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AN n-SILENT-VS.-NOISY DUEL WITH ARBITRARY ACCURACY FUNCTIONS

1. Introduction. The aim of this paper is to give a generalization of the game presented in [7]. The class of games of timing is described in details in [2]. Some new types of these games have been published in [1] and [3]-[6]. We consider the following model of a duel:

Two opponents, denoted by A and B, have n and one bullets, respectively. None of the n shots of A is heard by B. The shot of B is heard by A. The probability of hitting the opponent (an *accuracy function*) is a function of time, $P(t)$ and $Q(t)$ for A and B, respectively.

We assume that

(i) $P(t)$ and $Q(t)$ are differentiable, and $P'(t) > 0$ and $Q'(t) > 0$ for $t \in (0, 1)$;

(ii) $P(0) = Q(0) = 0$ and $P(1) = Q(1) = 1$.

Player A gets a value $+1$ if he hits B not being hit himself, and a value -1 if he is hit by B not hitting B before. The game is over when one of the opponents is hit. The situation is reverse for player B so that the game is a zero-sum game. The pure strategies for the players, i.e. the moments of shooting, will be denoted by x_i with $i = 1, 2, \ldots, n$, $x_i \in [0, 1]$, and by y with $y \in [0, 1]$ for A and B, respectively.

For this game we give optimal strategies for both players. At first, we define the spaces of mixed strategies for both players and evaluate the game pay-off function. Next, in Section 3, we formulate a system of integral and differential equations for the density functions of the optimal mixed strategies for A and B, and discuss the existence and uniqueness of its solution. Section 4 contains the equations for determination of constants appearing in the analytical form of the strategies found. Section 5 gives a proof of optimality, and in Section 6 we present a numerical example of the described game.

2. Pay-off function for the game of timing. It is easy to find the form of the pay-off function using the induction with respect to n. We prove that, for every natural $n \geqslant 2$, the pay-off function has the following form:

$$K(x_1, x_2, \ldots, x_n;\ y) \tag{1}$$

$$= \begin{cases} 1-[1+Q(y)]\prod_{j=1}^{n}[1-P(x_j)] & \text{if } x_1 < x_2 < \ldots < x_n < y, \\ 1-2Q(y)\prod_{j=1}^{i}[1-P(x_j)] & \text{if } x_i < y < x_{i+1}, \\ & \quad i = 1, 2, \ldots, n-1, \\ 1-2Q(y) & \text{if } y < x_1 < x_2 < \ldots < x_n. \end{cases}$$

Let $n = 2$; then we obtain

$$K(x_1, x_2; y) = \begin{cases} P(x_1)+[1-P(x_1)]\big[P(x_2)-\big(1-P(x_2)\big)Q(y)\big] & \\ \quad = 1-[1+Q(y)][1-P(x_1)][1-P(x_2)] & \text{for } x_1 < x_2 < y, \\ P(x_1)+[1-P(x_1)][-Q(y)+1-Q(y)] & \\ \quad = 1-2Q(y)[1-P(x_1)] & \text{for } x_1 < y < x_2, \\ -Q(y)+1-Q(y) = 1-2Q(y) & \text{for } y < x_1 < x_2. \end{cases}$$

Clearly, expression (1) is valid for $n = 2$. Now, we assume (1) to be valid for some $n = k$. We prove its validity for $n = k+1$, denoting by x_0 the first moment of shooting for A. We have to consider the following cases:

1° If $x_0 < x_1 < \ldots < x_k < y$, then

$$K(x_0, x_1, \ldots, x_k;\ y)$$
$$= P(x_0)+\big(1-P(x_0)\big)K(x_1, x_2, \ldots, x_k;\ y) = 1-[1+Q(y)]\prod_{j=0}^{n}[1-P(x_j)].$$

2° If $x_i < y < x_{i+1}$ and $i = 0, 1, 2, \ldots, k-1$, then

$$K(x_0, x_1, \ldots, x_k;\ y)$$
$$= P(x_0)+\big(1-P(x_0)\big)K(x_1, x_2, \ldots, x_k;\ y) = 1-2Q(y)\prod_{j=0}^{i}[1-P(x_j)].$$

3° If $y < x_0 < x_1 < \ldots < x_k$, then

$$K(x_0, x_1, \ldots, x_k;\ y) = -Q(y)+1-Q(y) = 1-2Q(y).$$

It is sufficient to renumerate the variables x_i to see that we have obtained form (1) for $n = k+1$.

Now, let us define a class of mixed strategies for both players in which we shall seek optimal solutions. They will represent density functions of random variables x_i $(i = 1, 2, \ldots, n)$ for player A, and $g(y)$ with weight $\beta = \mathrm{P}\{y = 1\}$ for player B.

Thus we have, for A, $\{f_i(x_i);\ x_i \in [a_i, a_{i+1})\}_{i=1,2,\ldots,n}$ with a normalization condition

$$(2) \qquad \int_{a_i}^{a_{i+1}} f_i(x_i)\,dx_i = 1, \quad i = 1, 2, \ldots, n \ (a_{n+1} = 1),$$

and, for B, a function $g(y)$, $y \in [a_1, 1)$ and $\beta > 0$, such that

$$(3) \qquad \int_{a_1}^{1} g(y)\,dy + \beta = 1.$$

3. System of integral-differential equations for the functions $f_i(x_i)$ and $g(y)$. We assume heuristically for the beginning that there exists a value of the game. We justify this assumption later in Section 4.

Thus, if $y \in \operatorname{supp} g$, then, by (1), we have the following expressions for the value v:

$$(4) \quad v = \int_{a_1}^{y} [1-2Q(y)[1-P(x_1)]]f_1(x_1)\,dx_1 + \int_{y}^{a_2} [1-2Q(y)]f_1(x_1)\,dx_1 \quad \text{if } y \in [a_1, a_2);$$

$$(5) \quad v = \int_{a_1}^{a_2} \ldots \int_{a_{i-1}}^{a_i} \Big[\int_{a_i}^{y} \Big[1-2Q(y)\prod_{j=1}^{i}[1-P(x_j)]\Big] f_i(x_i)\,dx_i + \int_{y}^{a_{i+1}} \Big[1-2Q(y)\prod_{j=1}^{i-1}[1-P(x_j)]f_i(x_i)\,dx_i\Big]\Big]\prod_{j=1}^{i-1} f_j(x_j)\,dx_j$$
$$\text{if } y \in [a_i, a_{i+1}) \text{ with } i = 2, 3, \ldots, n-1;$$

$$(6) \quad v = \int_{a_1}^{a_2} \ldots \int_{a_{n-1}}^{a_n} \Big[\int_{a_n}^{y} \Big[1-(1+Q(y))\prod_{j=1}^{n}[1-P(x_j)]\Big]f_n(x_n)\,dx_n + \int_{y}^{1} \Big[1-2Q(y)\prod_{j=1}^{n-1}[1-P(x_j)]f_n(x_n)\,dx_n\Big]\Big]\prod_{j=1}^{n-1} f_j(x_j)\,dx_j \,. \quad \text{if } y \in [a_n, 1].$$

