ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XV, 3 (1976)

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A MIXED GAME OF TIMING INVOLVING $(p+n)\times 1$ ACTIONS

0. Introduction. The game analyzed in this paper is one of the examples of games of timing. We consider the following model. Two opponents A and B duel. Player A has p silent and n noisy bullets ($p \geqslant 1$ and $n \geqslant 1$), and player B has 1 noisy bullet. A fires off his silent bullets first. The terms *noisy* or *silent* say if the players are able to hear the shot or not.

We assume that the *accuracy function* (probability of hitting the opponent at the time t) for both players equals t, $t\in[0,1]$.

The pay-off function is denoted by $W[S_1; S_2]$, where S_1 and S_2 are strategies for A and B, respectively, and it is equal to

$$(*)\qquad W[S_1;S_2] = \Pr\{A \text{ survives alone}\} - \Pr\{B \text{ survives alone}\} \quad \text{for adopted } S_1 \text{ and } S_2.$$

The above information is known to both duellists.

We show that the game has a value and we find an optimal strategy for player A and an ε-optimal strategy for player B.

In Section 1 we evaluate the pay-off function for pure strategies, we describe classes of mixed strategies and we find optimal strategies. In Section 2 we prove the optimality of the strategies found.

The case $p = 1$, $n = 1$ of our game with arbitrary accuracy functions for both duellists was solved by Smith in [4], and the case $n = 0$ by Styszyński in [5]. The converse duel, i.e. when player A shoots his noisy bullets first, was considered by the authors in [3]. Other types of games of timing are considered in [1] and [2].

1. Mixed strategies. Denote by $\bar{x}_p$ the vector $(x_1, x_2, \ldots, x_p)$, and by $\bar{z}_n$ the vector $(z_1, z_2, \ldots, z_n)$, where

$$0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_p \leqslant z_1 \leqslant z_2 \leqslant \ldots \leqslant z_n \leqslant 1.$$

Let $W[\bar{x}_p, \bar{z}_n; y]$ denote the pay-off function when player A fires his silent bullets at the moments determined by the vector $\bar{x}_p$, the noisy ets at the moments determined by the vector $\bar{z}_n$, and player B fires his noisy bullet at the moment y, $0 \leqslant y \leqslant 1$. Clearly, we assume that 1 for $z_n < y$, and $z_n = 1$ for $y < z_n$. The pay-off function is of the form

$$W[\bar{x}_p, \bar{z}_n; y]$$

$$\begin{array}{ll}
1-2y & \text{if } y < x_1, \\
1-2y\prod\limits_{i=1}^{s}(1-x_i) & \text{if } x_s < y < x_{s+1}\ (s = 1, 2, \ldots, p-1), \\
1-2y\prod\limits_{i=1}^{p}(1-x_i) & \text{if } x_p < y < z_1, \\
1-2y\prod\limits_{i=1}^{p}(1-x_i)\prod\limits_{i=1}^{s}(1-z_i) & \text{if } z_s < y < z_{s+1}\ (s = 1, 2, \ldots, n-1), \\
1-(2-z_s)y\prod\limits_{i=1}^{p}(1-x_i)\prod\limits_{i=1}^{s-1}(1-z_i) & \\
 & \text{if } y = z_s\ (s = 1, 2, \ldots, n-1), \\
1-\prod\limits_{i=1}^{p}(1-x_i)\prod\limits_{i=1}^{n-1}(1-z_i) & \text{if } y = z_n, \\
1-2\prod\limits_{i=1}^{p}(1-x_i)\prod\limits_{i=1}^{n}(1-z_i) & \text{if } y > z_n.
\end{array}$$

It is easy to obtain these equations directly from the definition of pay-off function.

We seek for an optimal strategy S_A for player A in the following s of strategies.

A shoots his i-th silent bullet at time x_i belonging to the interval $a_{i+1})$ according to the density function $f_i(x_i)$ $(i = 1, 2, \ldots, p)$. Funcs $f_i(x_i)$ satisfy the normalizing conditions

$$\int_{a_i}^{a_{i+1}} f_i(x_i)\,dx_i = 1 \quad \text{for } i = 1, 2, \ldots, p.$$

A will shoot his j-th noisy bullet at moment c_j $(j = 1, 2, \ldots, n)$ with bability 1 under the condition that B does not shoot his bullet until time $t = c_n$ yet. In the opposite case, A will shoot his last noisy bullet $= 1$.

We assume furthermore that

$$0 < a_1 < \ldots < a_p < a_{p+1} = c_1 < c_2 < \ldots < c_n < 1.$$

In order to describe the class of strategies for player B we define an accessory class of strategies in a noisy duel. In the duel two opponents C and D have n and 1 noisy bullets, respectively. The strategy for player D will be defined by induction with respect to the number r of noisy bullets of C.

1. For $r = 1$, the strategy $S(1)$ for player D is the following:

If D does not hear his opponent's shot until the time $t = c_n$, he will shoot at this moment. In the opposite case, he will wait to the moment $t = 1$ with his action.

2. For $r = k < n$, we assume that the strategy $S(k)$ for player D is known.

3. For $r = k+1$, the strategy $S(k+1)$ is obtained from $S(k)$ in the following manner:

If player D does not hear his opponent's first noisy shot until the time $t = c_{n-k}$, he will begin to shoot during the interval $(c_{n-k}, c_{n-k}+\varepsilon_{n-k})$ with probability density $1/\varepsilon_{n-k}$ $(0 < \varepsilon_i \leqslant \min[(c_{i+1}-c_i), \varepsilon/2^i]$, $i = 1, 2, \ldots, n-1)$. However, he breaks the shooting after he had heard the shot of C in the interval $(c_{n-k}, c_{n-k}+\varepsilon_{n-k})$. Then he follows the strategy $S(k)$ considering the second noisy shot of C as the first one. In the opposite case, i.e. if player D hears the shot of his opponent before the time $t \leqslant c_{n-k}$, then he will at once follow the strategy $S(k)$ considering the next shot as the first one.

Now we can describe the class of strategies in which we seek for an ε-optimal strategy for player B.

Player B shoots his bullet at time y belonging to the interval $[a_1, c_1)$ with probability density $g(y)$, and with probability β follows the above-defined strategy $S(n)$. The described strategy for player B will be denoted by S_B^ε.

Clearly, the following condition is satisfied:

$$\int_{a_1}^{c_1} g(y)\,dy + \beta = 1. \tag{3}$$

It follows from the definitions of S_A, S_B^ε and $W[S_1; S_2]$ that

$$W[S_A; y] = \int_{a_1}^{a_2}\int_{a_2}^{a_3}\cdots\int_{a_p}^{c_1} W[\bar{x}_p, \bar{z}_n; y]\prod_{i=1}^{p} f_i(x_i)\,dx_i \tag{4}$$

and

$$W[\bar{x}_p, \bar{z}_n; S_B] = \int_{a_1}^{c_1} W[\bar{x}_p, \bar{z}_n; y]g(y)\,dy + \beta W[\bar{x}_p, \bar{z}_n; S(n)]. \tag{5}$$

Let us assume that the following conditions (6)-(9) are satisfied:

$$W[S_A; y] = \text{const} = v \quad \text{for every } y \in [a_1, c_1), \tag{6}$$

$$W[\bar{x}_p^{**}, \bar{c}_n; S_B^\varepsilon] = v \quad \text{for every } \bar{x}_p^{**}, \tag{7}$$

where $\bar{x}_p^{**}$ denotes any vector $\bar{x}_p$ the components of which are restricted by $x_i \in [a_i, a_{i+1})$ for $i = 1, 2, \ldots, p$ and by $\bar{c}_n = (c_1, c_2, \ldots, c_n)$,

$$\lim_{y \to c_i^-} W[S_A; y] = v \quad \text{for } i = 2, 3, \ldots, n, \tag{8}$$

$$W[S_A; 1] = v. \tag{9}$$

We show that strategies S_A and S_B^ε obtained by (6)-(9) are the sought ones.

