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A NOVEL ANALYSIS OF NON-LINEAR TRANSFORMATION OF RANDOM SIGNALS AND ITS APPLICATION TO THE AMPLITUDE LIMITER

GLOSSARY OF SYMBOLS

- $x(t)$ — random process
- $s(t)$ — useful signal
- $n(t), n'(t)$ — noise
- $w_1(x, t), \varphi_1(x, t)$ — single-argument probability density distribution of instantaneous values of $x(t)$
- $E_k x(t)$ — initial k -th moment of the process $x(t)$
- $R_x(t_1, t_2), R_x(\tau)$ — correlation function of the process $x(t)$
- $G_x(\omega)$ — spectral power density of the process $x(t)$
- $K(i\omega)$ — transfer function of the linear system
- $g[x(t)]$ — characteristic of the non-linear system

1. INTRODUCTION

The object of the present paper is to determine the power of the useful signal S_0 , the power of the noise N_0 , the ratio S_0/N_0 and what is referred to as a *coefficient of improvement* $(S_0/N_0)/(S_i/N_i)$ at the output of the non-linear system which is known in electronic engineering as a *perfect amplitude limiter*. At the input the system is excited by a frequency modulated signal with power S_i and by the additive Gaussian white noise with power N_i , independent of the signal. It is assumed, moreover, that at the input and output of the limiter there are selective linear systems referred to as *band-pass filters*.

The output powers S_0 and N_0 will be determined in an indirect manner by determining, at the first stage of computation, the correlation function for output signals of the non-linear system. A new method will be applied for determining the correlation function, consisting in expanding the characteristic function of many arguments (involved in the expansion

of the correlation function) into a functional series. The terms of the latter will be determined by moments of the input process before the non-linear system. The coefficients will be found in terms of definite integrals, the integrands being products of the distribution function of the process at the input and the derivatives of the function expressing the non-linearity of the system. The integrals are easy to find by any numerical method. That procedure was submitted first by Shutterly [7], and then developed by Schmelovsky and Kempe [6].

The classical analysis of the amplitude limiter, excited by an unmodulated signal and noise, was the object of the work of Davenport [1] in which the method of characteristic functions of many arguments and Fourier transformations were applied. The solution was found, here also, in the form of a series, the terms of which were expressed in terms of the gamma function $\Gamma(x)$ and hypergeometric functions which are very inconvenient for calculation by digital computers.

2. ASSUMPTIONS

Fig. 1 represents a diagram of the system to be analyzed.

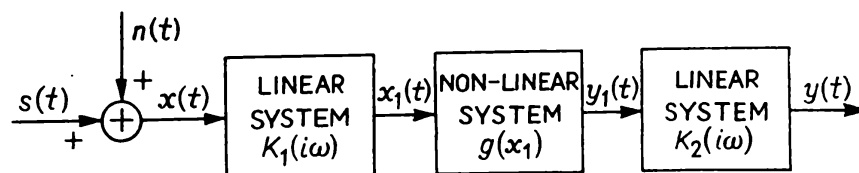


Fig. 1. Diagram of the system investigated

The characteristics of the system are

$$(1) \quad K_1(i\omega) = K_0 \exp \left[-\frac{\pi}{2} \left(\frac{\omega - \omega_0}{\Delta\omega} \right)^2 \right], \quad -\infty < \omega < \infty,$$

where K_0 is the amplification for the frequency $\omega_0/2\pi$, and $\Delta\omega/2\pi$ the effective band-width of the filter.

The amplitude characteristic of the non-linear system is determined by the relation

$$(2) \quad y_1(t) = g[x_1(t)] = \begin{cases} 1 & \text{for } x(t) > 0, \\ 0 & \text{for } x(t) = 0, \\ -1 & \text{for } x(t) < 0, \end{cases}$$

which is represented in Fig. 2, and

$$(3) \quad K_2(i\omega) = \begin{cases} \frac{1}{2} & \text{for } -\omega_0 - \frac{\Delta\omega}{2} < \omega < -\omega_0 + \frac{\Delta\omega}{2} \\ & \text{and } \omega_0 - \frac{\Delta\omega}{2} < \omega < \omega_0 + \frac{\Delta\omega}{2}, \\ 0 & \text{for all other } \omega. \end{cases}$$

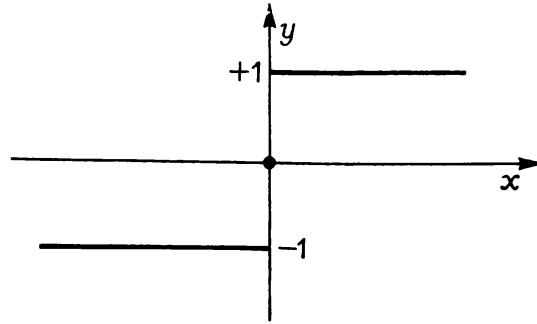


Fig. 2. Characteristic of the perfect amplitude limiter

The input of the system is acted on by a useful stationary [5] frequency modulated signal and by a perturbing stationary noise $n(t)$, statistically independent of the signal.

The signal $s(t)$ is expressed as

$$(4) \quad s(t) = A \cos[\omega_0 t + \varphi + \theta(t)] = A \sum_{r=-\infty}^{\infty} J_r(D) \cos(\omega_0 + r\omega_m)t,$$

where A is the amplitude of the signal, $\omega_0/2\pi$ the carrier frequency of the signal, φ the random initial phase angle of the signal with uniformly distributed probability density $w(\varphi) = 1/2\pi$ ($0 \leq \varphi \leq 2\pi$), and $\theta(t)$ the phase angle of the signal, which depends on the modulating signal,

$$(5) \quad \theta(t) = D \sin(\omega_m t + \psi),$$

where D is the modulation index, $\omega_m/2\pi$ the frequency of the modulating signal, and ψ the random initial phase angle of the signal with uniformly distributed probability density $w(\psi) = 1/2\pi$ ($0 \leq \psi \leq 2\pi$).

The statistical characteristics of the noise $n(t)$ are as follows:

It is a white stationary Gaussian noise with zero mean value and variance σ^2 .

Its correlation function is $R_n(\tau) = N_0 \delta(\tau)/2$.

Its spectral power density is $G_n(\omega) = N_0/2$, $-\infty < \omega < \infty$.

The total input signal is

$$x(t) = s(t) + n(t).$$

Assuming the ratio $p = S_i/N_i$ of the power of the signal S_i at the input to the non-linear system to the power of the noise N_i to be an independent variable, our task is to find the power of the useful signal at the output $S_o(p)$, the output noise $N_o(p)$, the ratio $S_o/N_o = f(p)$ and the coefficient of improvement $(S_o/N_o)/(S_i/N_i) = \varphi(p)$.