We use (2) to obtain

$$v = 1-2\Big[Q(y)-Q(y)\int_{a_1}^{y} P(x_1)f_1(x_1)\,dx_1\Big], \qquad y \in [a_1, a_2);$$

$$v = 1-2\prod_{j=1}^{j-1}\int_{a_j}^{a_j+1} [1-P(x_j)]f_j(x_j)\,dx_j\Big[Q(y)-Q(y)\int_{a_i}^{y} P(x_i)f_i(x_i)\,dx_i\Big],$$
$$y \in [a_i, a_{i+1}),\ i = 2, 3, \ldots, n-1;$$

$$v = 1-\prod_{j=1}^{n-1}\int_{a_j}^{a_j+1} [1-P(x_j)]f_j(x_j)\,dx_j\Big[(1+Q(y))\int_{a_n}^{y} (1-P(x_n))f_n(x_n)\,dx_n + 2Q(y)\int_{y}^{1} f_n(x_n)\,dx_n\Big], \qquad y \in [a_n, 1],$$

instead of (4), (5) and (6), respectively.

The differentiation by sides with respect to y of these expressions leads to

$$\text{(7)}\qquad \frac{P(y)f_1(y)}{1-\int\limits_{a_n}^{y}P(u)f_1(u)\,du} = \frac{Q'(y)}{Q(y)}, \quad y\in[a_1, a_2);$$

$$\text{(8)}\qquad \frac{P(y)f_i(y)}{1-\int\limits_{a_i}^{y}P(u)f_i(u)\,du} = \frac{Q'(y)}{Q(y)}, \quad y\in[a_i, a_{i+1}),\ i = 2, 3, \ldots, n-1;$$

$$\text{(9)}\qquad \frac{-\big(1+P(y)\big)f_n(y)}{1-\int\limits_{a_n}^{y}P(u)f_n(u)\,du+\int\limits_{y}^{1}f_n(u)\,du} = \frac{Q'(y)[1+P(y)]}{[1+Q(y)][1-P(y)]-2Q(y)},$$

$$y\in[a_n, 1).$$

Taking into account the boundary conditions

$$f_1(a_1) = \frac{Q'(a_1)}{P(a_1)Q(a_1)}, \quad f_i(a_i) = \frac{Q'(a_i)}{P(a_i)Q(a_i)}, \ i = 2, 3, \ldots, n-1,$$

$$f_n(a_n) = \frac{2Q'(a_n)}{2Q(a_n)-[1+Q(a_n)][1-P(a_n)]},$$

we have the following solutions of (7), (8) and (9):

$$f_1(x_1) = \frac{Q(a_1)Q'(x_1)}{P(x_1)Q^2(x_1)}, \quad x_1\in[a_1, a_2);$$

$$\text{(10)}\qquad f_i(x_i) = \frac{Q(a_i)Q'(x_i)}{P(x_i)Q^2(x_i)}, \quad x_i\in[a_i, a_{i+1}), \quad i = 2, 3, \ldots, n-1;$$

$$f_n(x_n) =$$

$$= \frac{2Q'(x_n)}{2Q(x_n)-[1+Q(x_n)][1-P(x_n)]}\exp\left\{\int\limits_{a_n}^{x_n}\frac{Q'(u)[1+P(u)]\,du}{[1+Q(u)][1-P(u)]-2Q(u)}\right\},$$

$$x_n\in[a_n, 1].$$

Assumptions (i) and (ii) assure the existence and uniqueness of these solutions.

Now, let $x_i\in\operatorname{supp} f_i$, $i = 1, 2, \ldots, n$. Then we have the following expression for the value of the game with respect to player B:

$$v = \int\limits_{a_1}^{x_1}[1-2Q(y)]g(y)\,dy+\sum_{k=1}^{n-1}\int\limits_{x_k}^{x_{k+1}}\Big[1-2Q(y)\prod_{s=1}^{k}[1-P(x_s)]\Big]g(y)\,dy+$$

$$+\int\limits_{x_n}^{1}\Big[1-\big(1+Q(y)\big)\prod_{s=1}^{n}[1-P(x_s)]\Big]g(y)\,dy+\beta\Big[1-2\prod_{j=1}^{n}[1-P(x_j)]\Big].$$

Requiring (3) to be satisfied, we obtain

$$(11)\quad v = 1-2\int_{a_1}^{x_1} Q(y)g(y)dy-2\sum_{k=1}^{n-1}\prod_{s=1}^{k}[1-P(x_s)]\Big\{\int_{x_k}^{a_{k+1}} Q(y)g(y)dy+$$

$$+\int_{a_{k+1}}^{x_{k+1}} Q(y)g(y)dy\Big\}-\prod_{j=1}^{n}[1-P(x_j)]\int_{x_n}^{1}[1+Q(y)]g(y)dy-2\beta\prod_{j=1}^{n}[1-P(x_j)].$$

Let us differentiate this expression by sides with respect to x_n. We get

$$\prod_{s=1}^{n-1}[1-P(x_s)]\Big\{2Q(x_n)g(x_n)-P'(x_n)\int_{x_n}^{1}[1+Q(y)]g(y)dy-$$

$$-[1-P(x_n)][1+Q(x_n)]g(x_n)-2\beta P'(x_n)\Big\}=0$$

or, equivalently,

$$(12)\quad \frac{-[1+Q(x_n)]g(x_n)}{2\beta+\int_{x_n}^{1}[1+Q(y)]g(y)dy}=\frac{P'(x_n)[1+Q(x_n)]}{[1+Q(x_n)][1-P(x_n)]-2Q(x_n)}.$$

The solution of the original integral equation (12) has the following form:

$$(13)\quad g(y)=$$

$$=\frac{2\beta P'(y)}{2Q(y)-[1+Q(y)][1-P(y)]}\exp\Big\{-\int_{y}^{1}\frac{P'(u)[1+Q(u)]du}{[1+Q(u)][1-P(u)]-2Q(u)}\Big\}.$$

Let us introduce

$$2K_n=2\int_{a_n}^{x_n}Q(y)g(y)dy+[1-P(x_n)]\int_{x_n}^{1}[1+Q(y)]g(y)dy+2\beta[1-P(x_n)].$$

By (12), we know that

$$\frac{dK_n}{dx_n}=0 \quad \text{for } x_n\in[a_n,1],$$

which means that $2K_n$ is constant in $[a_n, 1]$. In the special case where $x_n = a_n$, using (12) and (13), we have

$$(14)\quad 2K_n=2\beta[1-P(a_n)]\exp\Big\{\int_{a_n}^{1}\frac{P'(u)[1+Q(u)]du}{P(u)Q(u)+P(u)+Q(u)-1}\Big\}.$$

Thus relation (11) reduces, by (13), to

$$(15)\quad v = 1-2\int_{a_1}^{x_1} Q(y)g(y)dy-2\sum_{k=1}^{n-2}\prod_{s=1}^{k}[1-P(x_s)]\Big\{\int_{x_k}^{a_{k+1}} Q(y)g(y)dy+$$

$$+\int_{a_{k+1}}^{x_{k+1}} Q(y)g(y)dy\Big\}-2\prod_{j=1}^{n-1}[1-P(x_j)]\int_{x_{n-1}}^{a_n}Q(y)g(y)dy-2K_n\prod_{j=1}^{n-1}[1-P(x_j)].$$

