казать. Липшицевость — свойство (iii) — следует из несложной оценки интегралов по множествам $f_t^{-1}(q)$.

В связи с изложенным естественно сформулировать следующую задачу. Так как уравнение (7) надо решать в реальном времени (в ходе игры), а описанный процесс приближений сходится тем быстрее, чем меньше константа Липшица в формуле (11), то желательно уметь при заданном семействе F_t и компакте K строить семейство выметаний (9) с наименьшей константой или с константой, близкой к возможной нижней грани. Мне неизвестно, насколько описанный метод построения семейства m_t дает константу, близкую к желаемой.

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THE DUALITY OF CONVEX FUNCTIONS AND CESARI'S PROPERTY Q

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§ 1. The property Q

Recent work on the existence of optimal controls and related problems (v. refs. cited in [3]; also [6]) has employed an alteration made by Cesari (*op. cit.*) in the classical notion of the upper semicontinuity of families of closed sets that applies in the special case where the sets involved are convex.

What is done is to assign to each point (t, x) in some index set I belonging, typically, to $R \times E_m$, a closed convex set $K(t, x)$ in E_{n+1} and then to form, at each fixed point $(\bar{t}, \bar{x})$ in I, the set

$$Q(\bar{t}, \bar{x}) = \bigcap_{\delta>0} \operatorname{cl} \operatorname{co} \bigcup_{(t,x)\in\mathcal{N}_\delta} K(t, x).$$

Here, 'cl' and 'co' mean *closure* and *convex hull*, resp., and $\mathcal{N}_\delta$ denotes a δ-neighbourhood of $(\bar{t}, \bar{x})$.

If the resulting set $Q(\bar{t}, \bar{x})$, which is clearly closed and convex, coincides with $K(\bar{t}, \bar{x})$, then, in the terminology of Cesari, the sets $K(t, x)$, "have the property Q" at the point $(\bar{t}, \bar{x})$. As Cesari observed, omission of the 'co' from the formula converts it into the classical definition of closed or topological limes superior, and the classical notion of upper semicontinuity of closed sets emerges.

Cesari's operation can also be regarded as a limes superior—this time, in the lattice of closed convex sets, when they are ordered by inclusion and the lattice operations are defined by

$$\bigwedge_\iota K_\iota = \bigcap_\iota K_\iota,$$
$$\bigvee_\iota K_\iota = \operatorname{cl} \operatorname{co} \bigcup_\iota K_\iota.$$

Then the operation Q is just the order limes superior, when I is made into a directed set by preordering points according to their distance from $(\bar{t}, \bar{x})$. Since the order limes superior is preserved under order isomorphisms, we can expect to arrive at equivalent characterizations of the property Q when we replace the lattice of closed convex sets by any lattice isomorphic to it. That, indeed, is the underlying theme of this paper, but we shall try to relate it to corresponding developments in calculus of variations and control theory.

§ 2. The rôle of property Q in variational and control problems

In problems of the calculus of variations and optimal control theory, the property Q typically occurs in the following way. We are given a real-valued integrand $F(t, x, u)$, where t is a scalar and x and u are vectors, and we are asked to find

$$\min \int_{t_0}^{t_1} F(t, x, u)\,dt$$

over all absolutely continuous curves $x = x(t)$ that satisfy the control equation

$$\frac{dx}{dt} = G(t, x, u) \text{ a.e.}$$

along with some given initial and/or terminal conditions, where the control variable $u = u(t)$ is constrained to lie in a given control region U.

We "deparametrize" the problem by introducing the function

$$f(t, x, z) = \begin{cases} \inf\limits_u \{F(t, x, u) |\ G(t, x, u) = z,\ u \in U\}, \\ +\infty, \quad \text{if} \quad G(t, x, u) \neq z \text{ for any } u \in U, \end{cases}$$

where z is, for the moment, just a dummy variable. It is ordinarily assumed that the infimum is finite. We then seek to minimize

$$I[x] = \int_{t_0}^{t_1} f(t, x, \dot{x})\,dt$$

over all absolutely continuous curves $x = x(t)$ with derivative $dx/dt = \dot{x}$, satisfying the initial and terminal conditions.

