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A NEW FORMULATION AND SOLUTION OF THE SEQUENCING PROBLEM: MATHEMATICAL MODEL

Many sequencing problems have been formulated and solved lately by using disjunctive graphs. In this paper a more general model of sequencing problems leading to a new construction of the disjunctive graph is presented. The algorithm of solving this problem is based on the branch-and-bound method. The way of review of the graph-tree of solution is based on new properties of disjunctive graphs. These properties allow us to construct a relatively efficient algorithm for solving the problems of greater size. An example from the literature is solved and the results are compared.

This paper consists of two parts: in the first part the mathematical model of the problem is presented, and in the second one an implicit algorithm is given.

1. Mathematical formulation of the sequencing problem. We have n operations which should be carried out on q machines. Some of the operations should be carried out in a certain technological order. To solve the sequencing problem we have to find such a sequence of the operations that the total time of all operations is minimal and the following assumptions are satisfied:

(a) Every operation should be carried out on a particular machine and every operation can be carried out on not more than one machine; every machine can perform only one operation simultaneously.

(b) The operations should be carried out according to some required technological order.

(c) The sequence of operations on every machine is arbitrary.

(d) Every operation cannot be interrupted.

(e) The set-up times are equal to zero.

Now we can give the mathematical formulation of the sequencing problem. Let $N = \{1, \ldots, n\}$ be the set of operations (numbers of operations) which should be carried out by using the set of machines $Q = \{1, \ldots, q\}$.

Further, let $RT \subset N \times N$ be the set of relations expressing the technological requirements of the operation order. Let $N_x \subset N$ be the set of operations which have no predecessors,

$$N_x = \{j \in N \mid \forall i \in N \wedge \langle i, j\rangle \notin RT\},$$

and let $N_y \subset N$ be the set of operations which have no successors,

$$N_y = \{j \in N \mid \forall i \in N \wedge \langle j, i\rangle \notin RT\}.$$

Let $N^k \subset N$ be the set of operations which should be carried out on the machine k, and let the following relations be satisfied:

$$\bigcup_{k \in Q} N^k = N, \qquad N^k \cap N^l = \emptyset \qquad (k, l \in Q, k \neq l).$$

Assume that t_j^x denotes the starting time of the operation j, t_j^y — the finishing time of the operation j, c_j — the duration of the operation j ($c_j > 0$ for all $j \in N$), t_0 — the starting moment, and t_z — the finishing moment of all operations.

We may write the sequencing problem in the following form: find t_0, t_z, t_j^x, t_j^y, $j \in N$, satisfying

$$t_z = \min, \tag{1}$$

$$t_j^y - t_j^x \geqslant c_j \quad (j \in N), \tag{2}$$

$$t_j^x - t_j^y \geqslant 0 \quad (\langle i, j\rangle \in RT), \tag{3}$$

$$t_j^x - t_0 \geqslant 0 \quad (j \in N_x), \tag{4}$$

$$t_z - t_i^y \geqslant 0 \quad (i \in N_y), \tag{5}$$

$$t_0,\ t_z,\ t_j^x,\ t_j^y \geqslant 0 \quad (j \in N), \tag{6}$$

$$(t_j^x - t_i^y \geqslant 0) \vee (t_i^x - t_j^y \geqslant 0) \quad (i, j \in N^k,\ i \neq j,\ k \in Q). \tag{7}$$

Conditions (1)-(7) constitute the problem which will be called *Problem P*. Constraints (2) require that the difference between that starting time of the operations and the finishing time cannot be less than the duration of the operations. Conditions (3) give the required technological order of operations, and (4) and (5) state the condition that t_0 is the starting moment of all operations and t_z is the finishing moment of all operations. Constraints (7) are disjunctions, each of which requires that two operations cannot be carried out on the same machine at the same time.

Problem P without constraints (7) is known as the linear programming problem of finding the critical path of the graph. An example of a graph concerning this problem is shown in Fig. 1.