The differentiation of (15) by sides with respect to x_{n-1} gives

$$\prod_{j=1}^{n-1}[1-P(x_j)]\Big\{Q(x_{n-1})g(x_{n-1})-P'(x_{n-1})\int_{x_{n-1}}^{a_n}Q(y)g(y)\,dy-$$
$$-[1-P(x_{n-1})]Q(x_{n-1})g(x_{n-1})-P'(x_{n-1})K_n\Big\}=0$$

or

$$\frac{Q(x_{n-1})g(x_{n-1})}{K_n+\int_{x_{n-1}}^{a_n}Q(y)g(y)\,dy}=\frac{P'(x_{n-1})}{P(x_{n-1})},\qquad x_{n-1}\in[a_{n-1},a_n]. \tag{16}$$

As the solution of (16) we have

$$g(y)=\frac{l_{n-1}P'(y)}{Q(y)P^2(y)},\qquad \text{where } l_{n-1}=P(a_n)K_n,\ y\in[a_{n-1},a_n). \tag{17}$$

Now, using (17) in (15), we obtain a new expression

$$v=1-2\int_{a_1}^{x_1}Q(y)g(y)\,dy-$$
$$-2\sum_{k=1}^{n-3}\prod_{s=1}^{k}[1-P(x_s)]\Big\{\int_{x_k}^{a_{k+1}}Q(y)g(y)\,dy+\int_{a_{k+1}}^{x_{k+1}}Q(y)g(y)\,dy\Big\}-$$
$$-2\prod_{s=1}^{n-2}[1-P(x_s)]\int_{x_{n-2}}^{a_{n-1}}Q(y)g(y)\,dy-\prod_{s=1}^{n-2}[1-P(x_s)]l_{n-1}\frac{1-P(a_{n-1})}{P(a_{n-1})}.$$

We differentiate this expression with respect to x_{n-2} and continue step by step the procedure outlined above. As the result of this method we obtain

(18)

$$g(y)=\begin{cases}\dfrac{l_iP'(y)}{Q(y)P^2(y)} & \text{for } y\in[a_i,a_{i+1}),\ i=1,2,\ldots,n-1,\\[2ex] \dfrac{2\beta P'(y)}{P(y)Q(y)+P(y)+Q(y)-1}\exp\Big\{\displaystyle\int_y^1\frac{P'(u)[1+Q(u)]\,du}{P(u)Q(u)+P(u)+Q(u)-1}\Big\} & \text{for } y\in[a_n,1),\end{cases}$$

where

$$l_{i+1}=\frac{l_i}{1-P(a_{i+1})},\qquad i=1,2,\ldots,n-2,\ \text{and } l_{n-1}=P(a_n)K_n.$$

Relation (11) is reduced to

$$v = 1-2l_1\frac{1-P(a_1)}{P(a_1)}. \tag{19}$$

Relations (10) and (18) give an analytical form of the strategies belonging to the class defined above.

4. System of equations for constants l_i, a_i ($i = 1, 2, \ldots, n$), β and v. We assume that the strategies are given by (10) and (18) and evaluate corresponding integrals

$$Q(a_j)\int_{a_j}^{a_{j+1}} \frac{1-P(x_j)}{P(x_j)Q^2(x_j)}Q'(x_j)\,dx_j$$

$$= 1-Q(a_j)\left[\frac{1}{Q(a_j)}-\frac{1}{Q(a_{j+1})}\right] = \frac{Q(a_j)}{Q(a_{j+1})}, \quad j = 1, 2, \ldots, n-1,$$

and

$$\prod_{j=1}^{i-1}\frac{Q(a_j)}{Q(a_{j+1})} = \frac{Q(a_1)}{Q(a_i)}, \quad i = 1, 2, \ldots, n,$$

to obtain, instead of the expressions on page 207,

$$v = 1-2Q(a_1), \quad x_1 \in [a_1, a_2), \tag{20}$$

$$v = 1-2\frac{Q(a_1)}{Q(a_i)}Q(a_i) = 1-2Q(a_1), \tag{21}$$

$$x_i \in [a\ , a_{i+1}),\ i = 2, 3, \ldots, n-1.$$

Let us write

$$S(y) = [1+Q(y)]\int_{a_n}^{y}[1-P(x_n)]f_n(x_n)\,dx_n + 2Q(y)\int_{y}^{1} f_n(x_n)\,dx_n.$$

We shall prove that

$$S(y) = 2Q(a_n) \quad \text{for } y \in [a_n, 1) \tag{22}$$

which leads to

$$v = 1-\frac{Q(a_1)}{Q(a_n)}S(y). \tag{23}$$

We already know by (9) that $S'(y) = 0$ for $y \in [a_n, 1)$. Thus $S(y)$ is constant in the interval. It is sufficient to prove the existence and uniqueness of $a_n \in (0, 1)$ such that (22) is valid for arbitrary functions $P(t)$ and $Q(t)$ satisfying our assumptions (i) and (ii). It is easy to see that $S(y)$ is a continuous function and, for the constant a_n satisfying (22), the normalization condition (2) is also satisfied. If $y = 1$, then we have to prove that there exists a constant a_n such that

$$(24)\qquad \int\limits_{a_n}^{1} [1-P(x_n)]f_n(x_n)\,dx_n = Q(a_n), \quad 0 < a_n < 1.$$

First, we use the form of $f_n(x_n)$ given by (10) to evaluate the following integral:

$$(25)\quad \int\limits_{a_n}^{1} [1+P(x_n)]f_n(x_n)\,dx_n = 2-2\exp\Big\{-\int\limits_{a_n}^{1} \frac{Q'(u)\big(1+P(u)\big)du}{P(u)Q(u)+P(u)+Q(u)-1}\Big\}.$$

Adding by sides (24) and (25) with condition (2) satisfied, we obtain

$$(26)\qquad \ln\frac{Q(a_n)}{2} = -\int\limits_{a_n}^{1} \frac{Q'(u)\big(1+P(u)\big)du}{P(u)Q(u)+P(u)+Q(u)-1}.$$

Let

$$r(z) \overset{\text{df}}{=} \int\limits_{z}^{1} \frac{Q'(u)\big(1+P(u)\big)du}{P(u)Q(u)+P(u)+Q(u)-1} + \ln\frac{Q(z)}{2}, \quad z\in(0,1).$$

Then

$$r'(z) = -\frac{Q'(u)\big(1-P(u)\big)}{Q(u)\big(P(u)Q(u)+P(u)+Q(u)-1\big)} < 0, \quad z\in(t_0,1),$$

where $t_0 \neq 1$ is the unique root of the equation

$$P(t)Q(t)+P(t)+Q(t)-1 = 0 \quad \text{in } (0,1).$$

It is easy to check that $r(z)$ is a continuous function and

$$\lim_{t\to t_0^+} r(t) = \infty \quad \text{and} \quad r(1) = -\ln 2 < 0.$$

So, there exists a unique constant $a_n \in (t_0, 1)$ such that $r(a_n) = 0$ and (22) is valid. Thus, (23) gives $v = 1-2Q(a_1)$.