Now the problem has the form of a classical problem in the calculus of variations, with the effect of the control variable being felt as an infinite penalty. Integrals of this sort are to be found in papers of McShane from the thirties. Their deduction from control problems seems to have occurred to a number of people, of whom Zachrisson [11] was perhaps one of the first.

Classical criteria for the lower semicontinuity of $I[x]$ involve, at the least, (1) the convexity of f in the variable z (expressed classically by the non-negativity of the Weierstrass E-function), and (2) some sort of semicontinuity of f in t and x. A rather refined notion of the latter, which Cesari [3] attributes to McShane and Tonelli, is the idea of *weak seminormality*. Assuming that, for a certain range of t and x, f is everywhere finite as a function of z, this criterion takes the form:

f is weakly seminormal in z at the point $(\bar{t}, \bar{x}, \bar{z})$ if, for every $\varepsilon > 0$ there are numbers $\delta > 0$ and r, and a real vector y, such that

$$f(t, x, z) \geqslant r + \langle z, y\rangle$$

for all z and all (t, x) in a δ-neighbourhood of $(\bar{t}, \bar{x})$, and

$$f(\bar{t}, \bar{x}, \bar{z}) \leqslant r + \langle \bar{z}, y\rangle + \varepsilon.$$

In the case where the function f can take the value $+\infty$ for certain values of t and x, Cesari [3] has added a further condition which can be expressed in the language of convex analysis in the following way. Recall that the *domain* of a convex function, abbreviated simply by 'dom', is the set of points where the function takes finite values, and the *epigraph* of a function, 'epi', for short, is the set of points that lie on or above its graph. Consider now the sets

$$K(t, x) = \operatorname{epi}_z f(t, x, z) = \{(r, z) |\ r \geqslant f(t, x, z)\},$$

and form from them the set $Q(\bar{t}, \bar{x})$ according to the prescription given above. Then Cesari's supplemental conditions takes the form

if (r, w) belongs to $Q(\bar{t}, \bar{x})$, then w belongs to $\operatorname{dom}_z f(\bar{t}, \bar{x}, z)$.

Let us agree to call the notion of weak seminormality, together with Cesari's further condition, *weak seminormality in the extended sense.*

In [3] Cesari proved the following remarkable theorem.

CESARI'S THEOREM. *Let the function $f(t, x, z)$ be lower semicontinuous and convex in the variable z for t and x in a certain range, and suppose that f never takes the value $-\infty$. Then the sets $K(t, x) = \operatorname{epi}_z f(t, x, z)$ have the property Q at the point $(\bar{t}, \bar{x})$ if and only if the function f is weakly seminormal in the extended sense at the point $(\bar{t}, \bar{x})$.*

With this theorem Cesari has shown how his own use of the property Q in existence questions extends a classical notion in the calculus of variations. We personally regard this "harmonization" of the ideas of optimal control with those of the traditional calculus of variations as eminently desirable, particularly when it exposes the underlying *geometrical* character of the classical ideas, as in the present case.

§ 3. A characterization of the property Q in terms of the Hamiltonian

The new feature of the property Q that comes out of its equivalence with weak seminormality is the emergence of the dual space of the z-space in the scalar product $\langle z, y\rangle$. This suggests that the duality theory of convex functions could be applied to give a dual characterization of the property Q in terms of conjugate convex functions.

The *conjugate function*, also known as the "Young–Fenchel transform", of the function $f(t, x, z)$ is the function $f^*(t, x, y)$ defined for y in the dual space Y by the formula

$$(*) \qquad f^*(t, x, y) = \sup_z [\langle z, y\rangle - f(t, x, z)].$$

Here, the variables (t, x), of course, just play the rôle of parameters.