Nodes x_j and y_j represent the starting and finishing moments of the operation $j \in N$ and the variables t_j^x, t_j^y. Let X and Y be the sets of all nodes x_j and y_j, respectively. The node 0 represents the start of all operations, and the node z the end of all operations. The variables t_0 and t_z are connected with these nodes. The set of all nodes equals

$$A = X \cup Y \cup \{0\} \cup \{z\}. \tag{8}$$

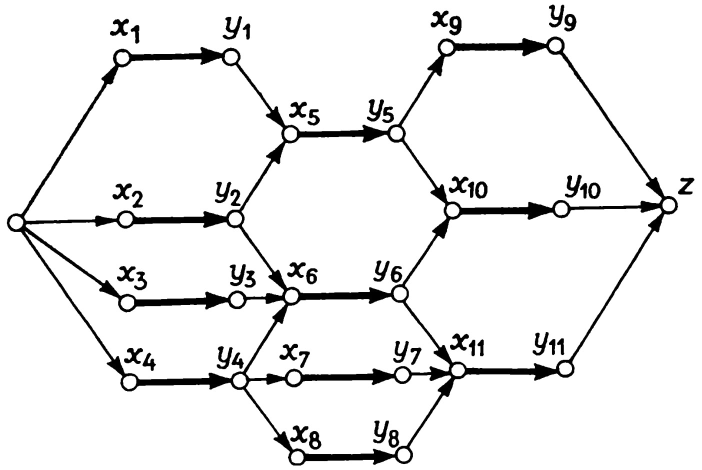

Fig. 1

Each arc $\langle x_j, y_j\rangle$ for $x_j \in X$, $y_j \in Y$, $j \in N$, defines the natural order of starting and finishing the operation j, expressed by constraints (2). The set of these arcs is denoted by U_1. To each arc let us correspond a real number c_j, called the *arc length*, representing the duration of the operation j.

Arcs of the form $\langle y_i, x_j\rangle$ for $y_i \in Y$, $x_j \in X$, $\langle i, j\rangle \in RT$, with zero length represent the technological order of relations and constraints $t_j^x - t_i^y \geqslant 0$. The set of these arcs is denoted by U_2.

Arcs $\langle 0, x_j\rangle$ for $x_j \in X$, $j \in N_x$, with zero length represent the relations between the starting moment of all operations and that of every operation from the set N_x expressed by the constraint $t_j^x - t_0 \geqslant 0$. The set of these arcs is denoted by U_3.

Arcs of the form $\langle y_i, z\rangle$ for $y_i \in Y$, $i \in N_y$, with zero length represent the relations between finishing moment of every operation from the set N_y and the finishing moment of all operations expressed by the constraint $t_z - t_i^y \geqslant 0$. The set of these arcs is denoted by U_4.

The set of all arcs equals

$$U = U_1 \cup U_2 \cup U_3 \cup U_4, \tag{9}$$

so the graph is of the form

$$D = \langle A, U\rangle. \tag{10}$$

To make our considerations easier we introduce the following notation. Let $C(x, y)$ denote the set of arcs of the longest path (if it exists) between nodes $x, y \in A$. The length of this path is denoted by $L(x, y)$.

Sometimes the set of arcs of the longest path of the graph will be denoted by C. Let $d(x, y)$ be the arc set of some paths (not necessarily the longest) between nodes $x, y \in A$. The length of this path is denoted by $l^d(x, y)$.

The disjunctive constraints (7) can be expressed as disjunctive pairs of arcs. Each constraint

$$(t_j^x - t_i^y \geqslant 0) \vee (t_i^x - t_j^y \geqslant 0)$$

will be associated with a disjunctive pair of arcs $\langle y_i, x_j\rangle$, $\langle y_j, x_i\rangle$. Such a pair of disjunctive arcs is shown in Fig. 2.