It follows from relations (1), (4) and (6) for $y \in [a_s, a_{s+1})$ and $s = 1, 2, \ldots, p$ that

$$v = \int_{a_1}^{a_2} \ldots \int_{a_{s-1}}^{a_s} \Big[\int_{a_s}^{y} \Big[1 - 2y \prod_{i=1}^{s} (1-x_i)\Big] f_s(x_s)\, dx_s + \\ + \int_{y}^{a_{s+1}} \Big[1 - 2y \prod_{i=1}^{s-1} (1-x_i)\Big] f_s(x_s)\, dx_s\Big] \prod_{i=1}^{s-1} f_i(x_i)\, dx_i. \tag{10}$$

Notice that $\prod_{i=1}^{0} (\cdot) \overset{\text{def}}{=} 1$.

Differentiating twice both sides of equation (10) with respect to y we have

$$\frac{f'(y)}{f_s(y)} = -\frac{3}{y} \quad \text{for } y \in [a_s, a_{s+1}),\ s = 1, 2, \ldots, p,$$

and

$$f_s(x_s) = \frac{k_s}{x_s^3} \quad \text{for } x_s \in [a_s, a_{s+1}),\ s = 1, 2, \ldots, p. \tag{11}$$

Putting (11) into (10) we obtain

$$v = \int_{a_1}^{a_2} \ldots \int_{a_{s-1}}^{a_s} \Big\{1 - 2 \prod_{i=1}^{s-1} (1-x_i) \Big[y\Big(1 - \frac{k_s}{a_s}\Big) + k_s \Big]\Big\} \prod_{i=1}^{s-1} \frac{k_i}{x_i^3}\, dx_i.$$

Since v does not depend on $y \in [a_s, a_{s+1})$, we have

$$k_s = a_s \quad \text{for } s = 1, 2, \ldots, p. \tag{12}$$

Therefore, for $s = 1, 2, \ldots, p$ we have

$$v = \int_{a_1}^{a_2} \ldots \int_{a_{s-1}}^{a_s} \Big[1 - 2k_s \prod_{i=1}^{s-1} (1-x_i)\Big] \prod_{i=1}^{s-1} \frac{k_i}{x_i^3}\, dx_i = 1 - 2k_s \prod_{i=1}^{s-1} k_i \int_{a_i}^{a_{i+1}} \frac{1-x_i}{x_1^3}\, dx_i$$

$$= 1 - 2k_s \frac{k_1}{a_2} \frac{k_2}{a_3} \ldots \frac{k_{s-1}}{a_s} = 1 - 2a_1.$$

In the last transformations we have used (2). Putting (11) and (12) into (2) we obtain

$$\frac{1}{a_i^2}-\frac{1}{a_{i+1}^2}=\frac{2}{a_i} \quad \text{for } i=1,2,\ldots,p. \tag{13}$$

Therefore, condition (6) holds if equations (11)-(13) are valid and

$$v=1-2a_1. \tag{14}$$

From (1), (5) and (7) we get

$$v=\sum_{i=1}^{p}\Big(\int_{a_i}^{x_i}\Big[1-2y\prod_{j=1}^{i-1}(1-x_j)\Big]g(y)dy+ \tag{15}$$
$$+\int_{x_i}^{a_{i+1}}\Big[1-2y\prod_{j=1}^{i}(1-x_j)\Big]g(y)dy\Big)+\beta\Big[1-\prod_{i=1}^{p}(1-x_i)\prod_{i=1}^{n-1}(1-c_i)\Big].$$

Differentiating twice both sides of equation (15) with respect to x_s for $1\leqslant s\leqslant p$ we have

$$\frac{g'(x_s)}{g(x_s)}=-\frac{3}{x_s} \quad \text{for } x_s\in[a_s,a_{s+1})$$

and

$$g(y)=\frac{l_s}{y^3} \quad \text{for } y\in[a_s,a_{s+1}),\ s=1,2,\ldots,p. \tag{16}$$

Putting (3) and (16) into (15) we obtain

$$v=1-2\Big[\sum_{i=1}^{p}l_i\prod_{j=1}^{i-1}(1-x_j)\Big(\frac{1}{a_i}-\frac{1}{a_{i+1}}-1+\frac{x_i}{a_{i+1}}\Big)+ \tag{17}$$
$$+\frac{\beta}{2}\prod_{i=1}^{p}(1-x_i)\prod_{i=1}^{n-1}(1-c_i)\Big].$$

From relations (1), (4) and (8) it follows for $s=2,3,\ldots,n$ that

$$v=\int_{a_1}^{a_2}\int_{a_2}^{a_3}\ldots\int_{a_p}^{c_1}\Big[1-2c_s\prod_{i=1}^{p}(1-x_i)\prod_{i=1}^{s-1}(1-c_i)\Big]\prod_{i=1}^{p}\frac{a_i}{x_i^3}dx_i$$
$$=1-2c_s\prod_{i=1}^{s-1}(1-c_i)\int_{a_1}^{a_2}\int_{a_2}^{a_3}\ldots\int_{a_p}^{c_1}\prod_{i=1}^{p}(1-x_i)\frac{a_i}{x_i^3}dx_i$$
$$=1-2c_s\prod_{i=1}^{s-1}(1-c_i)\frac{a_1}{a_{p+1}},$$

whence

$$v = 1-2c_s \prod_{i=1}^{s-1} (1-c_i) \frac{a_1}{a_{p+1}} = 1-2c_{s+1} \prod_{i=1}^{s} (1-c_i) \frac{a_1}{a_{p+1}}$$
$$\text{for } s = 1, 2, \ldots, n-1.$$

Therefore

$$c_{s+1} = \frac{c_s}{1-c_s} \quad \text{for } s = 1, 2, \ldots, n-1. \tag{18}$$

By relations (1), (4), (9) we have

$$v = 1-2 \prod_{i=1}^{n} (1-c_i) \frac{a_1}{a_{p+1}},$$

but also

$$v = 1-2c_n \prod_{i=1}^{n-1} (1-c_i) \frac{a_1}{a_{p+1}}$$

and, in this manner,

$$c_n = \frac{1}{2}. \tag{19}$$

It follows from formulas (18) and (19) that

$$c_i = \frac{1}{n-i+2} \quad \text{for } i = 1, 2, \ldots, n. \tag{20}$$

By relations (17) and (20) we obtain

$$\begin{aligned} v &= 1-2\Big[\sum_{i=1}^{p} l_i \prod_{j=1}^{i-1} (1-x_j)\Big(\frac{1}{a_i} - \frac{1}{a_{i+1}} - 1 + \frac{x_i}{a_{i+1}}\Big) + \beta c_1 \prod_{i=1}^{p} (1-x_i)\Big] \\ &= 1-2\Big[\sum_{i=1}^{p-1} l_i \prod_{j=1}^{i-1} (1-x_j)\Big(\frac{1}{a_i} - \frac{1}{a_{i+1}} - 1 + \frac{x_i}{a_{i+1}}\Big) + \\ &\quad + \prod_{i=1}^{p-1} (1-x_j)\Big(\frac{l_p}{a_p} - \frac{l_p}{a_{p+1}} - l_p + \frac{x_p l_p}{a_{p+1}} + \beta a_{p+1} - x_p \beta a_{p+1}\Big)\Big] = v. \end{aligned}$$

Note that $a_{p+1} = c_1$. The last expression does not depend on x_p if the following equation holds:

$$l_p = \beta a_{p+1}^2 = \frac{\beta}{(n+1)^2}. \tag{21}$$

We repeat step by step the procedure outlined above for the coefficients $p-2,\ p-3,\ldots,1$ and obtain

$$l_i = \frac{l_{i-1}}{1-a_i} \quad \text{for } i = 2,3,\ldots,p-1 \tag{22}$$

and

$$v = 1-2\left(\frac{l_1}{a_1}-l_1\right).$$

On the other hand, by (14), $v = 1-2a_1$ and, therefore,

$$l_1 = \frac{a_1^2}{1-a_1}. \tag{23}$$

By (21)-(23) we have

$$l_i = a_1^2\prod_{j=1}^{i}\frac{1}{1-a_j} \quad \text{for } i = 1,2,\ldots,p$$

and also

$$\beta = (n+1)^2 a_1^2\prod_{j=1}^{p}\frac{1}{1-a_j}. \tag{23a}$$

We have to prove the normalizing condition (3) which is equivalent to the equation

$$\frac{l_1}{a_1}+\frac{l_2}{a_2}+\ldots+\frac{l_p}{a_p}+\beta = 1,$$

and this to the following

$$a_1^2\sum_{i=1}^{p}\frac{1}{a_i}\prod_{j=1}^{i}\frac{1}{1-a_j}+\beta = 1. \tag{24}$$

