \dots, y_q) = \max_{-1 \leq x \leq 1} \Lambda_{np}(x; y_1^*, y_2^*, \dots, y_q^*).$$

In practice, it is difficult to solve this problem for arbitrary values of  $n$  and  $p$  ( $1 < n < p$ ). In this connection, the solution of (7.1) is approximated by the solution of a discrete problem which is formulated below. Let us introduce additional notation before.

Let  $\nu$  be an arbitrary natural number ( $\nu > 0$ ).

We define a discrete set  $X_\nu$  from the interval  $\langle -1, 1 \rangle$  as follows:

$$X_\nu = \{x_i \mid -1 \leq x_1 < x_2 < \dots < x_\nu \leq 1\} \subset \langle -1, 1 \rangle.$$

THE DISCRETE PROBLEM. Find the vector  $\bar{y} = (\bar{y}_1, \bar{y}_2, \dots, \bar{y}_q)$  such that

$$(7.2) \quad \min_{y_1, y_2, \dots, y_q} \max_{x_i \in X_\nu} \Lambda_{np}(x_i; y_1, y_2, \dots, y_q) = \max_{x_i \in X_\nu} \Lambda_{np}(x_i; \bar{y}_1, \bar{y}_2, \dots, \bar{y}_q).$$

If the set  $X_\nu$  covers sufficiently dense the interval  $\langle -1, 1 \rangle$ , then the quantity appearing on the right-hand side of equality (7.2) is a little smaller than the norm of the operator  $L_{np}$ . To do this, equality (7.2) can be written in the explicit form. We calculate from conditions (6.1) that part of parameters  $s_{kl}$  which is depending linearly from the free parameters  $y_j$  ( $j = 1, 2, \dots, q$ ):

$$s_{kl} = \beta_{kl} + \sum_{j=1}^q a_{klj} y_j.$$

Hence

$$\Lambda_{np}(x_i; y_1, y_2, \dots, y_q) = \sum_{l=0}^p \left| \sum_{k=0}^n \left( \beta_{kl} + \sum_{j=1}^q a_{klj} y_j \right) x_i^k \right| = \sum_{l=0}^p \left| \sum_{j=1}^q a_{lj}^{(i)} y_j + b_l^{(i)} \right|,$$

where

$$a_{lj}^{(i)} = \sum_{k=0}^n a_{klj} x_i^k \quad \text{and} \quad b_l^{(i)} = \sum_{k=0}^n \beta_{kl} x_i^k$$

$$(x_i \in X_\nu, i = 1, 2, \dots, \nu; j = 1, 2, \dots, q; l = 0, 1, \dots, p).$$

Finally, for fixed values of  $n$  and  $p$  and for step points  $t_l$ , the problem of finding the operator  $L_{np}$  with the smallest discrete norm defined as (7.2) is equivalent to the following one:

Find the vector  $\bar{y}$  such that

$$(7.3) \quad \min_{y_1, y_2, \dots, y_q} \max_{i=1, 2, \dots, \nu} \sum_{l=0}^p \left| \sum_{j=1}^q a_{lj}^{(i)} y_j + b_l^{(i)} \right| = \max_{i=1, 2, \dots, \nu} \sum_{l=0}^p \left| \sum_{j=1}^q a_{lj}^{(i)} \bar{y}_j + b_l^{(i)} \right|.$$

In [5] the description of a method allowing to solve problem (7.3) and an algorithm in the form of a procedure (written in the Algol 60 language) are given.

**8. Results of the calculations.** In the sequel, there are given examples of operators  $L_{np}$  which were mentioned in sections 6 and 7. They were calculated on the Odra 1204 computer. The considerable time of calculations necessary to find the solution of (7.3) and the small memory of this computer caused that the best operators (6.2) were found for certain (small) values of  $n$  and  $p$  only. For every of the operators we give the distribution of points  $t_l$  ( $l = 0, 1, \dots, p$ ;  $t_l \in \langle -1, 1 \rangle$ ) (due to theorem 2.2 we can assume  $-t_0 = t_p = 1$ ), the array containing the part of coefficients  $s_{kl}$  for  $k = 0, 1, \dots, n$ ;  $l = 0, 1, \dots, [p/2]$  (the remaining ones are such as it follows from (6.6)), and the norm of operator  $L_{np}$ . Moreover, graphs of the Lebesgue functions  $\Lambda_{np}(x)$  connected with the operators  $L_{np}$  (in virtue of (6.5) it suffices to restrict oneself to the part of the graph, for  $x \in \langle 0, 1 \rangle$ ) are given.