The *closure* of a proper convex function is the function whose epigraph is the closure of the epigraph of the original function. The property of being "closed", i.e., having a closed epigraph, is equivalent to the *lower semicontinuity* of a function. In the case of finite-dimensional spaces, the passage from a proper convex function to its closure amounts at most to the replacement of the values of the function at the relative boundary of its domain by the limits of the values assumed by the function as its argument proceeds along line segments extending from the relative interior of the domain to the relative boundary: in all, a minor regularization process. It can be shown that the conjugate function is *always* closed, even if the original function is not.

In the next section we shall show how general considerations concerning conjugate convex functions in locally convex spaces lead, as a very special case, to the following result.

THEOREM. *Under the assumptions of Cesari's Theorem, the sets $K(t, x)$ will have the property Q at the point $(\bar{t}, \bar{x})$ if and only if the conjugate function $f^*(t, x, y)$ satisfies*

$$f^*(\bar{t}, \bar{x}, \cdot) = \mathrm{cl}_y[\limsup_{(t,x)\to(\bar{t},\bar{x})} f^*(t, x, y)]$$

Here, 'cl_y' denotes the closure operation described above, when the term in brackets is finite or $+\infty$ for all y; otherwise, it assigns the value $-\infty$.

This theorem means that *the property Q is equivalent, modulo closure, to the upper semicontinuity of the conjugate function f^* in the parameters* (t, x). If it occurs that the lim sup is finite-valued for all y, as is frequently the case, then it is automatically closed, because it is convex and, therefore, continuous, in y. In this case, the property Q is equivalent to the upper semicontinuity of $f^*(t, x, y)$ in (t, x) for fixed values of y.

In the general case, we cannot omit the closure, but we can exploit elementary facts concerning the conjugacy operation to establish the following

COROLLARY. *The sets $K(t, x)$ will have the property Q at $(\bar{t}, \bar{x})$ if and only if*

$$f(\bar{t}, \bar{x}, \cdot) = [\limsup_{(t,x)\to(\bar{t},\bar{x})} f^*(t, x, y)]^*.$$

This expresses the property Q solely in terms of the pointwise lim sup and the conjugacy operation.

Now, it is known that when f is smooth and a Legendre condition is satisfied, the conjugacy relation (*) reduces to the Legendre transformation, and the function f^* is just the *Hamiltonian* corresponding to the *Lagrangian* f. The virtue of the formula (*), however, is that it eliminates these restrictions and defines the Hamiltonian globally. The Hamiltonian is thus available, now, in problems that could not have been treated classically with rigour because of the local character of the Legendre transformation and the need to consider only finite-valued Lagrangians.

It thus emerges that the classical notion of weak seminormality has its natural formulation as the upper semicontinuity of the Hamiltonian, modulo closure. Nevertheless, this simple formulation seems to have escaped the attention of classical workers in the field. If we seek an explanation that goes beyond an abhorrence of the non-rigorous way in which Hamiltonians were sometimes treated, it may lie, in part, in the fact that, even in free problems, the conjugate function f^* may well assume infinite values, and the crucial notion of closure for such functions had not been worked out prior to the 1949 paper of Fenchel [4]. If, on the other hand, coercivity conditions are placed upon f as a function of z, as is usual in existence theorems, thereby forcing f^* to be finite-valued, the weak seminormality of f gets converted into seminormality, and the simple formulation in terms of the Hamiltonian becomes obscured.

The foregoing suggests that it might be easier in existence questions to use the Hamiltonian directly, instead of arguing in terms of the Lagrangian. Recent work of Rockafellar bears this out.

§ 4. The property Q in locally convex spaces

In the present section, we shall establish a characterization of the property Q in terms of conjugate convex functions on dual locally convex topological vector spaces. Our use of such general spaces is not meant to be frivolous. They arise as soon as one looks at function spaces endowed with a weak topology. Moreover, they occur as the natural setting in the treatment of convex functions, and it is remarkable that everything that we want to do can be carried out in them without introducing any further restrictions.