By (13) we have

$$\frac{1}{a_p^2}-(n+1)^2 = \frac{2}{a_p}$$

which implies

$$a_p+a_p^2(n+1)^2 = 1-a_p,$$

whence

$$\frac{a_p^2}{a_p(1-a_p)}+(n+1)^2\frac{a_p^2}{1-a_p} = 1. \tag{25}$$

Note at first that equation (13) can be written in the form

$$\frac{1}{a_{i+1}^2}+\frac{1}{a_i} = \frac{1-a_i}{a_i^2} \quad \text{for } i = 1,2,\ldots,p.$$

Using this relation we transform equation (25) making $p-1$ operations, where the j-th operation is carried out in the following manner:

We multiply the equation obtained from (25) after $j-1$ operations by $1/a_{p-j+1}^2$; then we add to both sides $1/a_{p-j}$ and multiply by $a_{p-j}^2/(1-a_{p-j})$. For example, we show the first operation:

$$\frac{a_p^2}{a_p(1-a_p)}+\frac{(n+1)^2a_p^2}{1-a_p}=1,$$

$$\frac{1}{a_p(1-a_p)}+\frac{(n+1)^2}{1-a_p}=\frac{1}{a_p^2},$$

$$\frac{1}{a_{p-1}}+\frac{1}{a_p(1-a_p)}+\frac{(n+1)^2}{1-a_p}=\frac{1}{a_p^2}+\frac{1}{a_{p-1}}=\frac{1-a_{p-1}}{a_{p-1}^2};$$

$$\frac{a_{p-1}^2}{a_{p-1}(1-a_{p-1})}+\frac{a_{p-1}^2}{a_p(1-a_p)(1-a_{p-1})}+\frac{a_{p-1}^2(n+1)^2}{(1-a_p)(1-a_{p-1})}=1.$$

After $p-1$ operations we obtain (24).

By (23a), (24), (13) and (20) we have

$$0<\beta<1,\quad 0<a_1<\ldots<a_{p+1}=c_1<c_2<\ldots<c_n<1.$$

In this manner we have found all the parameters and density functions which define the strategies of both players. Now we have to show the optimality of them.

2. Proof of optimality of strategies S_A and S_B^ε. In this section we show that

$$\min_{0\leqslant y\leqslant 1} W[S_A;y]=v \tag{26}$$

and

$$\max_{0\leqslant x_1\leqslant\ldots\leqslant x_p\leqslant z_1\leqslant\ldots\leqslant z_n\leqslant 1} W[\bar{x}_p,\bar{z}_n;S_B^\varepsilon]\leqslant v+\varepsilon. \tag{27}$$

At first we prove equation (26) considering the following cases:

1. If $y<a_1$, then $W[S_A;y]=1-2y>1-2a_1=v$.
2. If $y\in[a_1,c_1)$, then, by (6), we have $W[S_A;y]=v$.
3. If $y\in(c_s,c_{s+1})$ for $s=1,2,\ldots,n-1$, then, by (8),

$$W[S_A;y]\geqslant\lim_{y\to c_{s+1}^-}W[S_A;y]=v.$$

4. If $y=c_s$ for $s=1,2,\ldots,n-1$, then

$$W[S_A;y]=\int_{a_1}^{a_2}\int_{a_2}^{a_3}\ldots\int_{a_p}^{c_1}\Big[1-(2-c_s)c_s\prod_{i=1}^{p}(1-x_i)\Big]\prod_{i=1}^{p}\frac{a_i}{x_i^3}dx_i$$

$$\geqslant \int_{a_1}^{a_2} \int_{a_2}^{a_3} \cdots \int_{a_p}^{c_1} \Big[1-2c_s \prod_{i=1}^{p} (1-x_i) \prod_{i=1}^{s-1} (1-c_i)\Big] \prod_{i=1}^{p} \frac{a_i}{x_i^3}\, dx$$

$$= \lim_{y \to c_s^-} W[S_A; y] = v.$$

5. If $y = c_n$, then, by (9),

$$W[S_A; c_n] = \int_{a_1}^{a_2} \cdots \int_{a_p}^{a_{p+1}} \Big[1-\prod_{i=1}^{p} (1-x_i) \prod_{i=1}^{n-1} (1-c_i)\Big] \prod_{i=1}^{p} \frac{a_i}{x_i^3}\, dx_i$$

$$= \int_{a_1}^{a_2} \cdots \int_{a_p}^{a_{p+1}} \Big[1-2\prod_{i=1}^{p} (1-x_i) \prod_{i=1}^{n} (1-c_i)\Big] \prod_{i=1}^{p} \frac{a_i}{x_i^3}\, dx_i$$

$$= W[S_A; 1] = v.$$

6. If $y > c_n$, then

$$W[S_A; y] \geqslant W[S_A; 1] = v.$$

This completes the proof of equation (26).

Let $\bar{x}_{p-s}$ and $\bar{z}_{n-s}$ denote the vectors $\bar{x}_p$ and $\bar{z}_n$, respectively, without s first components. Assume that $\bar{x}_p^*$ is composed from $\bar{x}_p$ by setting the components greater than c_1 equal to c_1 and the components smaller than a_1 equal to a_1, and suppose that $\bar{z}_n^*$ is composed from $\bar{z}_n$ by setting the components smaller than c_1 equal to c_1.

Before the proof of inequality (27) we give some lemmas.

LEMMA 1. a. *If $y > x_s$ for s, $1 \leqslant s \leqslant p$, then*

$$W[\bar{x}_p, \bar{z}_n; y] = 1-\prod_{i=1}^{s} (1-x_i) + \prod_{i=1}^{s} (1-x_i) W[\bar{x}_{p-s}, \bar{z}_n; y], \tag{28}$$

where $W[\bar{x}_{p-s}, \bar{z}_n; y]$ denotes the pay-off function in the duel "$p-s$ silent and n noisy bullets versus 1 noisy bullet".

b. *If $y > z_t$ for t, $1 \leqslant t \leqslant n$, then*

$$W[\bar{x}_p, \bar{z}_n; y] = 1-\prod_{i=1}^{p} (1-x_i) \prod_{i=1}^{t} (1-z_i) + \prod_{i=1}^{p} (1-x_i) \prod_{i=1}^{t} (1-z_i)\, W[\bar{z}_{n-t}; y]. \tag{29}$$

c. *The following equations hold:*

$$W[\bar{x}_p, \bar{z}_n; S(n)] = 1-\prod_{i=1}^{s} (1-x_i) \{1-W[\bar{x}_{p-s}, \bar{z}_n; S(n)]\} \quad \textit{for } s,\ 0 \leqslant s \leqslant p, \tag{30}$$

$$(31)\quad W[\bar{x}_p, \bar{z}_n; S(n)] = 1 - \prod_{i=1}^{p}(1-x_i)\prod_{i=1}^{t}(1-z_i)\{1 - W[\bar{z}_{n-t}; S(n-t)]\}$$
$$\textit{if } z_1 \leqslant c_1,\ z_2 \leqslant c_2, \ldots, z_t \leqslant c_t,\ 0 \leqslant t \leqslant n-1.$$

Proof. $1-\prod_{i=1}^{s}(1-x_i)$ denotes the probability of hitting the player B by player A with any of his s first silent bullets, and $\prod_{i=1}^{s}(1-x_i)$ is the probability of the contrary event. In the first case the pay-off for player A equals $+1$, and in the second case the pay-off equals $W[\bar{x}_{p-s}, \bar{z}_n; y]$. Thus by (*) we have relation (28).