Example 8.1.  $n = 2$ ,  $p = 3$ ;  $-t_0 = t_3 = 1$ ,  $-t_1 = t_2 = .0744$ .

TABLE 8.1

| $k$ | $s_{k0}$              | $s_{k1}$             |
|-----|-----------------------|----------------------|
| 0   | $-.3013432753_{10}-2$ | $.5030134327_{10}0$  |
| 1   | $-.4905114923_{10}0$  | $-.1225905388_{10}0$ |
| 2   | $.5030134327_{10}0$   | $-.5030134327_{10}0$ |

$$\|L_{2,3}\| = 1.245183.$$

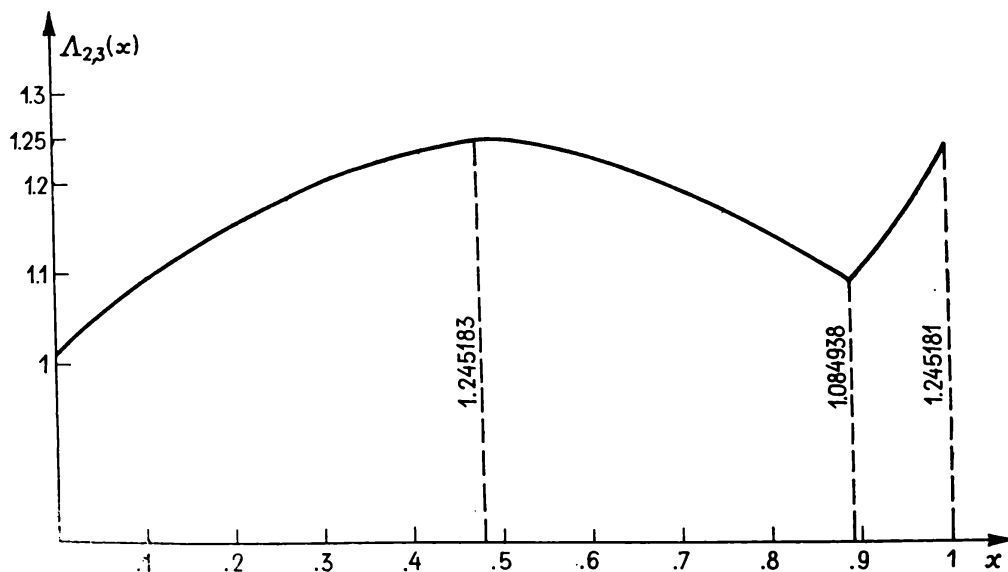


Fig. 8.1

Example 8.2.  $n = 2$ ,  $p = 4$ ;  $-t_0 = t_4 = 1$ ,  $-t_1 = t_3 = .866 \approx \sqrt{3}/2$ ,  $t_2 = 0$ .

TABLE 8.2

| $k$ | $s_{k0}$                | $s_{k1}$               | $s_{k2}$              |
|-----|-------------------------|------------------------|-----------------------|
| 0   | $-.5246021095_{10} - 1$ | $.6994782083_{10} - 1$ | $.9650247802_{10} 0$  |
| 1   | $-.3044060827_{10} 0$   | $-.2258538057_{10} 0$  | $.0$                  |
| 2   | $.3830719706_{10} 0$    | $.1559059848_{10} 0$   | $-.1077955911_{10} 1$ |