3. SOLUTION OF THE PROBLEM

3.1. Determination of the parameters of the useful signal and the noise at the input of the non-linear system. The transfer characteristic of the linear filter at the input of the system is determined by (1). The effective band-width of the filter $\Delta\omega/2\pi$ is assumed large enough, so that signal (4) is passed by the filter with no distortion [8]:

$$(6) \quad \begin{aligned} G_{n'}(\omega) &= G_n(\omega) |K_1(i\omega)|^2 \\ &= \frac{N_0 K_0^2}{2} \exp \left[-\pi \left(\frac{\omega - \omega_0}{\Delta\omega} \right)^2 \right]. \end{aligned}$$

Making use of the Wiener-Khinchin theorem interrelating the spectral density function $G(\omega)$ and the correlation function $R(\omega)$ for a stationary process, we find $R_{n'}(\tau)$ for the noise at the input to the non-linear system:

$$(7) \quad \begin{aligned} R_{n'}(\tau) &= \frac{1}{\pi} \int_0^\infty G_{n'}(\omega) \cos \omega \tau d\omega \\ &= \frac{1}{\pi} \int_0^\infty \frac{N_0 K_0^2}{2} \exp \left[-\pi \left(\frac{\omega - \omega_0}{\Delta\omega} \right)^2 \right] \cos \omega \tau d\omega. \end{aligned}$$

By changing variables according to the relation $\omega - \omega_0 = m$, we find

$$R_{n'}(\tau) = \frac{N_0 K_0^2}{2} \int_{-\omega_0}^\infty \exp \left[-\pi \left(\frac{m}{\Delta\omega} \right)^2 \right] \cos(\omega_0 + m) \tau dm.$$

If $\Delta\omega \ll \omega_0$, which is always the case for a real system, the lower integration limit $-\omega_0$ may be replaced by $-\infty$. We have then

$$(8) \quad R_{n'}(\tau) = \frac{N_0 K_0^2}{2} \int_{-\infty}^\infty \exp \left[-\pi \left(\frac{m}{\Delta\omega} \right)^2 \right] \cos(\omega_0 + m) \tau dm.$$

Integral (8) can be found in tables of integrals [3]. We have

$$(9) \quad \begin{aligned} R_{n'}(\tau) &= \frac{N_0 K_0^2}{2} \Delta\omega \exp \left[-\frac{1}{\pi} \left(\frac{\tau \Delta\omega}{2} \right)^2 \right] \cos \omega_0 \tau \\ &= \sigma_1^2 \varrho(\tau) \cos \omega_0 \tau, \end{aligned}$$

where the following notation has been introduced:

$$\sigma_1^2 = \frac{N_0 K_0^2}{2} \Delta\omega \quad \text{and} \quad \varrho(\tau) = \exp \left[-\frac{1}{\pi} \left(\frac{\tau \Delta\omega}{2} \right)^2 \right].$$

3.2. Derivation of the general expression describing the correlation function of the output signal of the non-linear system. Let the non-linear characteristic of the system be expressed, in a general manner, as $y(t) = g[x(t)]$. We assume that there exist the Laplace transforms

$$F(p) = \int_{-\infty}^{\infty} g(x) \exp(-px) dx$$

and

$$y = g(x) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} F(p) \exp(px) dp,$$

where $p = \sigma + i\lambda$ and $\sigma_1 < \sigma < \sigma_2$.

In the general case, the correlation function for a stationary signal $y(t)$ is expressed as

$$(10) \quad \begin{aligned} R_y(\tau) &= \mathbb{E}\{y(t)y(t+\tau)\} \\ &= \mathbb{E}\{g[x(t)]g[x(t+\tau)]\} = \mathbb{E}\{g(x_1)g(x_2)\} \\ &= \mathbb{E}\left\{ \frac{1}{(2\pi i)^2} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} F(p_1)F(p_2) \exp(x_1 p_1 + x_2 p_2) dp_1 dp_2 \right\} \\ &= \frac{1}{(2\pi i)^2} \int_L \int_L F(p_1)F(p_2) \varphi_L(-p_1, -p_2) dp_1 dp_2, \end{aligned}$$

where

$$(11) \quad \begin{aligned} \varphi_L(-p_1, -p_2) &= \mathbb{E}\{\exp(x_1 p_1 + x_2 p_2)\} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_2(x_1, x_2) \exp(+x_1 p_1 + x_2 p_2) dx_1 dx_2 \end{aligned}$$

is the Laplace transform of the two-argument probability distribution of the random variables x_1 and x_2 .

Equation (11) can be given in another form:

$$(12) \quad \varphi_L(p_1, p_2) = \varphi_{L_1}(p_1) \varphi_{L_2}(p_2) \varphi_{L_{1,2}}(p_1, p_2).$$

Moreover, $\varphi_{L_{1,2}}(p_1, p_2)$ can be expanded in a series

$$(13) \quad \varphi_{L_{1,2}}(p_1, p_2) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{p_1^\mu}{\mu!} \frac{p_2^\nu}{\nu!} \frac{\partial^{\mu+\nu}}{\partial p_1^\mu \partial p_2^\nu} \varphi_{L_{1,2}}(p_1, p_2) \Big|_{p_1=p_2=0}.$$

Substituting (12) and (13) into (10), we find

$$(14) \quad R_y(\tau) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{\partial^{\mu+\nu}}{\partial p_1^\mu \partial p_2^\nu} \varphi_{L_{1,2}}(-p_1, -p_2) \Big|_{p_1=p_2=0} \times \\ \times \frac{1}{2\pi i} \int_L \frac{p_1^\mu}{\mu!} F(p_1) \varphi_{L_1}(-p_1) dp_1 \cdot \frac{1}{2\pi i} \int_L \frac{p_2^\nu}{\nu!} F(p_2) \varphi_{L_2}(-p_2) dp_2.$$

It is known that

$$\begin{aligned} (-1)^{\mu+\nu} \frac{\partial^{\mu+\nu}}{\partial p_1^\mu \partial p_2^\nu} \varphi_L(-p_1, -p_2) \Big|_{p_1=p_2=0} &= \mathbb{E}\{x_1 x_2\} \\ &= (-1)^{\mu+\nu} \sum_{r=0}^{\mu} \sum_{s=0}^{\nu} \binom{\mu}{r} \binom{\nu}{s} \frac{\partial^r}{\partial p_1^r} \varphi_{L_1}(-p_1) \frac{\partial^s}{\partial p_2^s} \times \\ &\quad \times \varphi_{L_2}(-p_2) \frac{\partial^{\mu-r+\nu-s}}{\partial p_1^{\mu-r} \partial p_2^{\nu-s}} \varphi_{L_{1,2}}(-p_1, -p_2) \Big|_{p_1=p_2=0} \\ &= \sum_{r=0}^{\mu} \sum_{s=0}^{\nu} \binom{\mu}{r} \binom{\nu}{s} \mathbb{E}\{x_1^r\} \mathbb{E}\{x_2^s\} \mathbb{E}\{x_{1ac}^{\mu-r} x_{2ac}^{\nu-s}\} \end{aligned}$$

or

$$(15) \quad \mathbb{E}\{x_1^\mu x_2^\nu\} = \mathbb{E}\left\{ \sum_{r=0}^{\mu} \binom{\mu}{r} \mathbb{E}\{x_1^r\} x_{1ac}^{\mu-r} \cdot \sum_{s=0}^{\nu} \binom{\nu}{s} \mathbb{E}\{x_2^s\} x_{2ac}^{\nu-s} \right\}.$$

By virtue of (15), we find a recurrence equation for computing $x_{1ac}^{\mu-r}$ and $x_{2ac}^{\nu-s}$ which we call, after Shutterly [7], *ac quantities*:

$$(16) \quad x_1^\mu = \sum_{r=0}^{\mu} \binom{\mu}{r} \mathbb{E}\{x_1^r\} x_{1ac}^{\mu-r} \quad \text{and} \quad x_2^\nu = \sum_{s=0}^{\nu} \binom{\nu}{s} \mathbb{E}\{x_2^s\} x_{2ac}^{\nu-s}.$$

The first values of the quantity x_{ac} are expressed in terms of x as follows:

$$\begin{aligned}
x_{ac}^0 &= 1 \quad \text{for } \mu = 0 \text{ (it being assumed additionally that } \mathbf{E}\{x\} = 0), \\
x_{ac}^1 &= x \quad \text{for } \mu = 1, \\
x_{ac}^2 &= x^2 - \mathbf{E}\{x^2\} \quad \text{for } \mu = 2, \\
x_{ac}^3 &= x^3 - 3\mathbf{E}\{x^2\}x - \mathbf{E}\{x^3\} \quad \text{for } \mu = 3, \\
&\dots\dots\dots
\end{aligned}$$

Making use of the *ac* quantities just introduced, the correlation function (14) can now be rewritten as

$$\begin{aligned}
(17) \quad R_v(\tau) &= \mathbf{E} \left\{ \sum_{\mu=0}^{\infty} \frac{x_{1ac}^{\mu}}{2\pi i \mu!} \int_L p_1^{\mu} F(p_1) \varphi_{L_1}(-p_1) dp_1 \times \right. \\
&\quad \left. \times \sum_{\nu=0}^{\infty} \frac{x_{2ac}^{\nu}}{2\pi i \nu!} \int_L p_2^{\nu} F(p_2) \varphi_{L_2}(-p_2) dp_2 \right\}.
\end{aligned}$$

We shall now be concerned with the determination of the integral

$$\frac{1}{2\pi i} \int_L p_1^{\mu} F(p_1) \varphi_{L_1}(-p_1) dp_1$$

involved in (17). To this end, let us introduce a new auxiliary function

$$(18) \quad r(x) = \int_{-\infty}^{\infty} g(u) w[-(x-u)] du,$$

where $w(x)$ is the probability distribution of the variable x , and $g(x)$ is a function determining the non-linearity of the system analyzed.