We can similarly show relation (29) to be true. In an analogous way we prove equations (30) and (31). We show, for example, equation (31).

$W[\bar{x}_p, \bar{z}_n; S(n)]$ is the pay-off of player A when he shoots his bullets at the moments determined by $(\bar{x}_p, \bar{z}_n)$, and player B uses the strategy $S(n)$. From the definition of the strategy of player B it follows that if $z_1 \leqslant c_1$, $z_2 \leqslant c_2, \ldots, z_t \leqslant c_t$, then this player omits his opponent's t noisy shots, the $(t+1)$-st noisy shot he considers as the first one and follows the strategy $S(n-t)$. If player A hits B with any of his silent bullets or with any of his t first noisy bullets, the probability of this event equals

$$1-\prod_{i=1}^{p}(1-x_i)\prod_{i=1}^{t}(1-z_i)$$

and his pay-off is $+1$.

On the other hand, if player A does not hit B with any bullet of his $p+t$ first shots, the probability of this event equals

$$\prod_{i=1}^{p}(1-x_i)\prod_{i=1}^{t}(1-z_i)$$

and his pay-off is $W[\bar{z}_{n-t}; S(n-t)]$. Thus the proof of relation (31) is complete.

LEMMA 2. *If* $x_1 < a_1 \leqslant z_1$, *then*

$$W[\bar{x}_p, \bar{z}_n; S_B^\varepsilon] \leqslant W[(\underbrace{a_1, a_1, \ldots, a_1}_{s \text{ times}}, x_{s+1}, x_{s+2}, \ldots, x_p), \bar{z}_n; S_B^\varepsilon],$$

where s $(1 \leqslant s \leqslant p)$ *is the number of those components of the vector* $\bar{x}_p$ *which are smaller than* a_1.

Proof. By the definition of S_B^ε and by (28) and (30) we have

$$W[\bar{x}_p, \bar{z}_n; S_B^\varepsilon] = \int_{a_1}^{c_1} W[\bar{x}_p, \bar{z}_n; y]g(y)dy + \beta W[\bar{x}_p, \bar{z}_n; S(n)]$$

$$= \int_{a_1}^{c_1} \Big[1 - \prod_{i=1}^{s} (1-x_i)\{1 - W[\bar{x}_{p-s}, \bar{z}_n; y]\}\Big] g(y)\,dy +$$
$$+ \beta W[(\underbrace{a_1, a_1, \ldots, a_1}_{t \text{ times}}, x_{s+1}, x_{s+2}, \ldots, x_p), \bar{z}_n; S(n)]$$
$$\leqslant W[(\underbrace{a_1, a_1, \ldots, a_1}_{t \text{ times}}, x_{s+1}, x_{s+2}, \ldots, x_p), \bar{z}_n; S_B^{\varepsilon}].$$

In the above transformations we used obvious relations

$$1 - W[\bar{x}_{p-s}, \bar{z}_n; y] \geqslant 0 \quad \text{and} \quad 1 - W[\bar{x}_{p-s}, \bar{z}_n; S(n)] \geqslant 0.$$

Thus the proof of Lemma 2 is complete.

LEMMA 3. *If $c_1 < x_p$, then, for ε, $0 < \varepsilon < 1/10n(n+1)$,*

$$W[\bar{x}_p, \bar{z}_n; S_B^{\varepsilon}] \leqslant W[(x_1, x_2, \ldots, x_s, \underbrace{c_1, c_1, \ldots, c_1}_{p-s \text{ times}}), \bar{z}_n; S_B^{\varepsilon}],$$

where $p-s$ $(1 \leqslant p-s \leqslant p)$ is the number of those components of the vector $\bar{x}_p$ which are greater than c_1.

Proof. Let $x_{s+1} > c_1$, and $\gamma = \min(z_1, c_1 + \varepsilon_1)$. We consider two cases: $x_{s+1} \geqslant c_1 + \varepsilon_1$ and $c_1 < x_{s+1} < c_1 + \varepsilon_1$.

1. For $x_{s+1} \geqslant c_1 + \varepsilon_1$, by (5) and the definition of $S(n)$ we have

$$W[\bar{x}_p, \bar{z}_n; S_B^{\varepsilon}] = \int_{a_1}^{c_1} W[\bar{x}_p, \bar{z}_n; y] g(y)\,dy + \beta \int_{c_1}^{c_1+\varepsilon_1} \Big[1 - 2y \prod_{i=1}^{s} (1-x_i)\Big] \frac{1}{\varepsilon_1}\,dy$$
$$\leqslant \int_{a_1}^{c_1} W[(x_1, x_2, \ldots, x_s, \underbrace{c_1, c_1, \ldots, c_1}_{p-s \text{ times}}), \bar{z}_n; y] g(y)\,dy +$$
$$+ \beta \int_{c_1}^{c_1+\varepsilon_1} \Big[1 - 2y \prod_{i=1}^{s} (1-x_i)(1-c_1)^{p-s}\Big] \frac{1}{\varepsilon_1}\,dy$$
$$= W[(x_1, x_2, \ldots, x_s, \underbrace{c_1, c_1, \ldots, c_1}_{p-s \text{ times}}), \bar{z}_n; S_B^{\varepsilon}].$$

2. For x_{s+1}, $c_1 < x_{s+1} < c_1 + \varepsilon_1$, let us denote by r the number of those components of the vector $\bar{x}_p$ which are smaller than $c_1 + \varepsilon_1$. Therefore, $x_i < c_1 + \varepsilon_1$ for $i = 1, 2, \ldots, r$, and

$$W[\bar{x}_p, \bar{z}_n; S_B^{\varepsilon}] = \int_{a_1}^{c_1} W[\bar{x}_p, \bar{z}_n; y] g(y)\,dy + \beta \Big\{ \int_{c_1}^{\gamma} W[\bar{x}_p, \bar{z}_n; y] \frac{1}{\varepsilon_1}\,dy +$$
$$+ \int_{\gamma}^{c_1+\varepsilon_1} \Big[1 - \prod_{i=1}^{p} (1-x_i)(1-z_1) + \prod_{i=1}^{p} (1-x_i)(1-z_1) W[\bar{z}_{n-1}; S(n-1)]\Big] \frac{1}{\varepsilon_1}\,dy \Big\}.$$

Let us consider $W[\bar{x}_p, \bar{z}_n; S_B^\varepsilon]$ as a function of x_{s+1} and denote it by $H(x_{s+1})$. Then

$$H(x_{s+1}) = \int_{a_1}^{c_1} W[\bar{x}_p, \bar{z}_n; y]g(y)\,dy + \beta\Big\{\int_{c_1}^{x_{s+1}} 1-2y\prod_{i=1}^{s}(1-x_i)\frac{1}{\varepsilon_1}\,dy +$$

$$+\sum_{i=s+1}^{r-1}\int_{x_i}^{x_{i+1}}\Big[1-2y\prod_{j=1}^{i}(1-x_j)\Big]\frac{1}{\varepsilon_1}dy + \int_{x_r}^{\gamma}\Big[1-2y\prod_{j=1}^{r}(1-x_j)\Big]\frac{1}{\varepsilon_1}dy +$$

$$+\int_{c_1}^{c_1+\varepsilon_1}\Big[1-\prod_{i=1}^{p}(1-x_i)(1-z_1)\big[1-W[\bar{z}_{n-1}; S(n-1)]\big]\Big]\frac{1}{\varepsilon_1}\,dy\Big\}$$

$$= \int_{a_1}^{c_1} W[\bar{x}_p, \bar{z}_n; y]g(y)\,dy + \beta - \frac{\beta\prod_{j=1}^{s}(1-x_j)}{\varepsilon_1}\Big\{\int_{c_1}^{x_{s+1}} 2y\,dy +$$

$$+\sum_{i=s+1}^{r-1}\int_{x_i}^{x_{i+1}} 2y\prod_{j=s+1}^{i}(1-x_j)\,dy + \int_{x_r}^{\gamma} 2y\prod_{j=s+1}^{r}(1-x_i)\,dy +$$

$$+(c_1+\varepsilon-\gamma)\prod_{i=s+1}^{p}(1-x_i)(1-z_1)\big[1-W[\bar{z}_{n-1}; S(n-1)]\big]\Big\}.$$