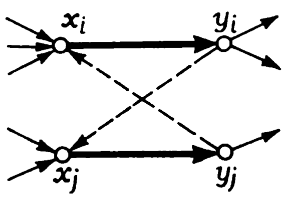

Fig. 2

We supplement the graph D of problem (1)-(6) by all pairs of disjunctive arcs $\langle y_i, x_j\rangle$, $\langle y_j, x_i\rangle$ such that $i, j \in N^k$, $i \neq j$, $k \in Q$. In this way we form the disjunctive graph $\bar{D}$. The set of all disjunctive arcs is denoted by V.

Now, we can represent Problem P by the disjunctive graph

$$\bar{D} = \langle A, U; V\rangle. \tag{11}$$

This definition of a disjunctive graph is more general than that given in [1]-[4].

The described disjunctive graph $\bar{D} = \langle A, U; V\rangle$ enables us to consider the cases where it is necessary, from the technological consideration point of view, to assemble several elements into one unit (Fig. 3a) and to split one part into its elements (Fig. 3b). In general, the graph $D = \langle A, U\rangle$ (in which every operation is represented by a pair of nodes with the linking them arc) can represent any structure of carried-out operations.

Besides, the disjunctive graph (11) allows us to consider problems in which instead of the set of operations N we have the set $\bar{N}$ of production tasks. Each production task $j \in \bar{N}$ consists of a set of operations which are to be carried out according to the required technological order. To solve the above problem we find such a sequence of the production tasks that the total time of all production tasks is minimal. Every production task $j \in \bar{N}$ can be represented in form of the graph $D^j = \langle A^j, U^j\rangle$, analogously as in the case of the graph $D = \langle A, U\rangle$. In the graph D^j, the nodes representing the starting and the finishing moments of the whole production task are denoted by 0^j and z^j, respectively. The disjunctive graph for the problem is similar to the graph $\bar{D}$. The nodes x_j and y_j in $\bar{D}$ have their counterparts in 0^j and z^j, and arcs $\langle x_j, y_j \rangle$ in $\bar{D}$ have their counterparts in the graph $D^j = \langle A^j, U^j \rangle$. The lengths of arcs $\langle x_j, y_j \rangle$ in $\bar{D}$ have their counterparts in the lengths of the critical paths of D_j. The disjunctive pairs of arcs $[\langle y_j, x_i \rangle, \langle y_i, x_j \rangle]$ in $\bar{D}$ have their counterparts in the pairs $[\langle z^j, 0^i \rangle, \langle z^i, 0^j \rangle]$ (Fig. 3c).