We shall use the existence of a_n to prove, by induction, the existence and uniqueness of $a_{n-1}, a_{n-2}, \ldots, a_1$. First, let us show that there exists a unique a_{n-1} such that (2) is satisfied. Let

$$s_{n-1}(z) \overset{\text{df}}{=} \int\limits_{z}^{a_n} \frac{Q'(x_{n-1})\,dx_{n-1}}{P(x_{n-1})Q^2(x_{n-1})} - \frac{1}{Q(z)}$$

and

$$s_i(z) \overset{\text{df}}{=} \int\limits_{z}^{a_{i+1}} \frac{Q'(x_i)\,dx_i}{P(x_i)Q^2(x_i)} - \frac{1}{Q(z)}.$$

We notice that $s_{n-1}(z)$ is continuous and

$$s'_{n-1}(z) = \frac{Q'(z)[P(z)-1]}{Q^2(z)P(z)} < 0, \quad z \in (0, a_n),$$

and

$$\lim_{z\to 0+} s_{n-1}(z) = \infty, \quad s_{n-1}(a_n) = -\frac{1}{Q(a_n)} < 0.$$

Thus, there exists a unique a_{n-1} such that (2) is valid and $a_{n-1} \in (0, a_n)$. Let us assume the existence and uniqueness of $a_{i+1} \in (0, a_{i+2})$. To prove that there exists a unique $a_i \in (0, a_{i+1})$ for (2) we notice that

$$s'_i(z) = \frac{Q'(z)[P(z)-1]}{P(z)Q^2(z)} < 0, \quad z \in (0, a_{i+1}),$$

and

$$\lim_{z\to 0+} s_i(z) = \infty, \quad s_i(a_{i+1}) = -\frac{1}{Q(a_{i+1})} < 0.$$

Thus we have proved the existence and uniqueness of $0 < a_1 < a_2 < a_3 < \ldots < a_n < 1$.

Requiring equations (20), (21), (23) and (19) to be consistent we must put

$$l_1 = \frac{P(a_1)Q(a_1)}{1-P(a_1)} \tag{27}$$

which gives an expression for $v = 1-2Q(a_1)$.

We take into account relations (18) and (27) to evaluate all coefficients l_i $(i = 1, 2, \ldots, n-1)$.

Now, we determine the coefficient β. To do this we divide by sides expressions (14) and (26) and obtain

$$\begin{aligned}\frac{2K_n}{Q(a_n)} &= \beta[1-P(a_n)]\exp\left\{\int_{a_n}^{1} \frac{P'(u)Q(u)+Q'(u)P(u)+P'(u)+Q'(u)}{P(u)Q(u)+P(u)+Q(u)-1}du\right\} \\ &= \frac{2\beta[1-P(a_n)]}{W(a_n)},\end{aligned}$$

where

$$W(t) = P(t)Q(t)+P(t)+Q(t)-1.$$

It is sufficient to use $l_{n-1} = P(a_n)K_n$ and (13) to get

$$\beta = \frac{W(a_n)l_{n-1}}{[1-P(a_n)]Q(a_n)P(a_n)} > 0. \tag{28}$$

It is clear that $0 < \beta < 1$.

It remains to check the normalization condition (2), i.e.

$$(29)\qquad \sum_{i=1}^{n-1}\int_{a_i}^{a_{i+1}} l_i\frac{P'(y)\,dy}{Q(y)P^2(y)}+$$
$$+\int_{a_n}^{1}\frac{2\beta P'(y)}{P(y)Q(y)+P(y)+Q(y)-1}\exp\left\{\int_{y}^{1}\frac{P'(u)[1+Q(u)]\,du}{P(u)Q(u)+P(u)+Q(u)-1}\right\}dy+$$
$$+\beta=1.$$

Integrating by parts and using (2), we have

$$l_i\int_{a_i}^{a_{i+1}}\frac{P'(y)\,dy}{Q(y)P^2(y)}=l_i\left[\frac{1-P(a_i)}{Q(a_i)P(a_i)}-\frac{1}{P(a_{i+1})Q(a_{i+1})}\right]$$

and

$$l_{i+1}=\frac{l_i}{1-P(a_{i+1})}\quad(i=1,2,\ldots,n-2),\qquad l_1=\frac{P(a_1)Q(a_1)}{1-P(a_1)},$$

we obtain, after reduction, instead of (29) the following relation to be proved:

$$(30)\qquad 1-\frac{l_{n-1}}{P(a_n)Q(a_n)}+$$
$$+\int_{a_n}^{1}\frac{2\beta P'(y)}{P(y)Q(y)+P(y)+Q(y)-1}\exp\left\{\int_{y}^{1}\frac{P'(u)[1+Q(u)]\,du}{P(u)Q(u)+P(u)+Q(u)-1}\right\}dy+$$
$$+\beta=1.$$

In the Appendix (p. 224) we prove the following relationship:

$$\int_{a_n}^{1}\frac{P'(y)}{P(y)Q(y)+P(y)+Q(y)-1}\exp\left\{\int_{y}^{1}\frac{P'(u)[1+Q(u)]\,du}{P(u)Q(u)+P(u)+Q(u)-1}\right\}dy$$
$$=-1+\exp\left\{\int_{a_n}^{1}\frac{P'(u)[1+Q(u)]\,du}{P(u)Q(u)+P(u)+Q(u)-1}\right\}-\frac{K_n}{2\beta}$$
$$=-1+\frac{K_n}{\beta[1-P(a_n)]}-\frac{K_n}{2\beta}.$$

Thus, the left-hand side of (30) is equal to

$$1-\frac{l_{n-1}}{P(a_n)Q(a_n)}+\left(-2\beta+\frac{2K_n}{1-P(a_n)}-K_n\right)+\beta=1-\frac{l_{n-1}}{P(a_n)Q(a_n)}-$$
$$-\frac{W(a_n)l_{n-1}}{[1-P(a_n)]P(a_n)Q(a_n)}+\frac{2l_{n-1}}{[1-P(a_n)]P(a_n)}-\frac{l_{n-1}}{P(a_n)}=1-\frac{l_{n-1}}{P(a_n)}\times$$
$$\times\frac{1-P(a_n)+P(a_n)+Q(a_n)+P(a_n)Q(a_n)-1-2Q(a_n)+Q(a_n)-P(a_n)Q(a_n)}{[1-P(a_n)]Q(a_n)}$$
$$=1.$$

Thus, the functions $g(y)$ given by (18) and β given by (28) define a probability distribution function which we denote by $G(y)$. The strategies described for players A and B will be denoted by S_A and S_B, respectively. We shall prove that these strategies are optimal. It is clear that in case where B has fired and not hit, player A will use the pure strategy at 1 with one of his remaining bullets. Further, the actual number of bullets in possession of A has no effect on the mixed strategy of player B during the game.