Remark that in the last part of this equation the first two terms do not depend on x_{s+1}. Therefore, for $r > s+1$ we have

$$H'(x_{s+1}) = \frac{-\beta\prod_{i=1}^{s}(1-x_i)}{\varepsilon_1}\Big\{3x_{s+1}^2 - x_{s+2}^2 -$$

$$-\Big(\sum_{i=s+2}^{r-1}\int_{x_i}^{x_{i+1}} 2y\prod_{j=s+2}^{i}(1-x_j)\,dy + \int_{x_r}^{\gamma} 2y\prod_{i=s+2}^{r}(1-x_i)\,dy\Big) -$$

$$-(c_1+\varepsilon_1-\gamma)\prod_{i=s+2}^{p}(1-x_i)(1-z_1)\big[1-W[\bar{z}_{n-1}; S(n-1)]\big]\Big\}$$

$$\leqslant \frac{-\beta\prod_{i=1}^{s}(1-x_i)}{\varepsilon_1}\Big\{3c_1^2 - (c_1+\varepsilon_1)^2 - \int_{c_1}^{c_1+\varepsilon_1} 2y\,dy - \varepsilon_1\Big\}$$

$$= \frac{-\beta\prod_{i=1}^{s}(1-x_i)}{\varepsilon_1}\{4c_1^2-2(c_1+\varepsilon_1)^2-\varepsilon_1\} < 0 \quad \text{for } 0<\varepsilon<\frac{1}{10n(n+1)}$$

since $c_1 = 1/(n+1)$, and $\varepsilon_1 \leqslant \varepsilon/2$.

For $r = s+1$, by similar transformations we obtain

$$H'(x_{s+1}) < \frac{-\beta\prod_{i=1}^{s}(1-x_i)}{\varepsilon_1}\{3c_1^2-\gamma^2-\varepsilon_1\} < 0 \quad \text{for } 0<\varepsilon<\frac{1}{10n(n+1)}.$$

Therefore, $H(x_{s+1})$ is a decreasing function of x_{s+1} in the interval $[c_1, x_{s+2})$ if $r > s+1$ and in the interval $[c_1, \gamma]$ if $r = s+1$, and

$$W[\bar{x}_p, \bar{z}_n; S_B^\varepsilon] \leqslant W[(x_1, x_2, \ldots, x_s, c_1, x_{s+2}, \ldots, x_p), \bar{z}_n; S_B^\varepsilon] \quad \text{for } r \geqslant s+1.$$

By (1) this inequality holds also for $r = s$. This completes the proof of Lemma 3.

LEMMA 4. *If $a_1 \leqslant z_1$, then*

$$W[\bar{x}_p, \bar{z}_n; S_B^\varepsilon] \leqslant W[\bar{x}_p, \bar{z}_n; S_B^\varepsilon] \quad \textit{for } 0<\varepsilon<\frac{1}{10n(n+1)}.$$

The lemma easily follows from Lemmas 2 and 3.

LEMMA 5. *Let s be the number of those components of the vector $\bar{z}_n$ which are smaller than a_1. Then*

$$W[\bar{x}_p, \bar{z}_n; S_B^\varepsilon] \leqslant W[\bar{x}_p, (\underbrace{a_1, a_1, \ldots, a_1}_{s \text{ times}}, z_{s+1}, \ldots, z_n); S_B^\varepsilon].$$

Proof. By (5), (29) and (31) we have

$$\begin{aligned}
W[\bar{x}_p, \bar{z}_n; S_B^\varepsilon] &= \int_{a_1}^{c_1} W[\bar{x}_p, \bar{z}_n; y]g(y)dy + \beta W[\bar{x}_p, \bar{z}_n; S(n)] \\
&= \int_{a}^{c_1}\Big[1-\prod_{i=1}^{p}(1-x_i)\prod_{i=1}^{s}(1-z_i)\{1-W[\bar{z}_{n-s}; y]\}\Big]g(y)dy + \\
&\quad + \beta\Big\{1-\prod_{i=1}^{p}(1-x_i)\prod_{i=1}^{s}(1-z_i)\big[1-W[\bar{z}_{n-s}; S(n-s)]\big]\Big\} \\
&\leqslant \int_{a_1}^{c_1}\Big[1-\prod_{i=1}^{p}(1-x_i)\prod_{i=1}^{s}(1-a_1)\{1-W[\bar{z}_{n-s}; y]\}\Big]g(y)dy + \\
&\quad + \beta\Big\{1-\prod_{i=1}^{p}(1-x_i)\prod_{i=1}^{s}(1-a_1)\big[1-W[\bar{z}_{n-s}; S(n-s)]\big]\Big\} \\
&= \int_{a_1}^{c_1} W[\bar{x}_p, (\underbrace{a_1, a_1, \ldots, a_1}_{s \text{ times}}, z_{s+1}, z_{s+2}, \ldots, z_n); y]g(y)dy +
\end{aligned}$$

$$+\beta W[\bar{x}_p, (\underbrace{a_1, a_1, \ldots, a_1}_{s \text{ times}}, z_{s+1}, z_{s+2}, \ldots, z_n); S(n)]$$

$$= W[\bar{x}_p, (\underbrace{a_1, a_1, \ldots, a_1}_{s \text{ times}}, z_{s+1}, z_{s+2}, \ldots, z_n); S_B^s].$$

LEMMA 6. *Let l be the number of those components of the vector $\bar{z}_n$ which are smaller than c_1. Then*

$$W[\bar{x}_p, \bar{z}_n; S_B^s] \leqslant W[\bar{x}_p, (\underbrace{c_1, c_1, \ldots, c_1}_{l \text{ times}}, z_{l+1}, \ldots, z_n); S_B^s]$$

$$\textit{for } 0 < \varepsilon < \frac{1}{10n(n+1)}.$$

Remark. Blackwell and Girshick have shown in [1] that $S(m)$ $(1 \leqslant m \leqslant n)$ is an ε_{n-m}-optimal strategy in the duel "m noisy bullets versus 1 noisy bullet" for the player having 1 bullet. The value of the game evaluated there equals $(m-1)/(m+1)$. Then

$$W[\bar{z}_{n-l}; S(n-l)] \leqslant \frac{n-l-1}{n-l+1} + \varepsilon_l. \tag{32}$$

Proof. By Lemma 5 we can assume that $z_l \in [a_s, a_{s+1})$. Let

$$b = \begin{cases} \max(z_{l-1}, a_s) & \text{if } l > 1, \\ \max(x_p, a_s) & \text{if } l = 1, \end{cases} \quad \text{and} \quad A = \prod_{i=1}^{p}(1-x_i)\prod_{i=1}^{l-1}(1-z_i).$$

Using (29), (31) and (22) we can write

$$W[\bar{x}_p, \bar{z}_n; S_B^s] = \int_{a_1}^{b} W[\bar{x}_p, \bar{z}_n; y]g(y)dy + \int_{b}^{z_l} [1-2yA]g(y)dy + \\ + \int_{z_l}^{c_1} [1-2y(1-z_l)A]g(y)dy + \beta W[\bar{x}_p, \bar{z}_n; S(n)]$$

which will be considered as a function $F(z_l)$ of z_l. Then

$$F(z_l) = \int_{a_1}^{b} W[\bar{x}_p, \bar{z}_n; y]g(y)dy + \int_{b}^{c_1} g(y)dy + \beta - 2A\int_{b}^{c_1} yg(y)dy - \\ -2A\left\{-z_l\int_{z_l}^{c_1} yg(y)dy + (1-z_l)\frac{\beta}{2}[1-W[\bar{z}_{n-l}; S(n-l)]]\right\}.$$

Observe that in this expression the first four terms do not depend on z_l.