$$\|L_{2,4}\| = 1.225864.$$

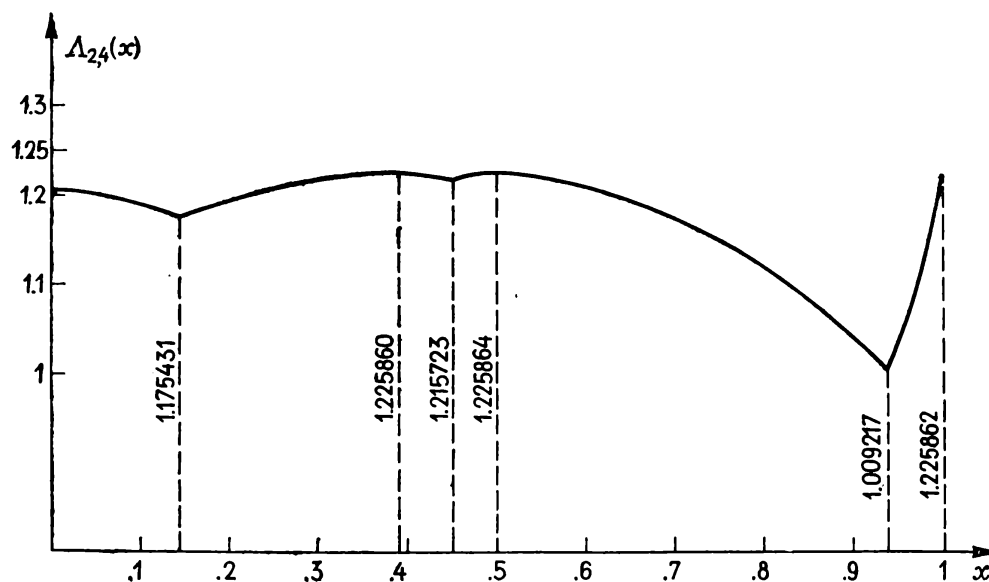


Fig. 8.2

Example 8.3.  $n = 2$ ,  $p = 5$ ;  $-t_0 = t_5 = 1$ ,  $-t_1 = t_4 = .06$ ,  $-t_2 = t_3 = .85$ .

TABLE 8.3

| $k$ | $s_{k0}$                | $s_{k1}$                | $s_{k2}$               |
|-----|-------------------------|-------------------------|------------------------|
| 0   | $-.4710430081_{10} - 1$ | $.4843212638_{10} 0$    | $.6278303703_{10} - 1$ |
| 1   | $-.3246435650_{10} 0$   | $-.5634781633_{10} - 1$ | $-.2023241954_{10} 0$  |
| 2   | $.4011279218_{10} 0$    | $-.5406090801_{10} 0$   | $.1395411583_{10} 0$   |

$$\|L_{2,5}\| = 1.225563.$$

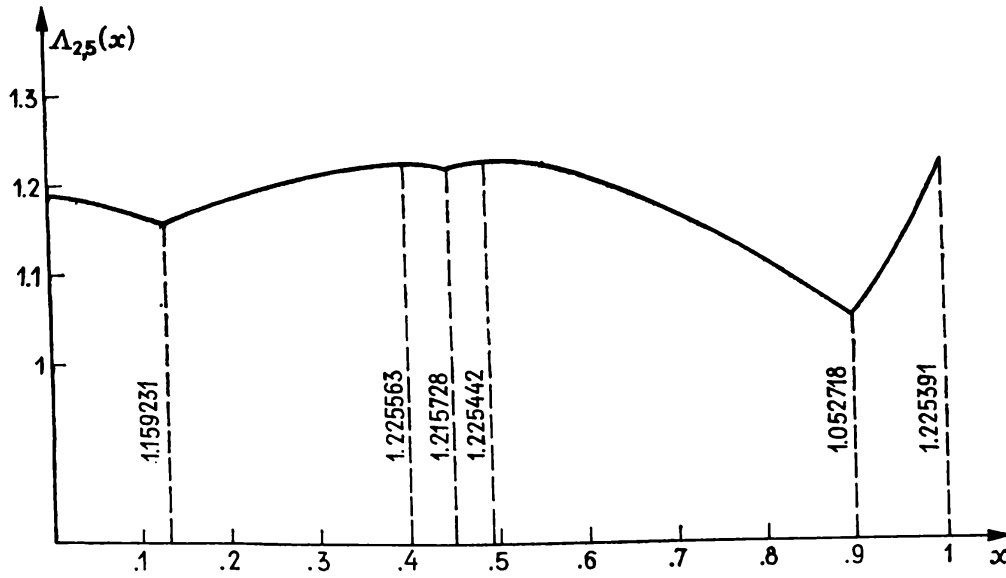


Fig. 8.3

**Example 8.4.**  $n = 2$ ,  $p = 6$ ;  $-t_0 = t_6 = 1$ ,  $-t_1 = t_5 = .7$ ,  $-t_2 = t_4 = .34$ ,  $t_3 = 0$ .

TABLE 8.4

| $k$ | $s_{k0}$                | $s_{k1}$                | $s_{k2}$                | $s_{k3}$              |
|-----|-------------------------|-------------------------|-------------------------|-----------------------|
| 0   | $-.2345598315_{10} - 1$ | $.6819042391_{10} - 1$  | $.3178197990_{10} - 1$  | $.8469671587_{10} 0$  |
| 1   | $-.4145638283_{10} 0$   | $-.5498239397_{10} - 1$ | $-.9534593968_{10} - 1$ | $.0$                  |
| 2   | $.4830924375_{10} 0$    | $-.1231728179_{10} 0$   | $.6356395980_{10} - 1$  | $-.8469671587_{10} 0$ |

