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ON SOME MACHINE SEQUENCING PROBLEMS (II)

The paper [1] has dealt with machine sequencing problems which appear, for example, in production processes without bins between successive machines. In Section 2 there was constructed the mathematical model of this problem and there was presented a theorem which stated the optimal processing sequence of an n-element parcel on m machines, and in Section 3 there were presented some suboptimal solutions of the extended problem: processing of N n-element parcels.

The present paper contains further results concerning the latter problem.

For the processing problem of N parcels, we supplement the set of constraints of the problem (see [1], Section 2) by the following one:

(e) The elements of the i-th parcel may be processed only after the elements of the $(i-1)$-th parcel have been processed ($i = 2, 3, \ldots, n$).

The objective remains the same: minimize the time required to complete all jobs. Taking into account constraints (a)-(e), like in [1], the minimization of the time required to complete all jobs is equivalent to the minimization of all breaks on the last machine. Analogically, let h_{ij} denote the break on machine m between the i-th and j-th elements ($i = 0, 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$). Expressions (4) and (5) from [1] for h_{ij} remain the same (In this paper we use the notation of paper [1] and the numeration of expressions is a continuation of that from paper [1].)

Let $\{k_1^{(i)}, k_2^{(i)}, \ldots, k_n^{(i)}\}$ be a permutation of the sequence $\{1, 2, \ldots, n\}$ which states a sequence of elements of the i-th parcel ($i = 1, 2, \ldots, N$).

The objective can be written in one of the following forms:

$$(11) \qquad Z_d = h_{0k_1^{(1)}} + \sum_{i=1}^{N} \sum_{j=1}^{n-1} h_{k_j^{(i)} k_{j+1}^{(i)}} + \sum_{i=1}^{N-1} h_{k_n^{(i)} k_1^{(i+1)}},$$

$$(12) \qquad Z_d = h_{0k_1^{(1)}} + \sum_{i=1}^{N-1} \sum_{j=1}^{n-1} h_{k_j^{(i)} k_{j+1}^{(i)}} + \sum_{i=1}^{N-1} h_{k_n^{(i)} k_1^{(i+1)}} + \sum_{j=1}^{n-1} h_{k_j^{(N)} k_{j+1}^{(N)}}.$$

Let Z_d^* denote the minimal value of Z_d.

If the processing sequence of elements is the same in all parcels, i. e. if

$$\{k_1^{(i)}, k_2^{(i)}, \ldots, k_n^{(i)}\} = \{k_1, k_2, \ldots, k_n\} \quad (i = 1, 2, \ldots, N),$$

then (11) and (12) are equal to (7) and (8) from [1] (like in [1], let V_d denote Z_d in this case).

Let the set I_H contain all minimal Hamiltonian cycles of the matrix H.

Remark. Next, we shall assume that each Hamiltonian cycle $\{k_1, k_2, \ldots, k_n, k_1\}$ of the matrix H satisfies the condition

$$\min_{1 \leqslant j \leqslant n} \{h_{0k_j} - h_{k_{j-1}k_j}\} = h_{0k_1} - h_{k_n k_1}, \quad \text{where } k_0 = k_n.$$

LEMMA. *For each two Hamiltonian cycles $\{i_1, i_2, \ldots, i_n, i_1\} \notin I_H$ and $\{l_1, l_2, \ldots, l_n, l_1\} \in I_H$ of the matrix H, there exists an integer number N^* such that, for every integer number $N \geqslant N^*$, we have*

$$V_d(\{i_1, i_2, \ldots, i_n\}) \geqslant V_d(\{l_1, l_2, \ldots, l_n\}). \tag{13}$$

Proof. Let c_l (c_i, respectively) denote the weight of the Hamiltonian cycle $\{l_1, l_2, \ldots, l_n, l_1\}$ ($\{i_1, i_2, \ldots, i_n, i_1\}$, respectively); then $c_l < c_i$ and $V_d(\{l_1, l_2, \ldots, l_n\}) = h_{0l_1} - h_{l_n l_1} + Nc_l$ and $V_d(\{i_1, i_2, \ldots, i_n\}) = h_{0i_1} - h_{i_n i_1} + Nc_i$.

If $h_{0l_1} - h_{l_n l_1} \leqslant h_{0i_1} - h_{i_n i_1}$, then inequality (13) is satisfied for each $N \geqslant 1$. If $h_{0l_1} - h_{l_n l_1} > h_{0i_1} - h_{i_n i_1}$, then

$$\begin{aligned} V_d(\{i_1, i_2, \ldots, i_n\}) - V_d(\{l_1, l_2, \ldots, l_n\}) \\ = N(c_i - c_l) - [(h_{0l_1} - h_{l_n l_1}) - (h_{0i_1} - h_{i_n i_1})]. \end{aligned} \tag{14}$$

The positive expression $[\ldots]$ does not depend on N, and $c_i - c_l > 0$, therefore, there exists an N^* such that, for each $N \geqslant N^*$, we have

$$N(c_i - c_l) > (h_{0l_1} - h_{l_n l_1}) - (h_{0i_1} - h_{i_n i_1}),$$

and so (13) holds.

Let V_d^* denote the minimal value of V_d. From the lemma we have the following theorem:

THEOREM. *For a sufficiently great integer number N, the value V_d^* is attained for that permutation $\{l_1, l_2, \ldots, l_n\}$ which satisfies the conditions*

1° *$\{l_1, l_2, \ldots, l_n, l_1\}$ is a minimal Hamiltonian cycle of the matrix H, and*

2° $\min_{i \in I_H} \{h_{0i_1} - h_{i_n i_1}\} = h_{0l_1} - h_{l_n l_1}$.