Using (32) we estimate $F'(z_l)$:

$$F'(z_l) = -2A\left[\frac{d}{dz_l}\left(-z_l\int\limits_{z_l}^{a_{s+1}} yg(y)dy - z_l\sum_{i=s+1}^{p}\int\limits_{a_i}^{a_{i+1}} yg(y)dy\right) - \right.$$

$$\left. -\frac{\beta}{2}\left(1-W[\bar{z}_{n-l};\, S(n-1)]\right)\right]$$

$$= -2A\left[\frac{l_s}{a_{s+1}} - \left(\frac{l_{s+1}}{a_{s+1}} - \frac{l_{s+1}}{a_{s+2}}\right) - \left(\frac{l_{s+2}}{a_{s+2}} - \frac{l_{s+2}}{a_{s+3}}\right) - \ldots - \right.$$

$$\left. - \left(\frac{l_p}{a_p} - \frac{l_p}{a_{p+1}}\right) - \frac{\beta}{2}\left(1-W[\bar{z}_{n-l};\, S(n-l)]\right)\right]$$

$$\geqslant -2A\left[\left(\frac{l_s}{a_{s+1}} - \frac{l_{s+1}}{a_{s+1}}\right) + \left(\frac{l_{s+1}}{a_{s+2}} - \frac{l_{s+2}}{a_{s+2}}\right) + \ldots + \left(\frac{l_{p-1}}{a_p} - \frac{l_p}{a_p}\right) + \right.$$

$$\left. + \frac{l_p}{a_{p+1}} - \frac{\beta}{2}\left(1 - \frac{n-l+1}{n-l-1} - \varepsilon_l\right)\right]$$

$$\geqslant -2A\left[-l_{s+1} - l_{s+2} - \ldots - l_p + \left(\frac{l_p}{a_{p+1}} - \frac{\beta}{n-l+1}\right) + \beta\varepsilon_{l+1}\right]$$

$$\geqslant -2A\beta\left[\frac{1}{(n+1)^3} - \frac{1}{n-l+1} + \varepsilon_{l+1}\right]$$

$$\geqslant -2A\beta\left[\frac{1}{(n+1)^3} - \frac{1}{n} + \varepsilon\right] > 0 \quad \text{for } 0 < \varepsilon < \frac{1}{10n(n+1)},$$

since $\varepsilon_{l+1} \leqslant \varepsilon/2^{l+1} < \varepsilon$.

Therefore, the function $F(z_l)$ increases in the interval $[a_s, a_{s+1})$ and, since it is continuous in the interval $[b, c_1)$, we conclude that it increases also in $[b, c_1)$. Hence

$$W[\bar{x}_p, \bar{z}_n;\, S_B^\varepsilon] \leqslant W[\bar{x}_p, (z_1, z_2, \ldots, z_{l-1}, c_1, z_{l+1}, \ldots, z_n);\, S_B^\varepsilon]$$

which completes the proof of Lemma 6.

LEMMA 7. $W[\bar{x}_p, \bar{z}_n;\, S_B^\varepsilon] \leqslant W[\bar{x}_p^*, \bar{z}_n^*;\, S_B^\varepsilon]$.

This lemma is an easy inference from Lemmas 2-6.

To complete the proof of the ε-optimality of the strategy S_B^ε of player B we must show that

$$W[\bar{x}_p^*, \bar{z}_n^*;\, S_B^\varepsilon] \leqslant W[\bar{x}_p^{**}, \bar{c}_n;\, S_B^\varepsilon] + \varepsilon, \qquad \text{where } \bar{c}_n = (c_1, c_2, \ldots, c_n).$$

Since condition (7) is valid, i.e. $W[\bar{x}_p^{**}, \bar{c}_n;\, S_B^\varepsilon] = v$, it will be sufficient for the ε-optimality of the strategy S_B^ε.

By relations (31) and (32) we have

$$\begin{aligned}
W[\bar{x}_p^*, \bar{z}_n^*; S_B^\varepsilon] &= \int_{a_1}^{c_1} W[\bar{x}_p^*, \bar{z}_n^*; y]g(y)dy + \beta\Big\{1 - \prod_{i=1}^{p}(1-x_i)\big[1 - W[\bar{z}_n^*; S(n)]\big]\Big\} \\
&\leqslant \int_{a_1}^{c_1} W[\bar{x}_p^*, \bar{z}_n^*; y]g(y)dy + \beta\Big\{1 - \prod_{i=1}^{p}(1-x_i)\Big(\frac{2}{n+1} - \varepsilon\Big)\Big\} \\
&\leqslant \int_{a_1}^{c_1} W[\bar{x}_p^*, \bar{c}_n; y]g(y)dy + \beta\Big\{1 - \prod_{i=1}^{p}(1-x_i)\prod_{i=1}^{n-1}(1-c_i)\Big\} + \varepsilon \\
&= W[\bar{x}_p^*, \bar{c}_n; S_B^\varepsilon] + \varepsilon,
\end{aligned}$$

since

$$\prod_{i=1}^{n-1}(1-c_i) = \frac{2}{n+1}.$$

The obtained result we formulate as

LEMMA 8. *$W[\bar{x}_p^*, \bar{z}_n^*; S_B^\varepsilon] \leqslant W[\bar{x}_p^*, \bar{c}_n; S_B^\varepsilon] + \varepsilon$ for $0 < \varepsilon < 1/10n(n+1)$.*

It follows from Lemma 8 that in order to complete the proof of inequality (27) it is sufficient to prove the condition

$$W[\bar{x}_p, \bar{c}_n; S_B^\varepsilon] \leqslant W[\bar{x}_p^{**}, \bar{c}_n; S_B^\varepsilon]. \tag{33}$$

The following lemma gives an accessory equation.

LEMMA 9. *If $x_{k-1} < a_k$ and $x_i \in [a_i, a_{i+1})$ $(i = k, k+1, \ldots, p)$, then*

$$\sum_{i=k}^{p}\Big\{\int_{a_i}^{x_i} y\prod_{j=1}^{i-1}(1-x_j)g(y)dy + \int_{x_i}^{a_{i+1}} y\prod_{j=1}^{i}(1-x_j)g(y)dy\Big\} + \\ + \beta c_1\prod_{i=1}^{p}(1-x_i) = \prod_{i=1}^{k-1}(1-x_i)\Big(\frac{l_k}{a_k} - l_k\Big). \tag{34}$$

Proof. Using (21) and (22) we prove this by induction with respect to the number k, the first induction step beginning from $k = p$.

1. For $k = p$ we have

$$\int_{a_p}^{x_p} y\prod_{j=1}^{p-1}(1-x_j)g(y)dy + \int_{x_p}^{a_{p+1}} y\prod_{j=1}^{p}(1-x_j)g(y)dy + \beta c_1\prod_{j=1}^{p}(1-x_j) \\ = \prod_{j=1}^{p-1}(1-x_j)\Big\{\int_{a_p}^{x_p} yg(y)dy + \int_{x_p}^{a_{p+1}} y(1-x_p)g(y)dy + \beta c_1(1-x_p)\Big\}$$

$$= \prod_{j=1}^{p-1}(1-x_j)\left\{\left(\frac{l_p}{a_p}-l_p-\frac{l_p}{a_{p+1}}+\beta c_1\right)+x_p\left(\frac{l_p}{a_{p+1}}-\beta c_1\right)\right\}$$

$$= \prod_{j=1}^{p-1}(1-x_j)\left(\frac{l_p}{a_p}-l_p\right).$$

2. For $k = r > 1$, we assume that the following equation is valid:

$$\sum_{i=r}^{p}\left\{\int_{a_i}^{x_i} y\prod_{j=1}^{i-1}(1-x_j)g(y)\,dy+\int_{x_i}^{a_{i+1}} y\prod_{j=1}^{i}(1-x_j)g(y)\,dy\right\}+\beta c_1\prod_{j=1}^{p}(1-x_j)$$
$$= \prod_{i=1}^{r-1}(1-x_i)\left(\frac{l_r}{a_r}-l_r\right).$$