It will also be assumed that there exists, for $r(x)$, the Laplace transform

$$\begin{aligned}
R(p) &= \int_{-\infty}^{\infty} r(x) \exp[-px] dx = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} g(u) w[-(x-u)] du \right\} \exp(-px) dx \\
&= \int_{-\infty}^{\infty} g(u) \left\{ \int_{-\infty}^{\infty} w[-(x-u)] \exp(-px) dx \right\} du.
\end{aligned}$$

Substituting $v = -x + u$, we have

$$(19) \quad R(p) = \int_{-\infty}^{\infty} g(u) e^{-pu} du \int_{-\infty}^{\infty} w(v) e^{pv} dv = F(p) \varphi_L(-p).$$

Making use of the inverse Laplace transform for the derivative $r^{(n)}(x)$, we obtain

$$(20) \quad r^{(n)}(x) \Big|_{x=0} = \frac{1}{2\pi i} \int_L p^n R(p) dp \Big|_{x=0} = \frac{1}{2\pi i} \int_L p^n F(p) \varphi_L(-p) dp.$$

With the aid of (20) each factor in the equation (17) can be expressed in the form

$$(21) \quad \sum_{\mu=0}^{\infty} \frac{x_{1ac}^{\mu}}{2\pi i \mu!} \int_L p_1^{\mu} F(p_1) \varphi_L(-p_1) dp_1 = \sum_{\mu=0}^{\infty} \frac{x_{1ac}^{\mu}}{\mu!} \frac{d^{\mu}}{dx^{\mu}} r(x) \Big|_{x=0}$$

$$= \sum_{\mu=0}^{\infty} \frac{x_{1ac}^{\mu}}{\mu!} \frac{d^{\mu}}{dx^{\mu}} \int_{-\infty}^{\infty} g(u) w[-(x-u)] du = \sum_{\mu=0}^{\infty} \frac{x_{1ac}^{\mu}}{\mu!} \int_{-\infty}^{\infty} g_l^{(\mu)}(u) w(u) du.$$

The following notation has been introduced in the latter integral:

$g_l^{(\mu)}(x) = g^{(\mu)}(x)$ for those values of x for which $g^{(\mu)}(x)$ exists;

$g_l^{(\mu)}(x) = g^{(\mu)}(x) = \delta^{(\mu-\gamma-1)}(x-x_n) \{g^{(\gamma)}(x_n+\varepsilon) - g^{(\gamma)}(x_n-\varepsilon)\}$ for those x_n for which the γ derivative still exists;

$\delta^{(n)}(x) = 0$ for $x \neq 0$;

$$\int_{-\infty}^{\infty} \delta^{(n)}(x-x_0) f(x) dx = (-1)^n f^{(n)}(x) \Big|_{x=x_0}.$$

By (21) the correlation function $R_y(\tau)$ can be expressed in the final form

$$(22) \quad R_y(\tau) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{1}{\mu! \nu!} \int_{-\infty}^{\infty} g_l^{(\mu)}(u) w(u) du \int_{-\infty}^{\infty} g_l^{(\nu)}(u) w(u) du E\{x_{1ac}^{\mu} x_{2ac}^{\nu}\}.$$

If the input signal of the non-linear system considered is a sum of statistically independent signals (a sum of the useful signal and the noise), $x(t) = x_1(t) + x_2(t)$, then

$$\varphi_{Lx_1+x_2}(p_1, p_2) = \varphi_{Lx_1}(p_1, p_2) \varphi_{Lx_2}(p_1, p_2),$$

and the correlation function is expressed, by virtue of (10), by the equation

$$(23) \quad R_y(\tau) = \frac{1}{(2\pi i)^2} \int_L \int_L F(p_1) F(p_2) \varphi_{Lx_1}(-p_1, -p_2) \varphi_{Lx_2}(-p_1, -p_2) dp_1 dp_2.$$

Expressions (22) and (23) are fundamental and will be used in what follows to solve the problem stated.

The determination of $R_y(\tau)$ for the output signal of the non-linear system consists in finding the quantities x_{ac} , the mean values $E\{x_{1ac}^{\mu} x_{2ac}^{\nu}\}$,

the integrals and sums involved in (22) from the prescribed probability densities $w(x)$ for the input signals and the characteristic $g(x)$ of the non-linear system.

3.3. Determination of $R_{y_1}(\tau)$ for the output signal of the non-linear system. At the input to the amplitude limiter statistically independent signals $s(t)$ and $n'(t)$ are superimposed. The correlation function for the signals $y_1(t)$ at the output of the limiter is determined by (23) as follows:

$$R_{y_1}(\tau) = \frac{1}{(2\pi i)^2} \int_L \int F(p_1) F(p_2) \varphi_{Ls}(-p_1, -p_2) \varphi_{Ln'}(-p_1, -p_2) dp_1 dp_2.$$

The transform $\varphi_{Ls}(-p_1, -p_2)$ is expressed by the relation

$$(24) \quad \begin{aligned} \varphi_{Ls}(-p_1, -p_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(s_1, s_2) \exp(p_1 s_1 + p_2 s_2) ds_1 ds_2 \\ &= \mathbb{E} \{ \exp(p_1 s_1 + p_2 s_2) \}, \end{aligned}$$

where

$$s_1 = s(t) = A \cos [\omega_0 t + \theta(t) + \varphi] = A \cos \alpha,$$

$$s_2 = s(t + \tau) = A \cos [\omega_0(t + \tau) + \theta(t + \tau) + \varphi] = A \cos \beta.$$

According to the Jacobi-Enger formula (see [2]), we have

$$\begin{aligned} \exp(p_1 A \cos \alpha) &= \sum_{a=0}^{\infty} \varepsilon_a I_a(p_1 A) \cos a\alpha, \\ \exp(p_2 A \cos \beta) &= \sum_{b=0}^{\infty} \varepsilon_b I_b(p_2 A) \cos b\beta, \end{aligned}$$

where $I(pA)$ is the modified Bessel function of the argument pA , and

$$\varepsilon_{a,b} = \begin{cases} 1 & \text{for } a, b = 0, \\ 2 & \text{for } a, b = 1, 2, \dots, \end{cases}$$

are Neumann coefficients.