5. Proof of optimality for S_A and S_B. We prove that S_A and S_B are optimal against each other, whence, by the general theorem, we have their absolute optimality.

Let player A follow his strategy S_A. We denote by $K(S_A, y)$ the value of the pay-off function when A applies a mixed strategy S_A, and B a pure strategy y. We define $K(x_1, x_2, \ldots, x_n;\ S_B)$ in a similar way. We have to show that

$$\min_{0\leqslant y\leqslant 1} K(S_A, y) = v = 1-2Q(a_1). \tag{31}$$

Let us consider the following cases:

1° If $y < a_1$, then $K(S_A, y) = 1-2Q(y) > 1-2Q(a_1) = v$.

2° If $y \in [a_1, 1]$, then, by (19) and (27), we have $K(S_A, y) = 1-2Q(a_1) = v$.

Hence it follows that S_B is relatively best against S_A and (31) is valid.

Now, let us assume that player B follows his strategy S_B. We have to show that

$$\max_{0\leqslant x_1\leqslant x_2\leqslant\ldots\leqslant x_n\leqslant 1} K(x_1, x_2, \ldots, x_n;\ S_B) = v = 1-2Q(a_1) \tag{32}$$

or, equivalently, that

$$\int_{a_1}^{1} K(x_1, x_2, \ldots, x_n;\ y)\,dG(y) \leqslant v \quad \text{for all } 0\leqslant x_1\leqslant x_2\leqslant\ldots\leqslant x_n\leqslant 1, \tag{33}$$

where $G(y)$ is the distribution determined by $g(y)$ and β is given by (18) and (28).

We prove (33) by induction with respect to the number of variables x_i ($i = 1, 2, \ldots, n$; $n = \text{const}$) lying in the corresponding intervals $[a_i, a_{i+1}]$. Let $\bar{x}_r$ be a vector $(x_1, x_2, \ldots, x_n)$ whose $n-r$ last components lie in their corresponding intervals $[a_i, a_{i+1}]$ for $i = r+1, r+2, \ldots, n$ ($a_{n+1} = 1$).

The first step in the induction procedure requires to check that (33) is valid in the case $r = 1$. According to the value taken by x_1 with $x_i \in [a_i, a_{i+1}]$ for $i = 2, 3, \ldots, n$, we have the following cases:

1° If $x_1 < a_1$, then

$$K(x_1, x_2, \ldots, x_n;\ y) = P(x_1) + [1-P(x_1)]K(x_2, x_2, \ldots, x_n;\ y).$$

Integrating with respect to $dG(y)$, we obtain

$$\int_{a_1}^{1} K(x_1, x_2, \ldots, x_n;\ y)\,dG(y)$$

$$= P(x_1)+[1-P(x_1)]\int_{a_1}^{a_2}[1-2Q(y)]\frac{l_1P'(y)\,dy}{Q(y)P^2(y)}+$$

$$+[1-P(x_1)]\int_{a_2}^{1}K(x_2, x_3, \ldots, x_n;\ y)\,dG(y).$$

The expression $\int_{a_2}^{1} K(x_2, x_3, \ldots, x_n;\ y)\,dG_1(y)$ represents the value $v = 1-2Q(a_2)$ of the game when A has $n-1$ bullets and if

$$dG_1(y) = \frac{P(a_2)Q(a_2)[1-P(a_1)]}{P(a_1)Q(a_1)}\,dG(y).$$

This is true because of the remark made at the end of Section 4 and the fact that $x_i \in \operatorname{supp} f_i$ $(i = 2, 3, \ldots, n)$ which implies the validity of the corresponding system of equations for constants l_i, a_i $(i = 2, 3, \ldots, n)$ and β.

Thus

$$\int_{a_1}^{1} K(x_1, x_2, \ldots, x_n;\ y)\,dG(y) = P(x_1)+[1-P(x_1)]\frac{1-P(a_1)-2Q(a_1)}{1-P(a_1)}$$

$$= \frac{1-P(a_1)-2Q(a_1)}{1-P(a_1)}+P(x_1)\frac{2Q(a_1)}{1-P(a_1)}.$$

The left-hand side of this expression is an increasing function of x_1 and it achieves the value $v = 1-2Q(a_1)$ at $x_1 = a_1$.

2° If $x_1 \in [a_1, a_2]$, then, by (11),

$$\int_{a_1}^{1} K(x_1, x_2, \ldots, x_n;\ y)\,dG(y) = 1-2Q(a_1) = v.$$

3° If $x_1 > a_2$, then $x_1 \in [a_2, x_2]$, since $x_1 \leqslant x_2$ and $x_2 \in [a_2, a_3]$.

Hence, taking into account relation (11), we can write in this case

$$\int_{a_1}^{1} K(x_1, x_2, \ldots, x_n;\ y)\,dG(y) = 1-2\Bigg\{\int_{a_1}^{a_2} Q(y)\,dG(y)+\int_{a_2}^{x_1} Q(y)\,dG(y)+$$

$$+[1-P(x_1)]\Bigg[\int_{x_1}^{x_2} Q(y)\,dG(y)+\sum_{k=2}^{n-1}\prod_{s=2}^{k}[1-P(x_s)]\int_{x_k}^{a_2} Q(y)\,dG(y)+$$

$$+\prod_{j=2}^{n}[1-P(x_j)]\int_{x_n}^{1}\frac{1+Q(y)}{2}\,dG(y)+\beta\prod_{j=2}^{n}[1-P(x_j)]\Bigg]\Bigg\}.$$

We attach new terms in this expression to obtain, after the integration,

$$\int_{a_1}^{1} K(x_1, x_2, \ldots, x_n;\ y)\,dG(y) = 1-2\left\{\frac{P(a_1)Q(a_1)}{1-P(a_1)}\left(\frac{1}{P(a_1)}-\frac{1}{P(a_2)}\right)+\right.$$

$$\left.+\int_{a_2}^{x_1} Q(y)\,dG(y)-[1-P(x_1)]\int_{a_2}^{x_1} Q(y)\,dG(y)+[1-P(x_1)]L(x_2, x_3, \ldots, x_n)\right\},$$

where

$$L(x_2, x_3, \ldots, x_n)$$

$$= \int_{a_2}^{x_2} Q(y)\,dG(y)+\sum_{k=2}^{n-1}\prod_{j=2}^{k}[1-P(x_j)]\int_{x_k}^{x_{k+1}} Q(y)\,dG(y)+$$

$$+\prod_{j=2}^{n}[1-P(x_j)]\int_{x_n}^{1}\frac{1+Q(y)}{2}\,dG(y)+\beta\prod_{j=2}^{n}[1-P(x_j)].$$

Comparing this expression with (19) and using the fact that $x_i \in \operatorname{supp} f_i$ $(i = 2, 3, \ldots, n)$ we see that

$$L(x_2, x_3, \ldots, x_n) = l_2\,\frac{1-P(a_2)}{P(a_2)}$$

and obtain the relation

$$\int_{a_1}^{1} K(x_1, x_2, \ldots, x_n;\ y)\,dG(y) = 1-2\left\{\frac{Q(a_1)}{1-P(a_1)}-\frac{P(a_1)Q(a_1)}{[1-P(a_1)]P(a_2)}+\right.$$

$$\left.+P(x_1)l_2\left[\frac{1}{P(a_2)}-\frac{1}{P(x_1)}\right]+l_2\,\frac{1-P(a_2)}{P(a_2)}-P(x_1)l_2\,\frac{1-P(a_2)}{P(a_2)}\right\}$$

$$= 1-2\left\{\frac{Q(a_1)}{1-P(a_1)}-l_2+P(x_1)l_2\right\}$$

which is a decreasing function of x_1 attaining the value $v = 1-2Q(a_1)$ at $x_1 = a_2$. Thus the first step in the induction is completed, i.e. (33) is valid in this case.