3. For $k = r-1$, we obtain

$$\sum_{i=r-1}^{p}\left\{\int_{a_i}^{x_i} y\prod_{j=1}^{i-1}(1-x_j)g(y)\,dy+\int_{x_i}^{a_{i+1}} y\prod_{j=1}^{i}(1-x_j)g(y)\,dy\right\}+\beta c_1\prod_{i=1}^{p}(1-x_i)$$
$$= \left\{\int_{a_{r-1}}^{x_{r-1}} y\prod_{j=1}^{r-2}(1-x_j)g(y)\,dy+\int_{x_{r-1}}^{a_r} y\prod_{j=1}^{r-1}(1-x_j)g(y)\,dy\right\}+$$
$$+\prod_{i=1}^{r-1}(1-x_j)\left(\frac{l_r}{a_r}-l_r\right)$$
$$= \prod_{j=1}^{r-2}(1-x_j)\left\{\frac{l_{r-1}}{a_{r-1}}-\frac{l_{r-1}}{a_r}+\frac{x_{r-1}l_{r-1}}{a_r}-l_{r-1}+\frac{l_r}{a_r}-l_r-x_{r-1}\left(\frac{l_r}{a_r}-l_r\right)\right\}$$
$$= \prod_{j=1}^{r-2}(1-x_j)\left\{\frac{l_{r-1}}{a_{r-1}}-\frac{l_{r-1}}{a_r}-l_{r-1}+\frac{l_r}{a_r}-l_r+x_{r-1}\left(\frac{l_{r-1}}{a_r}-\frac{l_r}{a_r}+l_r\right)\right\}$$
$$= \prod_{j=1}^{r-2}(1-x_j)\left(\frac{l_{r-1}}{a_{r-1}}-l_{r-1}\right).$$

This completes the proof.

Let $\bar{x}_p^{*r}$ be a vector $\bar{x}_p$ with components $x_i\in[a_i, a_{i+1}]$ for $i = r+1, r+2, \dots, p$ $(r = 0, 1, \dots, p)$. Consider any vector $\bar{x}_p^{*u}$ for $0 \leqslant u \leqslant p$. Inequality (33) can be replaced by

(35) $$W[\bar{x}_p^{*u}, \bar{c}_n; S_B^\varepsilon] \leqslant W[\bar{x}_p^{**}, \bar{c}_n; S_B^\varepsilon] \quad \text{for } 0 \leqslant u \leqslant p.$$

For $u = 0$ inequality (35) is true. Assume that $0 < u \leqslant p$. From this assumption it follows that

$$x_{u+1} \in [a_{u+1}, a_{u+2}],\ x_{u+2} \in [a_{u+2}, a_{u+3}],\ \ldots,\ x_p \in [a_p, a_{p+1}] \quad \text{if } u < p.$$

We have to consider the following cases:

(a) $x_u < a_u$ if $u \geqslant 2$,

(b) $x_u > a_{u+1}$ if $u \leqslant p-1$,

(c) $x_u \in [a_u, a_{u+1}]$.

It is clear that in case (c) there exists a vector $\bar{x}_p^{*u-1} = \bar{x}_p^{*u}$ such that

$$W[\bar{x}_p^{*u-1}, \bar{c}_n; S_B^\varepsilon] \geqslant W[\bar{x}_p^{*u}, \bar{c}_n; S_B^\varepsilon]. \tag{36}$$

We show now that also in cases (a) and (b) there exists a vector $\bar{x}_p^{*u-1}$ for which (36) is valid.

For case (b) we assume that $x_s > a_{u+1}$ and $x_{s-1} \leqslant a_{u+1}$ if $s > 1$, and specify two subcases: $u < p-1$ and $u = p-1$.

1. For $u < p-1$ we use (34) and obtain

$$W[\bar{x}_p^{*u}, \bar{c}_n; S_B^\varepsilon] = \int\limits_{a_1}^{a_{u+1}} K(\bar{x}_p^{*u}, \bar{c}_n; y)g(y)\,dy + \int\limits_{a_{u+1}}^{x_s} \Big[1-2y\prod_{i=1}^{s-1}(1-x_i)\Big]g(y)\,dy +$$

$$+\sum_{i=s}^{u}\int\limits_{x_i}^{x_{i+1}}\Big[1-2y\prod_{j=1}^{i}(1-x_j)\Big]g(y)\,dy + \int\limits_{x_{u+1}}^{a_{u+2}}\Big[1-2y\prod_{j=1}^{u+1}(1-x_j)\Big]g(y)\,dy +$$

$$+\sum_{i=u+2}^{p}\bigg(\int\limits_{a_i}^{x_i}\Big[1-2y\prod_{j=1}^{i-1}(1-x_j)\Big]g(y)\,dy + \int\limits_{x_i}^{a_{i+1}}\Big[1-2y\prod_{j=1}^{i}(1-x_j)\Big]g(y)\,dy\bigg) +$$

$$+\beta\Big[1-2c_1\prod_{i=1}^{p}(1-x_i)\Big] = \int\limits_{a_1}^{a_{u+1}} W[\bar{x}_p^{*u}, \bar{c}_n; y]g(y)\,dy + M -$$

$$-2\prod_{i=1}^{s-1}(1-x_i)\bigg\{\int\limits_{a_{u+1}}^{x_s} yg(y)\,dy + \sum_{i=s}^{u}\int\limits_{x_i}^{x_{i+1}} y\prod_{j=s}^{i}(1-x_j)g(y)\,dy +$$

$$+\int\limits_{x_{u+1}}^{a_{u+2}} y\prod_{j=s}^{u+1}(1-x_j)g(y)\,dy + \prod_{i=s}^{u+1}(1-x_i)\left(\frac{l_{u+2}}{a_{u+2}} - l_{u+2}\right)\bigg\} = \varphi(x_s),$$

$$\text{where } M = \int\limits_{a_{u+1}}^{a_{p+1}} g(y)\,dy + \beta,$$

$$\varphi'(x_s) = -2\prod_{i=1}^{s-1}(1-x_i)\bigg\{\frac{l_{u+1}}{x_{s+1}} - \sum_{i=s+1}^{u}\int\limits_{x_i}^{x_{i+1}}\prod_{j=s+1}^{i}(1-x_j)\,\frac{l_{u+1}}{y^2}\,dy -$$

$$- \int_{x_{u+1}}^{a_{u+2}} \prod_{j=s+1}^{u+1} (1-x_j) \frac{l_{u+1}}{y^2} dy - \prod_{i=s+1}^{u+1} (1-x_i) \left(\frac{l_{u+2}}{a_{u+2}} - l_{u+2} \right) \Bigg\}$$

$$< -2 \prod_{i=1}^{s-1} (1-x_i) \left\{ \frac{l_{u+1}}{x_{s+1}} - \int_{x_{s+1}}^{a_{u+2}} \frac{l_{u+1}}{y^2} dy - \frac{l_{u+2}}{a_{u+2}} + l_{u+2} \right\} = 0 .$$