Proof. From the lemma we infer that, for a sufficiently great integer number N^*, the minimization of V_d can be extended over the set of permutations $\{i_1, i_2, \ldots, i_n\}$ for which $\{i_1, i_2, \ldots, i_n, i_1\}$ is a minimal

Hamiltonian cycle of the matrix H. On the other hand, in view of (14), V_d^* is attained for the permutation $\{l_1, l_2, \ldots, l_n\}$ which satisfies condition 2°.

Assuming that the processing sequence of elements is not the same in all parcels, in paper [1] there was presented a solution with the value of the objective V_d^2 satisfying $V_d^* \geqslant V_d^2$. Obviously, $V_d^2 \geqslant Z_d^*$.

Now, we prove for the N-parcel processing problem a theorem which corresponds to the theorem proved for the one-parcel processing problem in [1].

Let us define the matrix $F = (f_{ij})$ $(i, j = 0, 1, 2, \ldots, p)$ as

$$F = \left|\begin{array}{cccccc} \infty & H_0 & & & & \\ & H & H & \infty & & \\ & & H & H & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \infty & & H & H \\ H_1 & & & & & H \end{array}\right|,$$

where $p = nN$, $H = (h_{ij})$ $(i, j = 1, 2, \ldots, n)$, $H_0 = [h_{01}, h_{02}, \ldots, h_{0n}]$ and $H_1^T = [0, 0, \ldots, 0]$ (n-element vector), and let us number the elements of the i-th parcel by integer numbers from the interval $[n(i-1)+1, in]$ $(i = 1, 2, \ldots, N)$.

THEOREM. *The permutation* $\{l_1, l_2, \ldots, l_p\}$ *minimizes* Z_d *if and only if* $\{0, l_1, l_2, \ldots, l_p, 0\}$ *is the minimal Hamiltonian cycle of the matrix* F.

Proof. First, we prove that to each Hamiltonian cycle of the matrix F with weight less than ∞ corresponds a feasible solution of the N-parcel processing problem which satisfies conditions (a)-(e).

Let $\{k_0, k_1, k_2, \ldots, k_p, k_0\}$ be a Hamiltonian cycle of the matrix F for which the weight is less than ∞, and let $k_0 = 0$. From the form of the matrix F we have:

(i) $i \leqslant k_1 \leqslant n$, i. e., the process is started with an element of the first parcel.

(ii) $n(N-1)+1 \leqslant k_p \leqslant p$, i. e., the process is ended by an element of the N-th parcel.

(iii) If $n(r-1)+1 \leqslant k_i \leqslant nr$ $(r = 1, 2, \ldots, N; i = 1, 2, \ldots, p)$, then either k_{i+1} belongs to the same interval, i. e., after an element k_i it follows an element of the same parcel, or $nr+1 \leqslant k_{i+1} \leqslant n(r+1)$, i. e., after an element k_i it follows an element of the next parcel. Hence, it appears that, for fixed r $(r = 1, 2, ,\ldots, N)$, the elements $k_{s+1}, \ldots, k_{s+n}$ satisfying $n(r-1)+1 \leqslant k_{s+j} \leqslant nr$ $(j = 1, 2, \ldots, n)$ occur after the elements $k_i \leqslant n(r-1)$ and before the elements $k_i \geqslant nr+1$, i. e., the elements $k_{s+1}, \ldots$ $\ldots, k_{s+n}$, which belong to the r-th parcel, are processed after the $(r-1)$-th parcel and before the $(r+1)$-th parcel.

The proof of the inverse correspondence is analogous and straightforward.

Let W denote the weight of the Hamiltonian cycle $\{0, k_1, \ldots, k_p, 0\}$ of the matrix F, let P_f be the set of Hamiltonian cycles of the matrix F with weights less than ∞, and let S be the set of feasible solutions of the N-parcel processing problem. We have proved that there exists a one-to-one mapping $\varphi : P_f \leftrightarrow S$. It is easy to verify that

$$W(\{0, k_1, \ldots, k_p, 0\}) = Z_d(\{k_1, k_2, \ldots, k_p\}),$$

i. e.,

$$\min_{P_f} W = \min_{S} Z_d.$$

We have obtained the solutions of the N-parcel processing problem for which the values of the objective satisfy the following inequalities:

$$V_d^* \geqslant V_d^2 \geqslant Z_d^*.$$

For practical, values of parameters n and N, especially for the sake of effectiveness of computations, we shall not use the result of the theorem. In practice, in an N-parcel process, the elements of all parcels are processed usually in the same sequence. Therefore, one can be interested to know how far the values V_d^* and V_d^2 are from Z_d^*.

Reference

[1] J. Grabowski and M. Sysło, *On some machine sequencing problems (I)*, Zastosow. Matem. 13 (1973), p. 339-345.

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O PEWNYM ZAGADNIENIU KOLEJNOŚCIOWYM (II)

STRESZCZENIE

W pracy [1] przedstawiono zagadnienie kolejnościowe, któremu w praktyce odpowiada proces obróbki na gorąco lub proces obróbki bez składowisk międzyoperacyjnych. I tak, w paragrafie 2 zbudowano model matematyczny zagadnienia i przed-

stawiono twierdzenie dające rozwiązanie problemu kolejności obróbki n-elementowej partii na m maszynach, a w paragrafie 3, dla rozszerzonego zagadnienia — obróbki N partii n-elementowych, podano pewne suboptymalne rozwiązania.

W pracy tej przedstawiono dalsze wyniki dotyczące rozwiązania tego ostatniego zagadnienia.