Secondly, let us assume that, for any fixed r $(1 \leqslant r \leqslant n)$,

$$\int_{a_1}^{1} K(\bar{x}_r;\ y)\,dG(y) \leqslant v. \tag{34}$$

We prove that, for each $\bar{x}_{r+1}$, there exists a vector $\bar{x}_r$ such that

$$\int_{a_1}^{1} K(\bar{x}_{r+1};\ y)\,dG(y) \leqslant \int_{a_1}^{1} K(\bar{x}_r;\ y)\,dG(y). \tag{35}$$

By the assumption, we have

$$\int_{a_1}^{1} K(\bar{x}_{r+1};\ y)\,dG(y) \leqslant v.$$

According to the value of x_{r+1}, we consider the following cases:

1° If $x_{r+1} < a_1$, then

$$K(\bar{x}_{r+1};\ y) = P(x_1) + \sum_{k=1}^{r} \prod_{j=1}^{k} [1-P(x_j)]P(x_{k+1}) + \\ + \prod_{j=1}^{r+1} [1-P(x_j)]K(x_{r+2}, \ldots, x_n;\ y).$$

Integrating with respect to $dG(y)$, we give

$$\int_{a_1}^{1} K(x_1, x_2, \ldots, x_n;\ y)\,dG(y) = P(x_1) + \sum_{k=1}^{r} \prod_{j=1}^{k} [1-P(x_j)]P(x_{k+1}) \\ + \prod_{j=1}^{r+1} [1-P(x_j)] \int_{a_1}^{1} K(x_{r+2}, \ldots, x_n;\ y)\,dG(y).$$

The left-hand side of this expression is an increasing function of x_{r+1} in the interval $(0, a_1)$.

2° Let $a_1 \leqslant x_{r+1} < a_{r+1}$ and $x_i < a_1$ $(i = 1, 2, \ldots, s)$. For definitiveness, let $x_{r+1} \in [a_k, a_{k+1})$ $(1 \leqslant k \leqslant r)$, $x_j \in [a_1, a_k)$ $(j = s+1,\ s+2, \ldots, p)$ and $x_l \in [a_k, x_{r+1}]$ $(l = p+1, p+2, \ldots, r)$. In this case

$$(36) \quad \int_{a_1}^{1} K(\bar{x}_{r+1}; y)\,dG(y) \\ = T(x_1, x_2, \ldots, x_r) + \int_{x_r}^{x_{r+1}} \Big[1-2Q(y)\prod_{t=1}^{r} [1-P(x_t)]\Big]dG(y) + \\ + \int_{x_{r+1}}^{a_{k+1}} \Big[1-2Q(y)\prod_{t=1}^{r+1} [1-P(x_t)]\Big]dG(y) + \\ + \int_{a_{k+1}}^{a_{r+2}} \Big[1-2Q(y)\prod_{t=1}^{r+1} [1-P(x_t)]\Big]dG(y) + \\ + \int_{a_{r+2}}^{1} dG(y) - 2\prod_{j=1}^{r+1} [1-P(x_j)] \int_{a_{r+2}}^{x_{r+2}} Q(y)\,dG(y) - \\ -2\sum_{s=r+2}^{n-1} \prod_{j=1}^{s} [1-P(x_j)] \int_{x_s}^{x_{s+1}} Q(y)\,dG(y) - \int_{x_n}^{1} [1+Q(y)]\,dG(y) \prod_{j=1}^{n} [1-P(x_j)],$$

where

$$T(x_1, x_2, \ldots, x_r) = P(x_1) + \sum_{k=1}^{s-1} \prod_{i=1}^{k} [1-P(x_i)]P(x_{k+1}) +$$
$$+ \int_{a_1}^{x_{s+1}} \Big[1-2Q(y)\prod_{i=1}^{s} [1-P(x_i)]\Big] dG(y) +$$
$$+ \sum_{n=s+1}^{r-1} \int_{x_j}^{x_{j+1}} \Big[1-2Q(y)\prod_{m=1}^{j} [1-P(x_m)]\Big] dG(y).$$

We notice that

$$\int_{a_{k+1}}^{a_{r+2}} Q(y)\,dG(y) = \sum_{i=k+1}^{r+1} l_i \left[\frac{1}{P(a_i)} - \frac{1}{P(a_{i+1})}\right] \tag{37}$$

and

$$P(x_{r+1}) \int_{x_{r+1}}^{a_{k+1}} Q(y)\,dG(y) = l_k P(x_{r+1}) \left[\frac{1}{P(x_{r+1})} - \frac{1}{P(a_{k+1})}\right], \tag{38}$$

$$l_{r+2}\, \frac{1-P(a_{r+2})}{P(a_{r+2})} = \frac{l_{r+1}}{P(a_{r+2})}. \tag{39}$$

Now, we add and subtract the term

$$\int_{a_k}^{x_r} \Big[1-2Q(y)\prod_{t=1}^{r} [1-P(x_t)]\Big] dG(y)$$

in (36) to get, using (37), (38) and (39), the relation

$$\int_{a_1}^{1} K(\bar{x}_{r+1};\, y)\,dG(y)$$
$$= T(x_1, x_2, \ldots, x_r) + \int_{a_k}^{1} dG(y) - \int_{a_k}^{x_r} \Big[1-2Q(y)\prod_{t=1}^{r} [1-P(x_t)]\Big] dG(y) -$$
$$-2 \prod_{j=1}^{r} [1-P(x_j)] \Big\{ l_k \int_{a_k}^{x_{r+1}} \frac{P'(y)}{P^2(y)}\,dy +$$
$$+ l_k [1-P(x_{r+1})] \int_{x_{r+1}}^{a_{k+1}} \frac{P'(y)}{P^2(y)}\,dy + [1-P(x_{r+1})] \int_{a_{k+1}}^{a_{r+2}} Q(y)\,dG(y) +$$
$$+ [1-P(x_{r+1})]\, S(x_{r+2}, x_{r+3}, \ldots, x_n) \Big\},$$

where

$$S(x_{r+2}, x_{r+3}, \ldots, x_n)$$
$$= \int_{a_{r+2}}^{x_{r+2}} Q(y)dG(y) + \sum_{k=r+2}^{n-1} \prod_{j=r+2}^{k} [1-P(x_j)] \int_{x_k}^{x_{k+1}} Q(y)dG(y) +$$
$$+ \prod_{j=r+2}^{n} [1-P(x_j)] \int_{x_n}^{1} \frac{1+Q(y)}{2} dG(y) + \beta \prod_{j=2}^{n} [1-P(x_j)].$$