$$\|L_{2,6}\| = 1.224788.$$

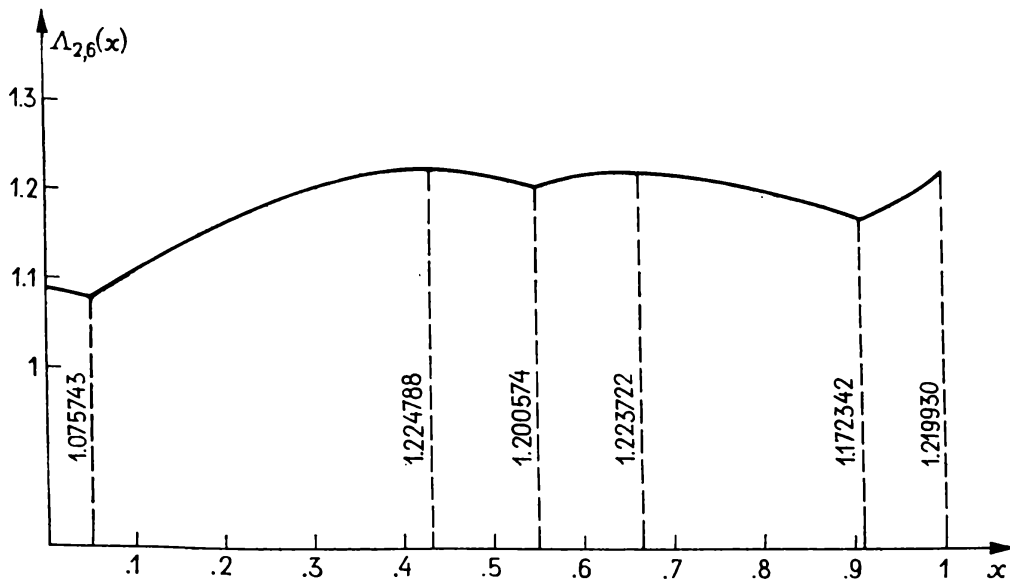


Fig. 8.4

Example 8.5.  $n = 3, p = 4; -t_0 = t_4 = 1, t_2 = 0, -t_1 = t_3 = .44.$

TABLE 8.5

| $k$ | $s_{k0}$               | $s_{k1}$              | $s_{k2}$              |
|-----|------------------------|-----------------------|-----------------------|
| 0   | $-.981461698_{10} - 1$ | $.5069533160_{10} 0$  | $.1823856920_{10} 0$  |
| 1   | $.1200396826_{10} 0$   | $-.1409181097_{10} 1$ | $.0$                  |
| 2   | $.5790867807_{10} 0$   | $-.4085060989_{10} 0$ | $-.3411613638_{10} 0$ |
| 3   | $-.6200396826_{10} 0$  | $.1409181097_{10} 1$  | $.0$                  |