Equation (24) can now be rewritten as

$$(25) \quad \begin{aligned} \varphi_{Ls}(-p_1, -p_2) &= \mathbb{E} \left\{ \sum_{a=0}^{\infty} \varepsilon_a I_a(p_1 A) \cos a\alpha \sum_{b=0}^{\infty} \varepsilon_b I_b(p_2 A) \cos b\beta \right\} \\ &= \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} I_a(p_1 A) I_b(p_2 A) \mathbb{E} \{ \varepsilon_a \cos a\alpha \cdot \varepsilon_b \cos b\beta \}. \end{aligned}$$

Since

$$\begin{aligned}
 \cos a\alpha &= \cos a[\omega_0 t + D \sin(\omega_m t + \psi) + \varphi] \\
 &= \operatorname{Re} \left\{ \exp[ia(\omega_0 t + \varphi)] \sum_{r=-\infty}^{\infty} J_r(aD) [\cos(\omega_m t + \psi)r + i \sin(\omega_m t + \psi)] \right\} \\
 &= \operatorname{Re} \sum_{r=-\infty}^{\infty} J_r(aD) \exp\{i[a(\omega_0 t + \varphi) + r(\omega_m t + \psi)]\} \\
 &= \sum_{r=-\infty}^{\infty} J_r(aD) \cos[a(\omega_0 t + \varphi) + r(\omega_m t + \psi)],
 \end{aligned}$$

where $J_r(aD)$ is the Bessel function of the r -th order, we have

$$\begin{aligned}
 \mathbb{E}\{\varepsilon_a \cos a\alpha \cdot \varepsilon_b \cos b\beta\} &= \frac{1}{2} \varepsilon_a \varepsilon_b \mathbb{E}\{\cos(a\alpha + b\beta) + \cos(a\alpha - b\beta)\} \\
 &= \begin{cases} \varepsilon_a^2 \mathbb{E}\{\cos a\alpha \cos a\beta\} & \text{for } a = b, \\ 0 & \text{for } a \neq b. \end{cases}
 \end{aligned}$$

Then expression (25) becomes

$$\begin{aligned}
 \varphi_{Ls}(-p_1, -p_2) &= \varepsilon_a^2 \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} J_r(aD) J_s(aD) \times \\
 &\times \mathbb{E}\{\cos[a(\omega_0 t + \varphi) + r(\omega_m t + \psi)] \cos[a(\omega_0(t + \tau) + \varphi) + s(\omega_m(t + \tau) + \psi)]\} \\
 &= \begin{cases} \varepsilon_a^2 \sum_{r=-\infty}^{\infty} J_r^2(aD) \mathbb{E}\{\cos[a(\omega_0 t + \varphi) + r(\omega_m t + \psi)] \times \\ \times \cos[a(\omega_0(t + \tau) + \varphi) + r(\omega_m(t + \tau) + \psi)]\} & \text{for } s = r, \\ 0 & \text{for } s \neq r, \end{cases} \\
 (26) \quad \varepsilon_a^2 \sum_{r=-\infty}^{\infty} J_r^2(aD) \mathbb{E}\{\cos[a(\omega_0 t + \varphi) + r(\omega_m t + \psi)] \times \\
 \times \cos[a(\omega_0(t + \tau) + \varphi) + r(\omega_m(t + \tau) + \psi)]\} \\
 = \varepsilon_a^2 \sum_{r=-\infty}^{\infty} J_r^2(aD) \frac{1}{2} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(a\omega_0 + r\omega_m) \tau d\varphi \right] d\psi \\
 = \sum_{r=-\infty}^{\infty} \varepsilon_a J_r^2(aD) \cos(a\omega_0 + r\omega_m) \tau.
 \end{aligned}$$

Expression (26) has been obtained for the distributions of the probability densities assumed for the angles φ and ψ . Substituting (26) into (25), we have

$$(27) \quad \varphi_{Ls}(-p_1, -p_2) = \sum_{a=0}^{\infty} \sum_{r=-\infty}^{\infty} \varepsilon_a I_a(p_1 A) I_a(p_2 A) J_r^2(aD) \cos(a\omega_0 + r\omega_m) \tau.$$

Making use of (27), we write the correlation function $R_{v_1}(\tau)$ in the form

$$(28) \quad R_{v_1}(\tau) = \frac{1}{(2\pi i)^2} \sum_{a=0}^{\infty} \sum_{r=-\infty}^{\infty} \varepsilon_a J_r^2(aD) \cos(a\omega_0 + r\omega_m)\tau \times \\ \times \int_L \int F(p_1) F(p_2) I_a(p_1 A) I_a(p_2 A) \varphi_{Ln'}(-p_1, -p_2) dp_1 dp_2.$$

The integral of (28) can be transformed in the following manner. It is known (see [3]) that

$$J_n(z) = \frac{(-i)^n}{2\pi} \int_{-\pi}^{\pi} \exp(iz \cos \psi) \cos n\psi d\psi,$$

$$I_n(z) = (-i)^n J_n(iz) \quad \text{and} \quad J_n(-z) = (-1)^n J_n(z),$$

therefore,

$$I_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[z \cos \psi] \cos n\psi d\psi.$$

Now

$$(29) \quad F(p_1) I_n(p_1 A) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x_1) e^{-p_1 x_1} dx_1 \int_{-\pi}^{\pi} \exp(p_1 A \cos a) \cos na da \\ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} g(x_1) \exp[-p_1(x_1 - A \cos a)] \cos na da dx_1.$$

Since in the considered case we have

$$x_1(t) = s(t) + n'(t) = A \cos a + n',$$

expression (29) can be rewritten in the form

$$(30) \quad F(p_1) I_n(p_1 A) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} g(A \cos a + n') e^{-pn'} \cos na da dn' \\ = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} g(A \cos a + n') \cos na da \right] e^{-pn'} dn' \\ = \int_{-\infty}^{\infty} g_a(n') e^{-pn'} dn' = F_a(p_1) = \mathcal{L}\{g_a(n')\}.$$

Expression (30) can be treated as a Laplace transform of the function

$$g_a(n') = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(A \cos a + n') \cos aa da.$$

Substituting (30) into (28), we have

$$(31) \quad R_{y_1}(\tau) = \frac{1}{(2\pi i)^2} \sum_{a=0}^{\infty} \sum_{r=-\infty}^{\infty} \varepsilon_a J_r^2(aD) \cos(a\omega_0 + r\omega_m) \tau \times \\ \times \int_L \int F_a(p_1) F_a(p_2) \varphi_{Ln'}(-p_1, -p_2) dp_1 dp_2.$$

The integral in (31) has exactly the same form as expression (10), therefore, making use of another form of (10), namely (22), we can replace expression (31) by

$$(32) \quad R_{y_1}(\tau) = \sum_{a=0}^{\infty} \sum_{r=-\infty}^{\infty} \varepsilon_a J_r^2(aD) \cos(a\omega_0 + r\omega_m) \tau \times \\ \times \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{1}{\mu! \nu!} \int_{-\infty}^{\infty} g_a^{(\mu)}(n') w(n') dn' \int_{-\infty}^{\infty} g_a^{(\nu)}(n') w(n') dn' E\{n_{1ac}^{\mu} n_{2ac}^{\nu}\},$$

where

$$g_a^{(\mu)}(n') = \frac{\partial^{\mu}}{\partial n'^{\mu}} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} g(n' + A \cos a) \cos aa da \right\}.$$

Writing

$$\int_{-\infty}^{\infty} g_a^{(\mu)}(n') w(n') dn' \int_{-\infty}^{\infty} g_a^{(\nu)}(n') w(n') dn' = h_{a\mu\nu},$$

we can express the correlation function $R_{y_1}(\tau)$ in the most general form

$$(33) \quad R_{y_1}(\tau) = \sum_{a=0}^{\infty} \sum_{r=-\infty}^{\infty} \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \varepsilon_a J_r^2(aD) \cos(a\omega_0 + r\omega_m) \tau \frac{h_{a\mu\nu}}{\mu! \nu!} E\{n_{1ac}^{\mu} n_{2ac}^{\nu}\}.$$

The correlation function (33) can be expressed in the form of a sum of three components, the first one corresponding to the useful output signal of the non-linear system and its harmonics, the second one to the

noise at the output, and the third one to the noise resulting from the combination of the signal and the noise due to the non-linearity.