Thus, following Moreau [8] and Brøndsted [2], we consider two linear spaces Z and Y over the reals $\boldsymbol{R}$, paired by a bilinear form $\langle z, y\rangle$ on $Z\times Y$ which separates points both of Z and of Y. Each space is endowed with a locally convex topology that makes the other into its topological dual with respect to the pairing. Such topologies, which are evidently Hausdorff (or *séparées*), are said to be *compatible* with the pairing. Any such topologies will do for the theory, since they all give the same closed convex sets. The direct sums $\boldsymbol{R}\oplus Z$ and $\boldsymbol{R}\oplus Y$ are placed in duality by the pairing $(r\oplus z,\ s\oplus y)\mapsto rs+\langle z, y\rangle$; the product topologies are compatible whenever the underlying topologies of Z and Y are. All compatible topologies on Z then yield the same closed convex sets in $\boldsymbol{R}\oplus Z$, and, thus, the same class of closed convex functions on Z, for they are just the extended real-valued convex functions whose epigraphs are closed in $\boldsymbol{R}\oplus Z$. Similar remarks, of course, hold for Y.

There is a subclass of the closed convex functions on Z which are said to be *regular* with respect to the pairing (cf. Asplund [1]). They are the ones which can be written as the supremum of extended real-valued affine functions:

$$f(z) = \sup_{\iota}[\langle z, y_\iota\rangle - c_\iota], \qquad \text{for all } z \text{ in } Z,$$

where the c_ι are extended real-valued constants and ι runs over some index set. Practically speaking, what this means is that f cannot take the value $-\infty$ without being constant. A similar definition of regularity applies to Y.

Restricting attention to the regular convex functions on Z, in their natural ordering, they become a complete lattice under the operations

$$(\bigvee_\iota f_\iota)(z) = \sup_\iota f_\iota(z),$$

$$(\bigwedge_\iota f_\iota)(z) = \sup_h [h(z) \mid h \leqslant f_\iota \text{ for all } \iota]$$

for all z in Z, where ι runs over any index set and h denotes a generic extended real-valued affine function.

If we try to compare these operations with the lattice operations already defined on closed convex sets in § 1, by setting $K_\iota = \operatorname{epi} f_\iota$, we find that

$$\bigwedge_\iota \operatorname{epi} f_\iota = \bigcap_\iota \operatorname{epi} f_\iota = \operatorname{epi} \bigvee_\iota f_\iota,$$

in all cases, while

$$\bigvee_\iota \operatorname{epi} f_\iota = \operatorname{cl}\operatorname{co} \bigcup_\iota \operatorname{epi} f_\iota = \operatorname{epi} \bigwedge_\iota f_\iota,$$

provided that $\bigcap_\iota \operatorname{epi} f_\iota$ does not extend to $-\infty$ in the $\boldsymbol{R}$-direction. If it does, then there is a discrepancy, which can be remedied by *redefining* $\operatorname{cl} K$ *to be* $\boldsymbol{R} \oplus Z$ *whenever* K *extends to* $-\infty$ *in the* $\boldsymbol{R}$ *direction.* Understanding 'cl' in this sense, the operations of § 1 again yield a complete lattice, and the passage from epigraphs of regular convex functions to the functions themselves is an anti-isomorphism of their respective lattices.

Similar remarks, of course, apply to the regular convex functions on Y.

We now introduce the *conjugate function* f^* on Y of a regular convex function f on Z by means of Fenchel's formula

$$f^*(y) = \sup_z [\langle z, y\rangle - f(z)].$$

Clearly, f^* is a regular convex function on Y. Dually, we have for the conjugate g^* of a regular convex function g on Y

$$g^*(z) = \sup_y [\langle z, y\rangle - g(y)].$$

It is known that this correspondence between regular convex functions is biunique.