By similar considerations as above for the function $L(x_2, x_3, \ldots, x_n)$ with the assumption $x_i \in [a_i, a_{i+1}]$ $(i = r+2, r+3, \ldots, n)$ we have

$$S(x_{r+2}, x_{r+3}, \ldots, x_n) = l_{r+2} \frac{1-P(a_{r+2})}{P(a_{r+2})} = \frac{l_{r+1}}{P(a_{r+2})}. \tag{40}$$

Hence

$$\int_{a_1}^{1} K(\bar{x}_{r+1}; y)dG(y)$$
$$= T(x_1, x_2, \ldots, x_r) - \int_{a_k}^{x_r} \Big[1-2Q(y)\prod_{l=1}^{r} [1-P(x_l)]\Big]dG(y) + \int_{a_k}^{1} dG(y) -$$
$$-2\prod_{j=1}^{r} [1-P(x_j)]\left\{\int_{a_k}^{a_{r+2}} Q(y)dG(y) - l_k + \frac{l_{r+1}}{P(a_{r+2})} - P(x_{r+1}) \sum_{s=k}^{r} \frac{l_{s+1}-l_s}{P(a_{s+1})}\right\}.$$

Thus, the left-hand side of this expression is an increasing function of x_{r+1}, since $l_{s+1} > l_s$ for every $s = k, k+1, \ldots, r$, and assumptions (i) and (ii) concerning the function $P(t)$ are valid. In the above-given consideration k is arbitrary, so $\int_{a_1}^{1} K(\bar{x}_{r+1}; y)dG(y)$ is an increasing function of x_{r+1} in the interval $[a_1, a_{r+1}]$.

3° If $x_{r+1} \in [a_{r+1}, a_{r+2}]$, then this leads directly to assumption (34).

4° Now, let $x_p, x_{p+1}, \ldots, x_r, x_{r+1}$ $(p = 1, 2, \ldots, r+1)$ lie in the interval $[a_{r+2}, x_{r+2}]$. In this case we prove that $\int_{a_1}^{1} K(\bar{x}_{r+1}; y)dG(y)$ is a decreasing function at each $x_p, x_{p+1}, \ldots x_r, x_{r+1}$.

Denoting by $T(x_1, x_2, \ldots, x_{p-1})$ the sum of the first terms depending only on $x_1, x_2, \ldots, x_{p-1}$, we have

$$\int_{a_1}^{1} K(\bar{x}_{r+1}; y)dG(y)$$
$$= T(x_1, x_2, \ldots, x_{p-1}) + \int_{a_{r+2}}^{x_p} \Big[1-2Q(y)\prod_{j=1}^{p-1} [1-P(x_j)]\Big] dG(y) +$$

$$+\sum_{t=p}^{r+1}\int_{x_t}^{x_{t+1}}\Big[1-2Q(y)\prod_{j=1}^{t}[1-P(x_j)]\Big]dG(y)+$$

$$+\int_{x_{r+2}}^{a_{r+3}}\Big[1-2Q(y)\prod_{j=1}^{r+2}[1-P(x_j)]\Big]dG(y)+$$

$$+\int_{a_{r+3}}^{x_{r+3}}\Big[1-2Q(y)\prod_{j=1}^{r+2}[1-P(x_j)]\Big]dG(y)+$$

$$+\sum_{s=r+3}^{n-1}\int_{x_s}^{x_{s+1}}\Big[1-2Q(y)\prod_{j=1}^{s}[1-P(x_j)]\Big]dG(y)+$$

$$+\int_{x_n}^{1}\Big[1-\big(1+Q(y)\big)\prod_{j=1}^{n}[1-P(x_j)]\Big]dG(y).$$

Now, we add and subtract a new term to obtain, after the integration,

$$(41)\quad \int_{a_1}^{1}K(\bar{x}_{r+1};y)\,dG(y)$$

$$=T(x_1,x_2,\ldots,x_{p-1})+\int_{a_{r+2}}^{1}dG(y)-2\prod_{j=1}^{p-1}[1-P(x_j)]\Big\{\int_{a_{r+2}}^{x_p}Q(y)\,dG(y)+$$

$$+\sum_{k=p}^{r}\prod_{j=p}^{k}[1-P(x_j)]\int_{x_k}^{x_{k+1}}Q(y)\,dG(y)\Big\}+2\prod_{j=1}^{r+1}[1-P(x_j)]\int_{a_{r+2}}^{x_{r+1}}Q(y)\,dG(y)-$$

$$-2\prod_{j=1}^{r+1}[1-P(x_j)]\Big\{\int_{a_{r+2}}^{x_{r+2}}Q(y)\,dG(y)+\sum_{k=r+2}^{n-1}\prod_{j=r+2}^{k}[1-P(x_j)]\int_{x_k}^{x_{k+1}}Q(y)\,dG(y)+$$

$$+\frac{1}{2}\prod_{j=r+2}^{n}[1-P(x_j)]\int_{x_n}^{1}\big(1+Q(y)\big)\,dG(y)\Big\}.$$

We use (40) in (41) and evaluate these integrals. Hence

$$(42)\quad \int_{a_1}^{1}K(\bar{x}_{r+1};y)\,dG(y)=T(x_1,x_2,\ldots,x_{p-1})+\int_{a_{r+1}}^{1}dG(y)+$$

$$+2l_{r+2}\prod_{j=1}^{r+1}[1-P(x_j)]\left[\frac{1}{P(a_{r+2})}-\frac{1}{P(x_{r+1})}\right]-$$

$$-2\prod_{j=1}^{r+1}[1-P(x_j)]\frac{l_{r+2}[1-P(a_{r+2})]}{P(a_{r+2})}-2l_{r+2}\prod_{j=1}^{p-1}[1-P(x_j)]\Big\{\frac{1}{P(a_{r+2})}-$$

$$-\frac{1}{P(x_p)}+\sum_{k=p}^{r}\prod_{j=p}^{k}[1-P(x_j)]\left[\frac{1}{P(x_k)}-\frac{1}{P(x_{k+1})}\right]\Big\}.$$

We use the following relation to reduce some of the terms in (42):

$$\prod_{j=p}^{i}[1-P(x_j)]\left[-\frac{1}{P(x_{i+1})}\right]+\prod_{j=p}^{i+2}[1-P(x_j)]\frac{1}{P(x_{i+1})}$$
$$=-\prod_{j=p}^{i}[1-P(x_j)],\qquad i=p,p+1,\ldots,r.$$