Introducing the notation

$$R_{sa}(\tau) = \sum_{r=-\infty}^{\infty} J_r^2(aD) \cos(a\omega_0 + r\omega_m)\tau,$$

$$R_{n'\mu\nu}(\tau) = \mathbb{E}\{n'_{1ac}{}^{\mu} n'_{2ac}{}^{\nu}\},$$

and bearing in mind that $R_{n'00}(\tau) = 1$ for $\mu = \nu = 0$, expression (33) can be written as

$$(34) \quad R_{y_1}(\tau) = \sum_{a=0}^{\infty} \varepsilon_a R_{sa}(\tau) h_{a00} + \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} R_{n'\mu\nu}(\tau) \frac{h_{0\mu\nu}}{\mu! \nu!} +$$

$$+ \sum_{a=1}^{\infty} \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} \varepsilon_a R_{sa}(\tau) R_{n'\mu\nu}(\tau) \frac{h_{a\mu\nu}}{\mu! \nu!}.$$

By applying the Wiener-Khinchin theorem to (34), we can determine the output frequency spectrum of the signal $y_1(t)$ of the amplitude limiter. It will be composed of the spectrum of the useful signal and of the noise. The spectrum of the useful signal should be identical (the lines of the spectrum may differ by a constant factor) with the spectrum of the useful signal $s(t)$ at the input to the amplitude limiter. The remaining components of the spectrum should be classified as the noise.

3.4. Determination of $R_y(\tau)$ for the output signal of the system. According to the scheme in Fig. 1 at the output of the amplitude limiter, there is a band-pass filter with a transfer function as determined by (3), therefore, the correlation function for the output signal $y(t)$ of the system is determined as follows:

$$(35) \quad R_y(\tau) = 2R_{s1}(\tau) h_{100} + \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} R_{n'\mu\nu}(\tau) \frac{h_{0\mu\nu}}{\mu! \nu!} +$$

$$+ 2 \sum_{a=1}^{\infty} \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} R_{sa}(\tau) R_{n'\mu\nu}(\tau) \frac{h_{a\mu\nu}}{\mu! \nu!}.$$

This is a general expression for $R_y(\tau)$. In subsequent numerical computations we shall confine ourselves to the summation indices $n = 1, 2$ and $\mu, \nu = 1, 2$. This is in agreement with the principles of correlation analysis, namely, those of confining the computation to the moments of the second order. Moreover, it follows from the subsequent analysis that the terms of the series decrease rapidly with increasing index which justifies also the simplification assumed.

Making use of relation (16) we shall find $R_{n'\mu\nu}(\tau) = E\{n'_{1ac}{}^{\mu} n'_{2ac}{}^{\nu}\}$. By the assumptions and the argument of Section 3.1, we have:

$$\begin{aligned}
 E\{n'(t)\} &= 0, \\
 E\{[n'(t)]^2\} &= \frac{N_0 K_0^2}{2} \Delta\omega = \sigma_1^2, \\
 R_{n'}(\tau) &= E\{n'(t)n'(t+\tau)\} = \sigma_1^2 \varrho(\tau) \cos \omega_0 \tau \\
 &= \frac{N_0 K_0^2}{2} \Delta\omega \exp\left[-\frac{1}{\pi} \left(\frac{\tau \Delta\omega}{2}\right)^2\right] \cos \omega_0 \tau, \\
 (36) \quad R_{n'11}(\tau) &= E\{n'_{1ac} n'_{2ac}\} = E\{n'(t)n'(t+\tau)\} = \sigma_1^2 \varrho(\tau) \cos \omega_0 \tau, \\
 R_{n'12}(\tau) &= E\{n'_{1ac} n'_{2ac}{}^2\} = E\{n'(t)[n'^2(t+\tau) - E\{n'^2(t+\tau)\}]\} \\
 &= E\{n'(t)n'^2(t+\tau) - n'(t)E\{n'^2(t+\tau)\}\} \\
 &= E\{n'(t)n'^2(t+\tau)\} - E\{n'(t)E\{n'^2(t+\tau)\}\} = 0, \\
 E\{n'(t)n'^2(t+\tau)\} &= 0
 \end{aligned}$$

(being an odd moment of the Gaussian distribution),

$$\begin{aligned}
 R_{n'21}(\tau) &= 0, \\
 (37) \quad R_{n'22}(\tau) &= E\{[n'^2(t) - E\{n'^2(t)\}][n'^2(t+\tau) - E\{n'^2(t+\tau)\}]\} \\
 &= E\{n'^2(t)n'^2(t+\tau)\} - E\{n'^2(t)\}E\{n'^2(t+\tau)\} \\
 &= \sigma_1^4[1 + 2\varrho^2(\tau)\cos^2\omega_0\tau] - \sigma_1^4 = \sigma_1^4\varrho^2(\tau)[1 + \cos 2\omega_0\tau],
 \end{aligned}$$

$$(38) \quad R_{s1}(\tau) = \sum_{r=-\infty}^{\infty} J_r^2(D) \cos(\omega_0 + r\omega_m)\tau,$$

$$(39) \quad R_{s2}(\tau) = \sum_{r=-\infty}^{\infty} J_r^2(2D) \cos(2\omega_0 + r\omega_m)\tau.$$

We now determine the coefficients $h_{a\mu\nu}$. We have

$$\begin{aligned}
 h_{100} &= \left\{ \int_{-\infty}^{\infty} g_1^{(0)}(n') w(n') dn' \right\}^2 \\
 &= \left\{ \int_{-\infty}^{\infty} \frac{1}{2\pi} \left[\int_{-\pi}^{\pi} g^{(0)}(n' + A \cos a) \cos a da \right] w(n') dn' \right\}^2 \\
 &= \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\int_{-\infty}^{\infty} g(n' + A \cos a) w(n') dn' \right] \cos a da \right\}^2.
 \end{aligned}$$

In agreement with (2),

$$g(n' + A \cos \alpha) = \begin{cases} 1 & \text{for } n' + A \cos \alpha > 0, \\ 0 & \text{for } n' + A \cos \alpha = 0, \\ -1 & \text{for } n' + A \cos \alpha < 0, \end{cases}$$

$$\begin{aligned} \int_{-\infty}^{\infty} g(n' + A \cos \alpha) w(n') dn' &= - \int_{-\infty}^{-A \cos \alpha} w(n') dn' + \int_{-A \cos \alpha}^{\infty} w(n') dn' \\ &= -\frac{1}{2} + \int_{-A \cos \alpha}^0 w(n') dn' + \int_{-A \cos \alpha}^{\infty} w(n') dn' \\ &= -2 \int_0^{-A \cos \alpha} w(n') dn'. \end{aligned}$$

Because

$$w(n') = \frac{1}{\sqrt{2\pi} \sigma_1} \exp \left[-\frac{n'^2}{2\sigma_1^2} \right],$$

$$h_{100} = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[-2 \int_0^{-A \cos \alpha} w(n') dn' \right] \cos \alpha d\alpha \right\}^2.$$

Substituting

$$u^{(1)} = \cos \alpha, \quad u = \sin \alpha,$$

$$v = \int_0^{-A \cos \alpha} w(n') dn', \quad v^{(1)} = w(-A \cos \alpha) A \sin \alpha,$$

and integrating by parts,

$$h_{100} = \left\{ -\frac{1}{\pi} \sin \alpha \int_0^{-A \cos \alpha} w(n') dn' \Big|_{-\pi}^{\pi} + \frac{A}{\pi} \int_{-\pi}^{\pi} \sin^2 \alpha w(-A \cos \alpha) d\alpha \right\}^2,$$

we obtain

$$\begin{aligned} (40) \quad h_{100} &= \frac{A^2}{\pi^2} \left\{ \int_{-\pi}^{\pi} \sin^2 \alpha w(-A \cos \alpha) d\alpha \right\}^2 \\ &= \frac{A^2}{\pi^2 \sigma_1^2} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sin^2 \alpha \exp \left[-\frac{A^2}{2\sigma_1^2} \cos^2 \alpha \right] d\alpha \right\}^2, \end{aligned}$$

$$\begin{aligned}
h_{011} &= \left\{ \int_{-\infty}^{\infty} g_0^{(1)}(n') w(n') dn' \right\}^2 \\
&= \left\{ \int_{-\infty}^{\infty} \frac{1}{2\pi} \left[\int_{-\pi}^{\pi} g^{(1)}(n' + A \cos \alpha) d\alpha \right] w(n') dn' \right\}^2 \\
&= \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\int_{-\infty}^{\infty} g^{(1)}(n' + A \cos \alpha) w(n') dn' \right] d\alpha \right\}^2.
\end{aligned}$$