$$\|L_{3,4}\| = 1.394371.$$

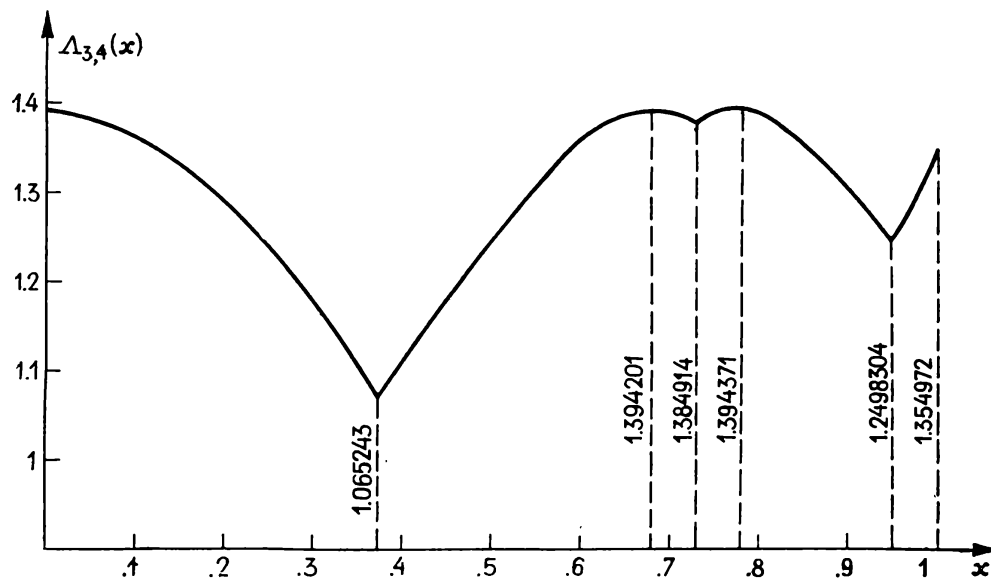


Fig. 8.5

Example 8.6.  $n = 4, p = 5; -t_0 = t_5 = 1, -t_1 = t_4 = .2, -t_2 = t_3 = .68.$

TABLE 8.6

| $k$ | $s_{k0}$               | $s_{k1}$                | $s_{k2}$              |
|-----|------------------------|-------------------------|-----------------------|
| 0   | $.1129877209_{10} - 1$ | $-.5450898159_{10} - 1$ | $.5432102079_{10} 0$  |
| 1   | $.3385543749_{10} 0$   | $-.1067548017_{10} 1$   | $-.7038857678_{10} 0$ |
| 2   | $-.4657933553_{10} 0$  | $.2183766130_{10} 1$    | $-.1717972771_{10} 1$ |
| 3   | $-.8173396015_{10} 0$  | $.9979520438_{10} 0$    | $.8670763233_{10} 0$  |
| 4   | $.9544945842_{10} 0$   | $-.2129257149_{10} 1$   | $.1174762561_{10} 1$  |

$$\|L_{4,5}\| = 1.541852.$$



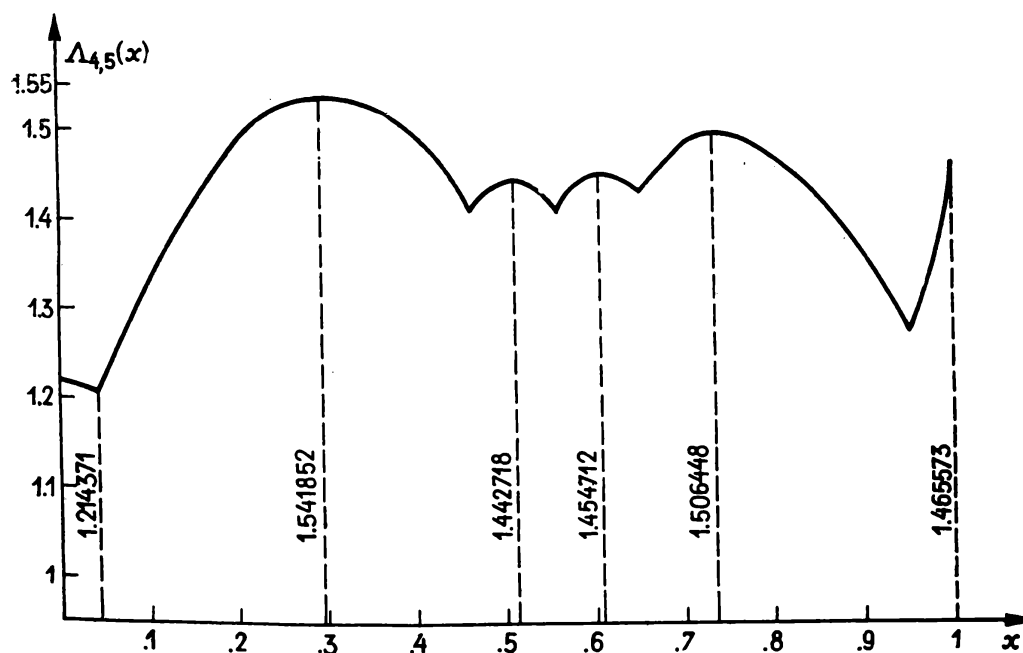


Fig. 8.6

**9. Unsolved problems.** The following questions are not answered:

1. Is the minimal projection unique?
2. Does there exist a discrete projection among minimal projections?
3. What are the necessary and sufficient conditions for the operator  $L_{np}$  to have the smallest norm?
4. Do for projections hold theorems analogous to Tchebycheff's theorem about the alternant and to Vallée-Poussin's or Remes' theorems for the error of the optimal polynomial approximation? In particular, does for every operator  $L_{np}$  hold the inequality