2. For $u = p-1$ we have

$$\varphi(x_s) = W[\bar{x}_p^{*u}, \bar{c}_n; S_B^\varepsilon] = \int_{a_1}^{a_p} W[\bar{x}_p^{*u}, \bar{c}_n; y] g(y) dy +$$

$$+ \int_{a_p}^{x_s} \Big[1 - 2y \prod_{i=1}^{s-1} (1-x_i)\Big] g(y) dy + \sum_{i=s}^{p-1} \int_{x_i}^{x_{i+1}} \Big[1 - 2y \prod_{j=1}^{i} (1-x_j)\Big] g(y) dy +$$

$$+ \int_{x_p}^{a_{p+1}} \Big[1 - 2y \prod_{j=1}^{u+1} (1-x_j)\Big] g(y) dy + \beta \Big[1 - 2c_1 \prod_{i=1}^{p} (1-x_i)\Big]$$

$$= \int_{a_1}^{a_p} W[\bar{x}_p^{*u}, \bar{c}_n; y] g(y) dy + M - 2 \prod_{i=1}^{s-1} (1-x_i) \Big\{ \int_{a_p}^{x_s} y g(y) dy +$$

$$+ \sum_{i=s}^{p-1} \int_{x_i}^{x_{i+1}} y \prod_{j=s}^{i} (1-x_j) g(y) dy + \int_{x_p}^{a_{p+1}} y \prod_{j=s}^{u+1} (1-x_j) g(y) dy + \beta c_1 \prod_{i=s}^{p} (1-x_i) \Big\},$$

where $M = \int_{a_p}^{a_{p+1}} g(y) dy + \beta$,

$$\varphi'(x_s) = -2 \prod_{i=1}^{s-1} (1-x_i) \left\{ \frac{l_p}{x_{s+1}} - \sum_{i=s+1}^{p-1} \int_{x_i}^{x_{i+1}} \prod_{j=s+1}^{i} (1-x_j) \frac{l_p}{y^2} dy - \right.$$

$$\left. - \int_{x_p}^{a_{p+1}} \prod_{j=s+1}^{p} (1-x_j) \frac{l_p}{y^2} dy - \beta c_1 \prod_{i=s+1}^{p} (1-x_i) \right\}$$

$$< -2 \prod_{i=1}^{s-1} (1-x_i) \left\{ \frac{l_p}{x_{s+1}} - \int_{x_{s+1}}^{a_{p+1}} \frac{l_p}{y^2} dy - \beta c_1 \right\}$$

$$= -2 \prod_{i=1}^{s-1} (1-x_i) \left\{ \frac{l_p}{a_{p+1}} - \beta c_1 \right\} = 0 .$$

Thus $\varphi(x_s)$ is a decreasing function of x_s. The expression $W[\bar{x}_p^{*u}, \bar{c}_n; S_B^\varepsilon]$ increases for $x_s = a_{u+1}$. We have the same fact for $x_s = a_{u+1}, x_{s+1} = a_{u+1}, \ldots, x_u = a_{u+1}$. Therefore, in case (b) we can find a vector $\bar{x}_p^{*u-1}$ for which (36) is valid.

Let us consider now case (a) in two parts. Assume that $a_s \leqslant x_u < a_{s+1}$ and $b = \max(x_{u-1}, a_s)$.

1. In the case $x_u < a_u$ for $2 \leqslant u < p$ we have

$$W[\bar{x}_p^{*u}, \bar{c}_n; S_B^\varepsilon] = \int_{a_1}^{b} W[\bar{x}_p^{*u}, \bar{c}_n; y]g(y)dy +$$
$$+\int_{b}^{x_u}\Big[1-2y\prod_{i=1}^{u-1}(1-x_i)\Big]\frac{l_s}{y^3}dy + \int_{x_u}^{a_{s+1}}\Big[1-2y\prod_{i=1}^{u}(1-x_i)\Big]\frac{l_s}{y^3}dy +$$
$$+\sum_{i=u+1}^{p}\left(\int_{a_i}^{x_i}\Big[1-2y\prod_{j=1}^{i-1}(1-x_j)\Big]\frac{l_i}{y^3}dy\right) + \int_{x_i}^{a_{i+1}}\Big[1-2y\prod_{j=1}^{i}(1-x_j)\Big]\frac{l_i}{y^3}dy +$$
$$+\beta\Big[1-2c_1\prod_{i=1}^{p}(1-x_i)\Big] + \sum_{i=s+1}^{u}\int_{a_i}^{a_{i+1}}\Big[1-2y\prod_{j=1}^{u}(1-x_j)\Big]\frac{l_i}{y^3}dy$$
$$= \int_{a_1}^{b} W[\bar{x}_p^{*u}, \bar{c}_n; y]g(y)dy + \int_{b}^{a_{p+1}} g(y)dy + \beta - 2\prod_{i=1}^{u-1}(1-x_i)\left\{\frac{l_s}{b} - \frac{l_s}{a_{s+1}} -\right.$$
$$-l_s + \frac{l_s x_u}{a_{s+1}} + (1-x_u)\left(\frac{l_{u+1}}{a_{u+1}} - l_{u+1}\right) + (1-x_u)\left(\frac{l_{s+1}}{a_{s+1}} - \frac{l_{s+1}}{a_{s+2}} + \frac{l_{s+2}}{a_{s+2}} -\right.$$
$$\left.\left.-\frac{l_{s+2}}{a_{s+3}} + \ldots + \frac{l_u}{a_u} - \frac{l_u}{a_{u+1}}\right)\right\} = \psi(x_u),$$

$$\psi'(x_u) = -2\prod_{i=1}^{u-1}(1-x_i)\left\{\frac{l_s}{a_{s+1}} - \frac{l_{u+1}}{a_{u+1}} + l_{u+1} - \frac{l_{s+1}}{a_{s+1}} + \frac{l_{s+1}}{a_{s+2}} -\right.$$
$$\left.-\frac{l_{s+2}}{a_{s+1}} + \frac{l_{s+2}}{a_{s+3}} - \ldots - \frac{l_u}{a_u} + \frac{l_u}{a_{u+1}}\right\}$$
$$= -2\prod_{i=1}^{u-1}(1-x_i)\left\{\left(\frac{l_s}{a_{s+1}} - \frac{l_{s+1}}{a_{s+1}}\right) +\right.$$
$$\left.+\left(\frac{l_{s+1}}{a_{s+2}} - \frac{l_{s+2}}{a_{s+2}}\right) + \ldots + \left(\frac{l_{u-1}}{a_u} - \frac{l_u}{a_u}\right) + \left(\frac{l_u}{a_{u+1}} - \frac{l_{u+1}}{a_{u+1}}\right) + l_{u+1}\right\}$$
$$= -2\prod_{i=1}^{u-1}(1-x_i)\{-l_{s+1} - l_{s+2} - \ldots - l_{u+1} + l_{u-1}\} > 0.$$

2. The case $x_u < a_u$ for $u = p$ can be proved in the same way as the condition $\psi'(x_u) > 0$ in case 1. Thus $\psi(x_u)$ is an increasing function of x_u in the interval $[b, a_{s+1})$. Therefore, also in case (a) there exists a vector $\bar{x}_p^{*(u-1)}$, arisen from $\bar{x}_p$ by setting $x_u = a_u$, such that (36) is valid.

Hence we have shown that for every vector $\bar{x}_p^{*u}$ $(u \geqslant 1)$ there exists $\bar{x}_p^{*(u-1)}$ such that inequality (36) is valid. Then it follows that

$$W[\bar{x}_p^{*u}, \bar{c}_n; S_B^\varepsilon] \leqslant W[\bar{x}_p^{**}, \bar{c}_n; S_B^\varepsilon] \quad \text{for } 0 \leqslant u \leqslant p$$

which completes the proof of (33). Thus S_B^ε is an ε-optimal strategy for player B.

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Received on 14. 9. 1974

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GRA CZASOWA MIESZANA TYPU $(p+n)\times 1$

STRESZCZENIE

W pracy rozpatrzono jeden z przykładów gier czasowych. Dwóch uczestników gry A i B toczy pojedynek. Gracz A ma p kul cichych i n głośnych $(p \geqslant 1,\ n \geqslant 1)$, a gracz B ma jedną kulę głośną. A strzela kule ciche przed głośnymi. Obu przeciwnikom przypisano jednakowe funkcje celności $P(t) = t$, $t \in [0, 1]$. Oznacza to, że w danej chwili t prawdopodobieństwo trafienia jednego gracza przez drugiego jest jednakowe z obu stron. Założenia te znane są przeciwnikom. Przyjmujemy następującą funkcję wypłaty dla gry:

$W[S_1; S_2] =$ Pr $\{A$ sam przeżyje cały pojedynek$\}$ $-$ Pr $\{B$ sam przeżyje cały pojedynek$\}$.

Zadaniem A jest maksymalizowanie danej funkcji wypłaty, zadaniem B zaś minimalizowanie jej.

Wprowadzamy randomizację, pokazujemy, że gra ma wartość i udowadniamy optymalność strategii gracza A i ε-optymalność strategii gracza B.