Since

$$g^{(1)}(n' + A \cos \alpha) = 2 \delta(n' + A \cos \alpha),$$

$$\begin{aligned}
\int_{-\infty}^{\infty} g^{(1)}(n' + A \cos \alpha) w(n') dn' \\
= 2 \int_{-\infty}^{\infty} \delta(n' + A \cos \alpha) w(n') dn' = 2 w(-A \cos \alpha),
\end{aligned}$$

and then

$$\begin{aligned}
(41) \quad h_{011} &= \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} 2 w(-A \cos \alpha) d\alpha \right\}^2 = \left\{ \frac{1}{\pi} \int_{-\pi}^{\pi} w(-A \cos \alpha) d\alpha \right\}^2 \\
&= \frac{1}{\pi^2 \sigma_1^2} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \exp \left[-\frac{A^2}{2\sigma_1^2} \cos^2 \alpha \right] d\alpha \right\}^2,
\end{aligned}$$

$$h_{022} = 0,$$

$$h_{111} = 0,$$

$$\begin{aligned}
h_{122} &= \left\{ \int_{-\infty}^{\infty} g_1^{(2)}(n') w(n') dn' \right\}^2 \\
&= \left\{ \int_{-\infty}^{\infty} \frac{1}{2\pi} \left[\int_{-\pi}^{\pi} g^{(2)}(n' + A \cos \alpha) \cos \alpha d\alpha \right] w(n') dn' \right\}^2 \\
&= \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\int_{-\infty}^{\infty} g^{(2)}(n' + A \cos \alpha) w(n') dn' \right] \cos \alpha d\alpha \right\}^2.
\end{aligned}$$

Because

$$g^{(2)}(n' + A \cos \alpha) = 2 \delta^{(1)}(n' + A \cos \alpha),$$

$$\begin{aligned}
\int_{-\infty}^{\infty} g^{(2)}(n' + A \cos \alpha) w(n') dn' &= 2 \int_{-\infty}^{\infty} \delta^{(1)}(n' + A \cos \alpha) w(n') dn' \\
&= 2(-1)^1 w^{(1)}(n'),
\end{aligned}$$

$$\frac{d}{dn'} w(n') = \frac{d}{dn'} \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left[-\frac{n'^2}{2\sigma_1^2}\right] = -\frac{1}{\sqrt{2\pi}\sigma_1} \frac{n'}{\sigma_1^2} \exp\left[-\frac{n'^2}{2\sigma_1^2}\right],$$

$$w^{(1)}(n') = w^{(1)}(-A \cos \alpha) = \frac{A}{\sigma_1^2} \cos \alpha w(-A \cos \alpha),$$

we have

$$(42) \quad h_{122} = \left\{ -\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{A}{\sigma_1^2} \cos \alpha w(-A \cos \alpha) \cos \alpha d\alpha \right\}^2$$

$$= \frac{A^2}{\pi^2 \sigma_1^6} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \cos^2 \alpha \exp\left[-\frac{A^2}{2\sigma_1^2} \cos^2 \alpha\right] d\alpha \right\}^2,$$

$$(43) \quad h_{211} = \left\{ \int_{-\infty}^{\infty} g_2^{(1)}(n') w(n') dn' \right\}^2$$

$$= \left\{ \int_{-\infty}^{\infty} \frac{1}{2\pi} \left[\int_{-\pi}^{\pi} g^{(1)}(n' + A \cos \alpha) \cos 2\alpha d\alpha \right] w(n') dn' \right\}^2$$

$$= \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\int_{-\infty}^{\infty} g^{(1)}(n' + A \cos \alpha) w(n') dn' \right] \cos 2\alpha d\alpha \right\}^2$$

$$= \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} 2w(-A \cos \alpha) \cos 2\alpha d\alpha \right\}^2$$

$$= \frac{1}{\pi^2 \sigma_1^2} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \cos 2\alpha \exp\left[-\frac{A^2}{2\sigma_1^2} \cos^2 \alpha\right] d\alpha \right\}^2,$$

$$h_{222} = 0.$$

We now determine the products $R_{sa}(\tau) R_{n'\mu\nu}(\tau)$.

For $a = 1$ and $\mu = \nu = 2$, we have

$$R_{s1}(\tau) R_{n'22}(\tau)$$

$$= \sum_{r=-\infty}^{\infty} J_r^2(D) \cos(\omega_0 + r\omega_m)\tau \cdot \sigma_1^4 \varrho^2(\tau) [1 + \cos 2\omega_0 \tau]$$

$$= \sum_{r=-\infty}^{\infty} \sigma_1^4 \varrho^2(\tau) J_r^2(D) \cos(\omega_0 + r\omega_m)\tau +$$

$$+ \frac{1}{2} \sigma_1^4 \varrho^2(\tau) \sum_{r=-\infty}^{\infty} J_r^2(D) [\cos(3\omega_0 + r\omega_m)\tau + \cos(\omega_0 - r\omega_m)\tau].$$

For $a = 2$ and $\mu = \nu = 1$, we have

$$\begin{aligned} R_{s2}(\tau) R_{n'11}(\tau) &= \sum_{r=-\infty}^{\infty} J_r^2(2D) \cos(2\omega_0 + r\omega_m)\tau \cdot \sigma_1^2 \varrho(\tau) \cos \omega_0 \tau \\ &= \frac{1}{2} \sigma_1^2 \varrho(\tau) \sum_{r=-\infty}^{\infty} J_r^2(2D) [\cos(3\omega_0 + r\omega_m)\tau + \cos(\omega_0 + r\omega_m)\tau]. \end{aligned}$$

Due to the presence of band-pass filter at the amplitude limiter output, transferring frequencies in the neighbourhood of the frequency ω_0 , the following components will be used for the subsequent computation

$$\begin{aligned} R_{s1}(\tau) R_{n'22}(\tau) &\cong \sum_{r=-\infty}^{\infty} J_r^2(D) \sigma_1^4 \varrho^2(\tau) \cos(\omega_0 + r\omega_m)\tau + \\ &\quad + \sum_{r=-\infty}^{\infty} J_r^2(D) \frac{\sigma_1^4}{2} \varrho^2(\tau) \cos(\omega_0 - r\omega_m)\tau, \\ R_{s2}(\tau) R_{n'11}(\tau) &\approx \sum_{r=-\infty}^{\infty} J_r^2(2D) \frac{\sigma_1^2}{2} \varrho(\tau) \cos(\omega_0 + r\omega_m)\tau. \end{aligned}$$

Write

$$\frac{S_i}{N_i} = \frac{A^2}{2\sigma_1^2} = p,$$

$$\left\{ \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \exp[-p \cos^2 \alpha] d\alpha \right\}^2 = C_{01}^2(p),$$

$$\left\{ \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sin^2 \alpha \exp[-p \cos^2 \alpha] d\alpha \right\}^2 = C_{10}^2(p),$$

$$\left\{ \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \cos^2 \alpha \exp[-p \cos^2 \alpha] d\alpha \right\}^2 = C_{12}^2(p),$$

$$\left\{ \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \cos 2\alpha \exp[-p \cos^2 \alpha] d\alpha \right\}^2 = C_{22}^2(p).$$