$$\min_{-1 \leq x \leq 1} \Delta_{np}(x) \leq \|L_{np}^*\|$$

(where by  $L_{np}^*$  we denote the operator  $L_{np}$  which for fixed values  $n$  and  $p$  has the smallest norm)?

5. Is the operator  $L_{np}$  with the smallest norm unique if  $-t_0 = t_p = 1$ ?
6. Does there exist the operator  $L_{np}$  with norm which is smaller than  $\|S_n\|$  for every  $n > 1$  and every  $p$  ( $p > n$ ) (by  $S_n$  we denote the projection whose value  $S_n f$  is identical with the  $n$ -th partial sum of the Tchebycheff series for the function  $f \in C_{(-1,1)}$ )?

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## References

- [1] E. W. Cheney and K. H. Price, *Minimal projections*, Proceedings of a Symposium Held at Lancaster, July 1969, ed. by A. Talbot, London 1970, p. 261-289.
- [2] — *Minimal interpolating projections*, Proceedings of the Conference on Approximation Theory, Oberwolfach, June 1969, p. 115-121.
- [3] I. K. Daugavet (И. К. Даугавет), *Об одном свойстве вполне непрерывных операторов в пространстве  $C$* , Усп. Мат. Наук 18 (1963), p. 157-158.
- [4] F. W. Luttman and T. J. Rivlin, *Some numerical experiments in the theory of polynomial interpolation*, IBM J. Develop. 9 (1965), p. 187-191.
- [5] E. Neuman, *The calculation of the minimum of a certain function of several variables*, this fascicle, p. 143-164.
- [6] S. Paszkowski, *Wielomiany i szeregi Czebyszewa* (to appear).

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## OPERATORY RZUTOWE W APROKSYMACJI JEDNOSTAJNEJ WIELOMIANOWEJ

### STRESZCZENIE

Praca poświęcona jest operatorom liniowym  $L_n$  ( $n$  całkowite dodatnie) takim, że  $L_n: C_{\langle a, b \rangle} \rightarrow W_n$  (gdzie  $C_{\langle a, b \rangle}$  oznacza przestrzeń funkcji rzeczywistych i ciągłych w przedziale  $\langle a, b \rangle$ ,  $W_n$  zaś klasę wielomianów algebraicznych stopnia co najwyżej  $n$ ) i dodatkowo spełniającym warunek (2.2). Takie operatory nazwano *rzutowymi*. W przytoczonym twierdzeniu 2.1 podano m. in. ogólną postać operatora rzutowego  $L_n$  (równość (2.3)), wzór (2.5) dla normy takiego operatora oraz nierówność (2.7) o jakości aproksymacji funkcji  $f \in C_{\langle a, b \rangle}$  przez element  $L_n f \in W_n$ . Ze wspomnianej nierówności wynika, że wielkość szacująca z góry błąd aproksymacji funkcji  $f$  przez wielomian  $L_n f$  będzie tym mniejsza, im mniejsza będzie norma operatora  $L_n$ . W związku z tym wprowadzono pojęcie *operatora minimalnego*, tj. operatora rzutowego, który dla ustalonej wartości  $n$  ma najmniejszą normę spośród wszystkich operatorów rzutowych. Wobec twierdzenia 2.2 przyjmuje się, że przedział aproksymacji  $\langle a, b \rangle$  jest w dalszym ciągu równy  $\langle -1, 1 \rangle$ . Paragraf 2 jest zakończony przykładem operatora rzutowego (2.9), który nazwano *dyskretnym*. Szczególnym przypadkiem takiego operatora jest operator interpolacyjny (2.11). Operatorom interpolacyjnym poświęcono paragrafy 3, 4 i 5.