By means of Fenchel's formula, lattice operations on families of regular convex functions on one space are transformed into dual operations on their conjugate functions. These conjugate operations are given by

The Fenchel–Brøndsted formulas:

$$[\bigwedge_\iota f_\iota]^* = \bigvee_\iota f_\iota^*,$$
$$[\bigvee_\iota f_\iota]^* = \bigwedge_\iota f_\iota^*.$$

Thus conjugacy acts as an anti-isomorphism between the two function lattices. This means that we can arrive at a *lattice isomorphism by placing in correspondence the epigraphs of the regular convex functions on one space with the conjugate functions themselves on the dual space.*

Now suppose that the index set I is a set directed to the left by an order relation $\prec$ and possessing (as we may assume without loss of generality) a least element ω. Then we can define the order limes superior in the lattice of epigraphs of regular convex functions on Z:

$$\operatorname{o-lim\,sup}_{\iota\in I} \operatorname{epi} f_\iota = \bigwedge_{\lambda\neq\omega} \bigvee_{\iota\prec\lambda} \operatorname{epi} f_\iota.$$

This is just Cesari's operation Q, the element ω playing the role of $(\bar{t}, \bar{x})$ and the remaining elements of the index set being preordered by their distances from $(\bar{t}, \bar{x})$. *The functions* f_ι *thus have the property* Q *at* ω *if and only if* $\operatorname{epi} f_\omega$ *is given by the above expression.*

The isomorphism just mentioned carries the order limes superior in the lattice of epigraphs onto the order limes superior of their images in the lattice of conjugate functions on Y. Thus we have proved the first part of the following

THEOREM. *Let* $I, \prec$ *be a directed set with least element* ω, *and let* f_ι, $\iota \in I$, *be a net of regular convex functions on* Z. *Then* f_ι *has the property* Q *at* ω *if and only if*

$$f_\omega^* = \bigwedge_{\lambda\neq\omega} \bigvee_{\iota\prec\lambda} f_\iota^*, \quad \text{or, equivalently,} \quad f_\omega = [\bigwedge_{\lambda\neq\omega} \bigvee_{\iota\prec\lambda} f_\iota^*]^*.$$

The second part of the theorem is an immediate consequence of the involutary nature of the conjugate transform: $[f^*]^* = f$.

In the special case treated by Cesari, the index set is *totally* preordered, i.e.) any two elements are comparable. In this case, the formulas of the theorem can be simplified. The functions

$$g_\lambda = \bigvee_{\iota\prec\lambda} f_\iota^*$$

are then not only monotonic in their dependence upon λ, but also totally ordered. As Brøndsted ([2], p. 21) has observed, this means that the function $\inf_\lambda g_\lambda$ is convex, We may therefore appeal to the following

LEMMA. *If the directed set* I *is totally preordered by* $\prec$, *then*

$$\operatorname{cl} \inf_\lambda g_\lambda = \bigwedge_\lambda g_\lambda.$$

To prove the lemma, we observe, to begin with, that Brøndsted's observation implies that

$$\operatorname{epi} \inf_\lambda g_\lambda \supset \operatorname{co} \bigcup_\lambda \operatorname{epi} g_\lambda.$$

It follows that the same inclusion holds when we pass to the closures of the sets in question, even when the closure is taken in the emended sense described above. Consequently,

$$\operatorname{cl} \inf_\lambda g_\lambda \leqslant \bigwedge_\lambda g_\lambda,$$

and we clearly have equality whenever $\bigcup_\lambda \operatorname{epi} g_\lambda$ extends to $-\infty$ in the $\boldsymbol{R}$-direction.

To see that we also have equality in the contrary case, it is enough to show that

$$\operatorname{epi} \inf_\lambda g_\lambda \subset \operatorname{cl}\operatorname{co} \bigcup_\lambda \operatorname{epi} g_\lambda,$$

for then we shall have equality when we take the closure of the set at left. Now, if this inclusion did not hold, we could strictly separate some point in the set at left from the closed convex set at right by means of a non-vertical closed hyperplane. Hence, for some constant c and some point y we would have

$$\inf_{\lambda} g_\lambda(y) < c < g_\lambda(y)$$

for all λ in I. Taking the infimum over λ at right then leads to a contradiction.