The following approximate expression can be written for the correlation function at the output of the system:

$$\begin{aligned}
(44) \quad R_y(\tau) \cong & \frac{4p}{\pi^2} \sum_{r=-\infty}^{\infty} J_r^2(D) \cos(\omega_0 + r\omega_m) \tau C_{10}^2(p) + \\
& + \frac{\varrho(\tau)}{\pi^2} \cos \omega_0 \tau C_{01}^2(p) + \frac{\varrho^2(\tau)}{\pi^2} p \sum_{r=-\infty}^{\infty} J_r^2(D) \cos(\omega_0 + r\omega_m) \tau C_{12}^2(p) + \\
& + \frac{\varrho^2(\tau)p}{2\pi^2} \sum_{r=-\infty}^{\infty} J_r^2(D) \cos(\omega_0 - r\omega_m) \tau C_{12}^2(p) + \\
& + \frac{\varrho(\tau)}{\pi^2} \sum_{r=-\infty}^{\infty} J_r^2(2D) \cos(\omega_0 + r\omega_m) \tau C_{21}^2(p).
\end{aligned}$$

3.5. Determination of the power of the useful signal and the power of the noise at the system output. The total power of the output signal P_{out} is determined by the relation

$$\begin{aligned}
(45) \quad P_{\text{out}} = R_y(0) &= S_0(p) + N_0(p) \\
&\cong \frac{4}{\pi^2} p C_{10}^2(p) + \frac{1}{\pi^2} C_{01}^2(p) + \frac{3}{2\pi^2} p C_{12}^2(p) + \frac{1}{\pi^2} C_{21}^2(p).
\end{aligned}$$

Quantity (45) has been found since

$$\sum_{r=-\infty}^{\infty} J_r^2(aD) = 1 \quad \text{and} \quad \varrho(0) = 1.$$

The power of the useful signal $S_0(p)$, that of the noise $N_0(p)$, the ratio $S_0(p)/N_0(p)$ and the coefficient of improvement $(S_0/N_0)/(S_i/N_i)$ are determined in function of p by the final relations

$$\begin{aligned}
(46) \quad S_0(p) &= \frac{4}{\pi^2} p C_{10}^2(p), \\
N_0(p) &\cong \frac{1}{\pi^2} [C_{01}^2(p) + C_{21}^2(p)] + \frac{3}{2\pi^2} p C_{12}^2(p), \\
\frac{S_0(p)}{N_0(p)} &= \frac{8p C_{10}^2(p)}{2[C_{01}^2(p) + C_{21}^2(p)] + 3p C_{12}^2(p)}, \\
\frac{S_0(p)/N_0(p)}{p} &= \frac{8C_{10}^2(p)}{2[C_{01}^2(p) + C_{21}^2(p)] + 3p C_{12}^2(p)}.
\end{aligned}$$

3.6. Numerical computations. The numerical computation of the coefficients $C_{10}^2(p)$, $C_{01}^2(p)$, $C_{12}^2(p)$, $C_{22}^2(p)$, as determined by equations (40)-(43) of the quantities $S_0(p)$, $N_0(p)$, $S_0(p)/N_0(p)$ and of the coefficient

of improvement, has been performed on the ODRA 1204 digital computer. The results are represented in Table 1 and, graphically, in Figs. 3-5.

TABLE 1

p	S_0	N_0	$p_0 = S_0/N_0$	p_0/p
0.01	0.006334	0.632649	0.0100	1.001
0.02	0.012606	0.628694	0.0201	1.003
0.05	0.031050	0.616931	0.0503	1.007
0.08	0.048952	0.605325	0.8009	1.011
0.10	0.060595	0.597679	0.1014	1.014
0.20	0.115491	0.560599	0.2060	1.030
0.50	0.251640	0.461776	0.5449	1.090
0.80	0.354384	0.381570	0.9288	1.161
1.00	0.408921	0.337451	1.2118	1.212
2.00	0.577836	0.197367	2.9277	1.464
5.00	0.723128	0.078402	9.2233	1.845
8.00	0.757859	0.049145	15.4210	1.928
10.00	0.768816	0.039477	19.4751	1.948
20.00	0.790030	0.019971	39.5596	1.978
50.00	0.802422	0.008056	99.6005	1.992
80.00	0.805487	0.005047	159.6099	1.995
100.00	0.806506	0.004040	199.6130	1.996

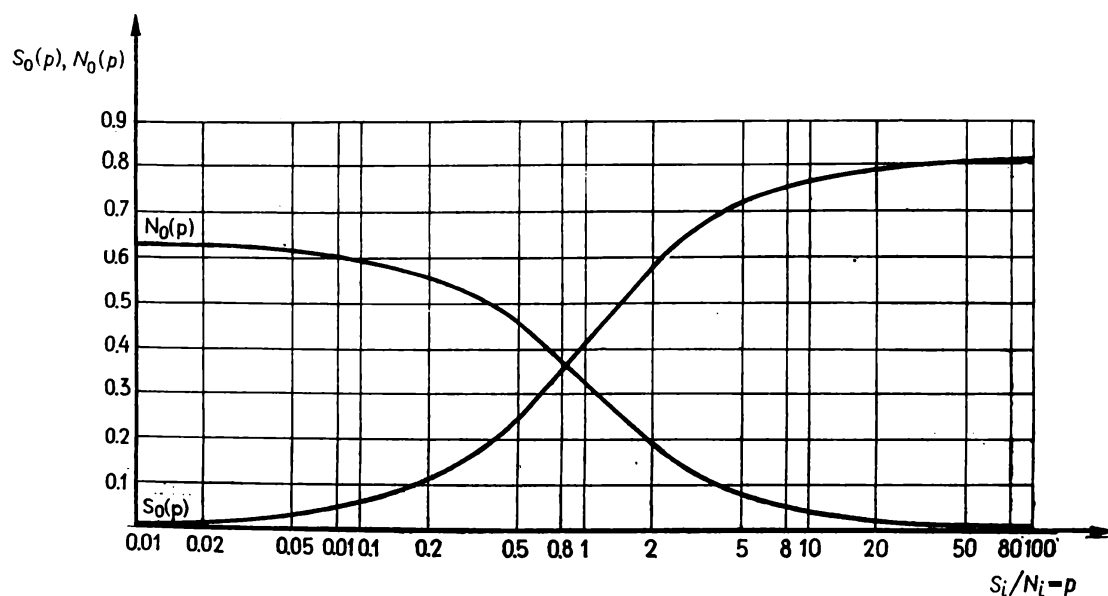


Fig. 3. Useful output signal power $S_0(p)$ and output noise power $N_0(p)$ of the amplitude limiter in function of the signal-to-noise power ratio S_i/N_i at the input

It was assumed for computation that the value of the ratio $p = S_i/N$ of the power of the useful signal to the power of the noise at the input to the non-linear system varies within the limits from 0.01 to 100. This range of values for p is in accordance with practical needs.