Thus, we have

$$(43)\qquad \int\limits_{a_1}^{1}K(\bar{x}_{r+1};y)\,dG(y)$$
$$=T(x_1,x_2,\ldots,x_{p-1})+\int\limits_{a_{r+2}}^{1}dG(y)+$$
$$+2l_{r+2}\prod_{j=1}^{p-1}[1-P(x_j)]\Bigg\{\prod_{j=p}^{r}[1-P(x_j)][1-P(x_{r+1})]-$$
$$-\prod_{j=p}^{r}[1-P(x_j)]\frac{1}{P(x_{r+1})}+\prod_{j=p}^{r}[1-P(x_j)]-\frac{1}{P(a_{r+2})}-\frac{1}{P(x_p)}-$$
$$-\sum_{k=p}^{r}\prod_{j=p}^{k}[1-P(x_j)]\left[\frac{1}{P(x_j)}-\frac{1}{P(x_{j+1})}\right]\Bigg\}$$
$$=T(x_1,x_2,\ldots,x_{p-1})+\int\limits_{a_{r+2}}^{1}dG(y)+2l_{r+2}\prod_{j=1}^{p-1}[1-P(x_j)]\Bigg\{\prod_{j=p}^{r+1}[1-P(x_j)]+$$
$$+\prod_{j=p}^{r}[1-P(x_j)]+\prod_{j=p}^{r-1}[1-P(x_j)]+\ldots+[1-P(x_p)]-\frac{1}{P(a_{r+2})}+1\Bigg\}.$$

We see that the left-hand side of (43) is a decreasing function at every $x_p, x_{p+1}, \ldots, x_{r+1}$ which lie in $[a_{r+2}, a_{r+3}]$ and it achieves the greatest value if all $x_p, x_{p+1}, \ldots, x_{r+1}$ are equal to a_{r+2}.

Thus, we have proved that (35) is valid and the second step in the induction is completed. This means that (34) is valid for every $1 \leqslant r \leqslant n$. This, in turn, implies that (32) and (33) are also satisfied; and whence the strategy S_A is relatively optimal against S_B.

Taking into account (31) and (32) we see that the strategies S_A and S_B for players A and B are optimal. This completes the proof of optimality for the strategies S_A and S_B.

6. Numerical example of the game of timing. As an example of the presented game let us consider two cases:

I. $n=3,\ P(t)=Q(t)=t,\ t\in[0,1]$.

II. $n=3,\ P(t)=t^2,\ Q(t)=t,\ t\in[0,1]$.

In case I, according to (10) and (18), for player A we have the optimal strategies

$$\begin{aligned} f_1(x_2) &= a_1 x_1^{-3} && \text{for } x_1 \in [a_1, a_2), \\ f_2(x_2) &= a_2 x_2^{-3} && \text{for } x_2 \in [a_2, a_3), \\ f_3(x_3) &= \sqrt{2}\, a_3 [x_3^2 + 2x_3 - 1]^{3/2} && \text{for } x_3 \in [a_3, 1], \end{aligned}$$

and, for player B,

$$g(y) = \begin{cases} l_1 y^{-3} & \text{for } y \in [a_1, a_2), \\ l_2 y^{-3} & \text{for } y \in [a_2, a_3), \\ 2\sqrt{2}\beta [y^2 + 2y - 1]^{-3/2} & \text{for } y \in [a_3, a_4),\ a_4 \equiv 1, \end{cases}$$

where

$$l_1 = \frac{a_1^2}{1 - a_1}, \quad l_2 = \frac{a_1^2}{(1 - a_1)(1 - a_2)}$$

and

$$\beta = \frac{a_1^2}{2(1 - a_1)(1 - a_2)(1 - a_3)} = \mathrm{P}\{y = 1\}.$$

In case II for player A we have the optimal strategies

$$\begin{aligned} f_1(x_1) &= a_1 x_1^{-4} && \text{for } x_1 \in [a_1, a_2), \\ f_2(x_2) &= a_2 x_2^{-4} && \text{for } x_2 \in [a_2, a_3), \\ f_3(x_3) &= \frac{2}{t^3 + t^2 + t - 1} \exp\left\{ - \int_{a_3}^{x_3} \frac{(1 + u^2)\,du}{u^3 + u^2 + u - 1} \right\} && \text{for } x_3 \in [a_3, 1], \end{aligned}$$

and, for player B,

$$g(y) = \begin{cases} 2 l_1 y^{-4} & \text{for } y \in [a_1, a_2), \\ 2 l_2 y^{-4} & \text{for } y \in [a_2, a_3), \\ \dfrac{4\beta y}{y^3 + y^2 + y - 1} \exp\left\{ \displaystyle\int_y^1 \frac{2u(1+u)\,du}{u^3 + u^2 + u - 1} \right\} & \text{for } y \in [a_3, 1), \end{cases}$$

where

$$l_1 = \frac{a_1^3}{1 - a_1^2}, \quad l_2 = \frac{a_1^3}{(1 - a_1^2)(1 - a_2^2)}$$

and

$$\beta = \frac{[a_3^3 + a_3^2 + a_3 - 1]\, a_1^3}{(1 - a_1^2)(1 - a_2^2)(1 - a_3^2)\, a_3^3} = \mathrm{P}\{y = 1\}.$$

The values of the constants appearing in the analytical form of the strategies found, i.e. a_1, a_2, a_3, β and v, have been evaluated according to the formulas given in Section 4 and are listed in the following table:

	I	II
a_1	0.2182	0.3640
a_2	0.2907	0.4350
a_3	0.4494	0.5700
β	0.0773	0.0790
v	0.5636	0.2720

It is interesting to notice that player A needs 6 bullets more in case II to assure the value of the game equal to the value in case I.

The problem of the general mixed game of timing, for instance of an n-noisy-vs.-silent duel, is still open.

Appendix. We prove that

$$\int_{a_n}^{1} g(u)\,du = -2\beta + \frac{2K_n}{1-P(a_n)} - K_n. \tag{44}$$

From (12) we obtain

$$2\beta + \int_{y}^{1} (1+Q(u))g(u)du = 2\beta \exp\left\{\int_{y}^{1} \frac{P'(u)(1+Q(u))du}{P(u)Q(u)+P(u)+Q(u)-1}\right\}.$$

Thus, for $y = a_n$, we have

$$\int_{a_n}^{1} g(u)du = -2\beta + 2\beta\exp\left\{\int_{y}^{1} \frac{P'(u)(1+Q(u))du}{P(u)Q(u)+P(u)+Q(u)-1}\right\} - \int_{a_n}^{1} Q(u)g(u)du.$$

Let us put $x_n = 1$ in the definition of K_n on page 209. It leads to the equation

$$\int_{a_n}^{1} Q(u)g(u)du = K_n.$$

Taking into account equation (14) we obtain (44).

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GRA CZASOWA TYPU $n\times 1$ O RÓŻNYCH FUNKCJACH SUKCESU

STRESZCZENIE

W pracy rozpatruje się model gry czasowej typu pojedynku, gdzie przeciwnik A ma n cichych kul, a przeciwnik B — jedną kulę głośną. Prawdopodobieństwo trafienia przeciwnika jest funkcją czasu $P(t)$ i $Q(t)$ dla $t\in[0, 1]$ odpowiednio dla pierwszego i drugiego gracza. Podano postać strategii mieszanych dla obu graczy oraz wykazano ich jednoznaczność i optymalność. Przedstawiono również dwa przykłady liczbowe opisanej gry czasowej.