With this lemma in hand, the first part of our theorem yields the

COROLLARY. *If, in addition, I is totally preordered by $\prec$, then the net f_ι, $\iota \in I$, has the property Q at ω if and only if*

$$f_\omega^* = \operatorname{cl} \limsup_{\iota\to\omega} f_\iota^*.$$

Indeed, replacing I in the lemma by $I\setminus\{\omega\}$ and using the definition of g_λ given above, yields

$$\bigwedge\nolimits_{\lambda\neq\omega} \bigvee\nolimits_{\iota\prec\lambda} f_\iota^* = \operatorname{cl} \inf_{\lambda\neq\omega} \sup_{\iota\prec\lambda} f_\iota^* = \operatorname{cl} \limsup_{\iota\to\omega} f_\iota^*,$$

by definition of the limes superior.

The second part of the theorem can be put into a nicer form by observing that $g^* = [\operatorname{cl} g]^*$ whenever g is convex. Proof: $g^{**} = \operatorname{cl} g$ by [2], p. 15, and $g^{***} = g^*$ by [2], p. 12. Consequently,

COROLLARY. *When I is totally preordered by $\prec$, the net f_ι, $\iota \in I$, has the property Q at ω if and only if*

$$f_\omega = [\limsup\nolimits_{\iota\to\omega} f_\iota^*]^*.$$

When specialized, these two corollaries give the results cited in § 3.

The first corollary allows us to give another proof, valid in locally convex spaces, of Cesari's Theorem, by establishing directly the equivalence of weak seminormality in the extended sense with the formula given in the corollary. For lack of space, we shall not enter into details here.

§ 5. A limes inferior and notion of convergence related to property Q

We have seen in the preceding section how Cesari's property Q is related to the order limes superior in certain complete lattices. It is both possible, and tempting, to introduce the order limes inferior (by inverting the order of the lattice operations) as a companion notion, and thus to define a convergence when these two partial limits coincide.

Nevertheless, experience with the topological limits of closed sets teaches us to be cautious, and to define the limes inferior in a different way (cf. [5]). Thus let $\mathscr{I}$ denote the class of all cofinal subsets I' of the directed set I, and let K_ι, $\iota \in I$, be a net of closed convex epigraphs over the space Z. We then define

$$\liminf\nolimits_{\iota\to\omega} K_\iota = \bigwedge\nolimits_{I'\in\mathscr{I}} \bigwedge\nolimits_{\lambda'\neq\omega} \bigvee\nolimits_{\iota'\prec\lambda'} K_{\iota'},$$

where, once again, ω is the minimal element of I. In other words, we take the limit superior over all cofinal subsets of I, and then their intersection.

We say that a net K_ι, $\iota \in I$, *converges* if the limes inferior and the limes superior agree. In the present case, this means that the limes superior does not change when we pass to cofinal subsets of I.

Our order isomorphism from the lattice of epigraphs of regular convex functions on Z to the lattice of their conjugate functions on Y preserves these operations and gives us, as a necessary and sufficient condition for the convergence of a net of epigraphs, the stability of the order limes superior of the conjugate functions under passage to cofinal subsets of I.

Whenever the directed set I is totally preordered, the limes inferior, defined above, when expressed in terms of the conjugate functions, can be shown to take the form

$$\bigwedge\nolimits_{I'\in\mathscr{I}} \bigwedge\nolimits_{\lambda'\neq\omega} \bigvee\nolimits_{\iota'\prec\lambda'} f_{\iota'}^* = \operatorname{cl}\operatorname{co}[\liminf\nolimits_{\iota\to\omega} f_\iota^*],$$

where the liminf is taken pointwise, for fixed elements y. Thus, the notion of convergence here is, essentially, pointwise convergence of the conjugate functions, followed by closure.

It is of interest to note that just this notion of convergence (under hypotheses which render unnecessary the taking of the closure) has been applied under the name of "G-convergence", by de Giorgi and his associates, to the conjugate functions of energy integrals in order to study the convergence of sequences of uniformly elliptic differential operators, v. [7].

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