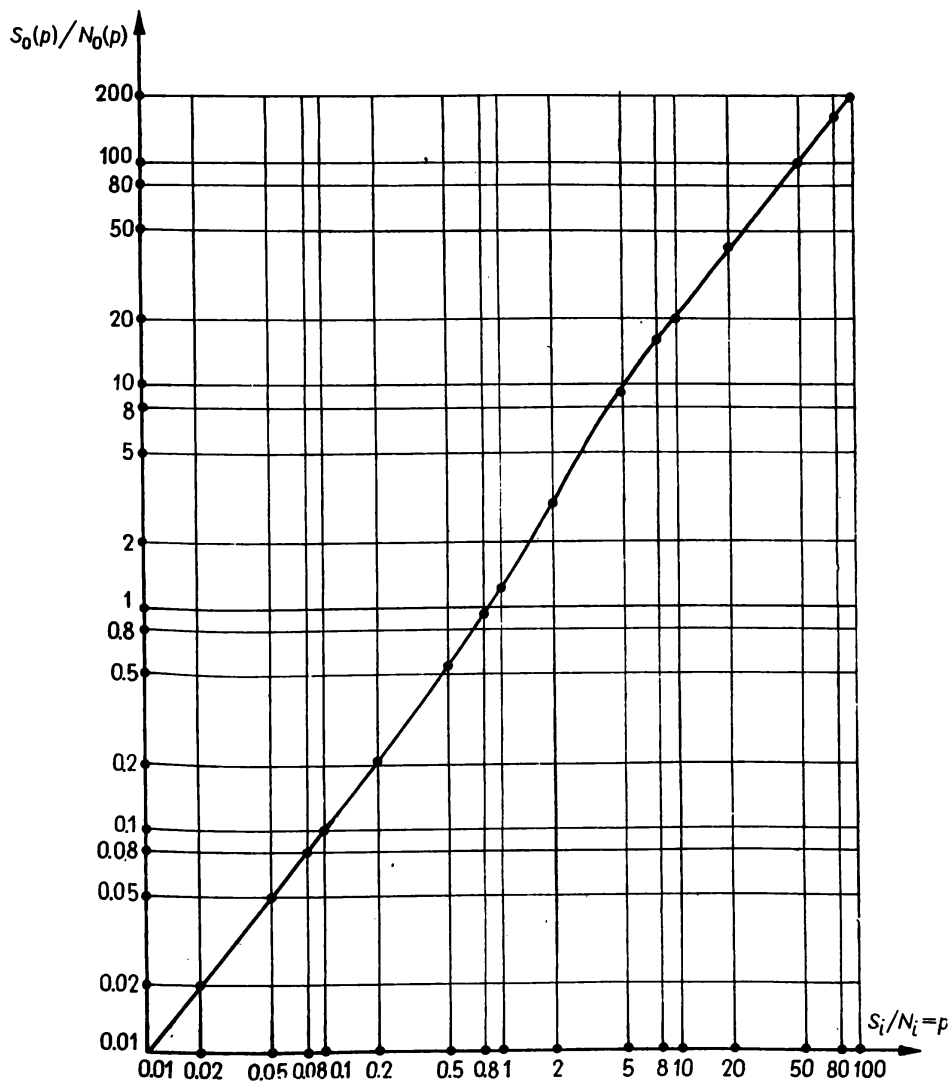


Fig. 4. Output signal-to-noise power ratio $S_0(p)/N_0(p)$ in function of the input signal-to-noise power ratio of the amplitude limiter

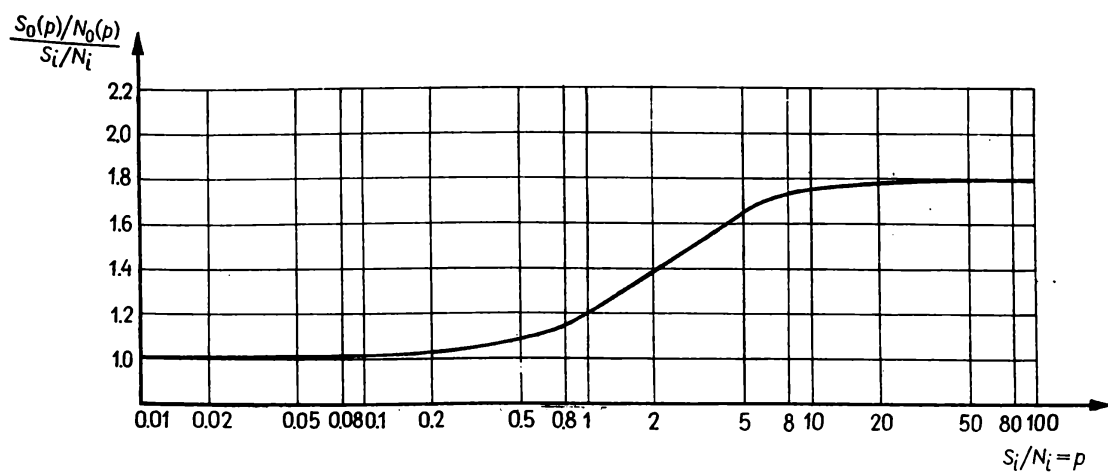


Fig. 5. The coefficient of improvement of the output signal-to-noise power ratio in function of the input signal-to-noise power ratio S_i/N_i

4. CONCLUSIONS

The general result obtained for the correlation functions and the spectrum of the output signal of the non-linear system, i.e. equation (34), has the form of a complicated series. However, the coefficients of that series can easily be computed. In practice, the result obtained can be used by confining ourselves to the numerical values of a few terms of the series, the coefficients of successive terms of the series decreasing rapidly.

The power of the output noise of the non-linear system is higher than that of the input noise. For the most effective separation of the useful signal from the noise one, it is necessary to provide at the output of the non-linear system a zonal filter of a frequency band just necessary for transmitting the useful signal without distortion. Such a zonal filter is used in the system analyzed in the present paper.

In view of the obtained numerical results, it can be stated that, for low values of the ratio $p = S_i/N_i$, the ratio of the corresponding values S_0/N_0 remains unchanged, that is, $(S_0/N_0)/(S_i/N_i) \approx 1$. For $p \geq 1$, there is an increase in S_0/N_0 , and for $p \rightarrow \infty$ we have $(S_0/N_0)/(S_i/N_i) \rightarrow 2$. Thus, the system under consideration is seen to serve its principal purpose, of limiting the amplitude of the output signal, and does not essentially change the ratio of the power of the useful signal to that of the noise. The value of the coefficient of improvement is contained within the interval $[1, 2]$.

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**NOWA METODA ANALIZY UKŁADÓW NIELINIOWYCH
PRZY POBUDZANIU SYGNAŁAMI PRZYPADKOWYMI
I JEJ ZASTOSOWANIE DO ANALIZY OGRANICZNIKA AMPLITUDOWEGO**

STRESZCZENIE

Treścią pracy jest obliczenie wartości mocy sygnału użytecznego S_0 , mocy szumu zakłócającego N_0 , stosunku S_0/N_0 oraz tzw. współczynnika poprawy $(S_0/N_0)/(S_i/N_i)$ na wyjściu układu nieliniowego, zwanego w radiotechnice *idealnym ogranicznikiem amplitudowym*, przy założeniu, że na jego wejście działają sygnał z modulacją częstotliwości o mocy S_i i addytywny, niezależny do sygnału, biały szum gaussowski o mocy N_i . Dodatkowo przyjęto, że przed i za ogranicznikiem znajdują się układy liniowe o selektywnych charakterystykach częstotliwościowych, zwane *filtrami pasmowymi*.

Wartości mocy wyjściowych S_0 i N_0 obliczono pośrednio przez określenie w pierwszym kroku obliczeń funkcji korelacji dla sygnałów na wyjściu układu nieliniowego. Dla określenia funkcji korelacji zastosowano nową metodę, która polega na zastąpieniu wieloargumentowej funkcji charakterystycznej, występującej w wyrażeniu określającym funkcje korelacji, przez szereg funkcyjny. Wyrazy szeregu są określone przez momenty funkcji reprezentującej proces na wejściu układu nieliniowego, a współczynniki przy wyrazach wyznaczone są przez całki oznaczone, w których integrandem są iloczyny funkcji rozkładu gęstości prawdopodobieństwa procesu na wejściu i pochodne funkcji określającej nieliniowość badanego układu. Całki są łatwe do obliczenia dowolnymi metodami numerycznymi. Metoda ta była po raz pierwszy przedstawiona przez Shutterly'ego [7] i następnie rozwinięta przez Schmelowskiego i Kempego [6].