Pewne elementarne własności funkcji Lebesgue'a (2.10) dla operatora interpolacyjnego podano w paragrafie 3.

Następny paragraf w całości poświęcony jest zachowaniu się maksimów lokalnych funkcji Lebesgue'a dla operatora interpolacyjnego przy zmianie węzłów. Twier-

dzenia 4.1-4.4 opisują zmianę odpowiednich maksimów wspomnianej funkcji przy przesunięciu jednego węzła. We wniosku 4.2 (który jest prostą konsekwencją twierdzenia 4.1) podano warunki dostateczne na to, aby operator interpolacyjny z węzłami  $x_k^*$  ( $k = 0, 1, \dots, n$ ) był *lokalnie minimalny*, tzn. taki, który dla ustalonego  $n$  ma najmniejszą normę spośród wszystkich operatorów interpolacyjnych, których węzły  $x_k$  należą do dostatecznie małych otoczeń odpowiednich węzłów  $x_k^*$ . Wspomniany warunek orzeka, że operator interpolacyjny jest lokalnie minimalny, jeśli wszystkie maksima odpowiadające mu funkcji Lebesgue'a są identyczne.

Twierdzenie 4.5 mówi o przyrostach lokalnych wszystkich maksimów funkcji Lebesgue'a, przy jednoczesnej lokalnej zmianie wszystkich węzłów interpolacji.

W paragrafie 5 podano warunki dostateczne na to, aby operator interpolacyjny  $L_n$  (dla  $n \leq 3$ ) był minimalny w klasie operatorów interpolacyjnych.

W następnym paragrafie podano konstrukcję pewnych nieinterpolacyjnych dyskretnych operatorów rzutowych. Wobec (2.3) wartość operatora rzutowego zależy od funkcji  $g_k$  ( $k = 0, 1, \dots, n$ ) o wahanu skończonym w przedziale  $\langle -1, 1 \rangle$ . Jako  $g_k$  przyjęto funkcje przedziałami stałe, z nieciągłościami pierwszego rodzaju w takich punktach  $t_l$ , że  $-1 \leq t_0 < t_1 < \dots < t_p \leq 1$ , gdzie  $p > n$ . Wobec takiego wyboru funkcji  $g_k$ , wartość operatora rzutowego (który dalej oznaczamy przez  $L_{np}$ ) wyraża się wzorem (6.2). Operator  $L_{np}$  jest wyznaczony przez punkty  $t_l$  i liczby (nazwane dalej parametrami)  $s_{kl}$  ( $k = 0, 1, \dots, n$ ;  $l = 0, 1, \dots, p$ ), spełniające układ równań (6.1). Na punkty  $t_l$  narzucono warunki (6.4), na parametry  $s_{kl}$  zaś warunki (6.6). O tym, że takie żądanie nie spowoduje wzrostu normy operatora  $L_{np}$ , świadczy przytoczone w pracy pewne ogólne twierdzenie o operatorach rzutowych [6].

W paragrafie 7 opisano metodę wyznaczania parametrów  $s_{kl}$ , które dla ustalonych wartości  $n$  i  $p$  ( $p > n > 1$ ) i liczb  $t_l$  ( $l = 0, 1, \dots, p$ ) dają operator  $L_{np}$  o najmniejszej normie dyskretnej, identycznej z prawą stroną równości (7.2). Taka norma dla pewnych warunków, o których mówimy w paragrafie 7, niewiele różni się od normy (7.1).

W następnym paragrafie podano przykłady operatorów  $L_{np}$  o najmniejszej normie, znalezionych dla pewnych (niedużych) wartości  $n$  i  $p$ . Operatory te wyznaczono na maszynie cyfrowej Odra 1204. Równocześnie podano normy tych operatorów (równe maksymalnej wartości odpowiadających im funkcji Lebesgue'a), punkty  $t_l$ , część parametrów  $s_{kl}$  (tablice 8.1-8.6) ( $k = 0, 1, \dots, n$ ;  $l = 0, 1, \dots, [p/2]$ ), przy czym pozostałe można obliczyć za pomocą równania (6.6), i wykresy 8.1-8.6 funkcji Lebesgue'a omawianych operatorów (wobec (6.5) można ograniczyć się do części wykresu, np. do przedziału  $\langle 0, 1 \rangle$ ).

Pracę kończy wykaz pytań, na które dotychczas nie odpowiedziano.