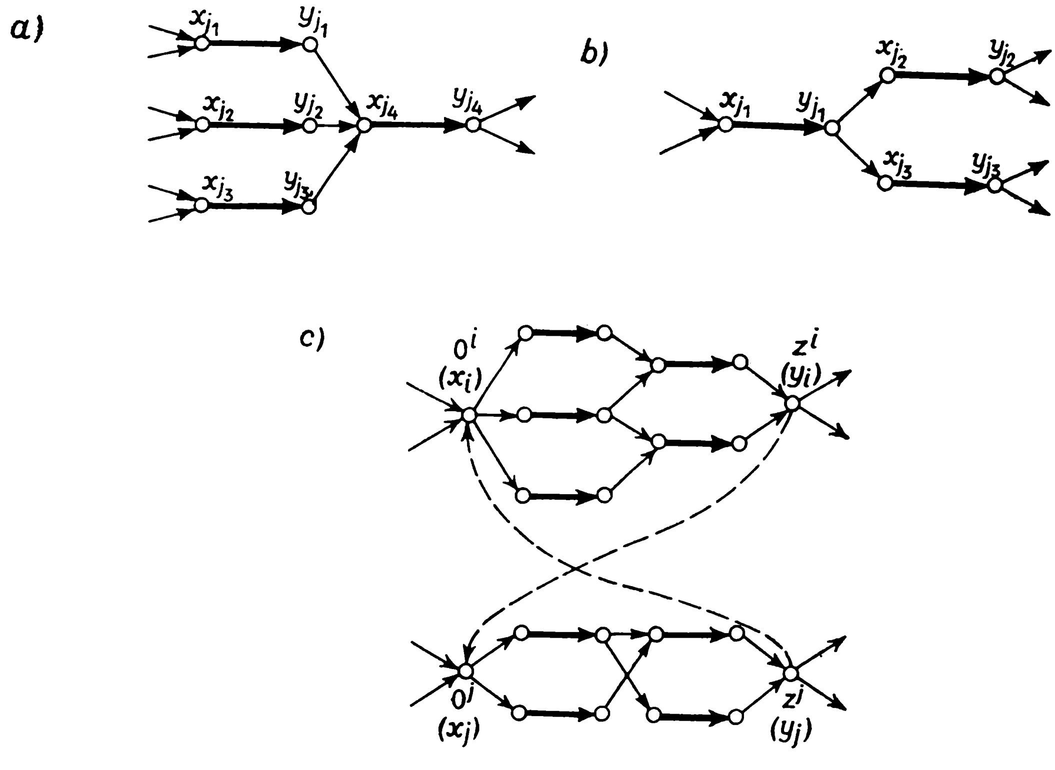

Fig. 3

Therefore, all considerations in this paper referring to the graph $\bar{D}$ include also the above problem after adjustment to new notation.

The disjunctive graph described in [1]-[4] does not enable to solve all such problems. It makes only possible to take into account those cases where it is necessary to assemble several elements into one unit (Fig. 3a).

An example of the disjunctive graph for

$$N^1 = \{1, 4, 8\}, \quad N^2 = \{2, 5, 7\}, \quad N^3 = \{3, 8\},$$

$$V^1 = \{\langle y_1, x_4 \rangle, \langle y_4, x_1 \rangle, \langle y_1, x_8 \rangle, \langle y_8, x_1 \rangle, \langle y_4, x_8 \rangle, \langle y_8, x_4 \rangle\},$$

$$V^2 = \{\langle y_2, x_5 \rangle, \langle y_5, x_2 \rangle, \langle y_2, x_7 \rangle, \langle y_7, x_2 \rangle, \langle y_5, x_7 \rangle, \langle y_7, x_5 \rangle\},$$

$$V^3 = \{\langle y_3, x_8 \rangle, \langle y_8, x_3 \rangle\}$$

is shown in Fig. 4.

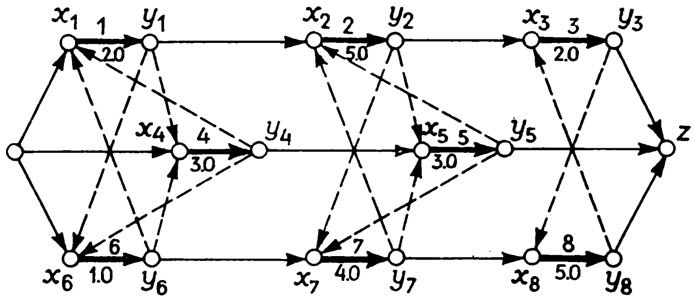

Fig. 4

2. Properties of the mathematical model. Let be given the disjunctive graph $\bar{D} = \langle A, U; V\rangle$ of Problem P. This graph has the following properties.

PROPERTY 1. The graph $D = \langle A, U\rangle$ has the source node 0 and the sink node z and for each $x \in A - [\{0\} \cup \{z\}]$ there exists a path from the node 0 to the node x and from x to z. The graph D has no circuits.

PROPERTY 2. (a) There exist in D two subsets of nodes $X \subset A$ and $Y \subset A$ such that

$$\begin{gathered} \Gamma(X) = Y, \quad \Gamma^{-1}(Y) = X, \\ \underset{x\in X}{\forall}\ \underset{y \in Y}{\exists}\,!(y = \Gamma x) \wedge \underset{y\in Y}{\forall}\ \underset{x\in X}{\exists}\,!(x = \Gamma^{-1} y), \end{gathered} \tag{12}$$

and each arc $\langle x, y\rangle$ $(x \in X, y \in Y)$ has the length $c(x, y) > 0$.

Condition (12) states that each node of the set X has exactly one successor in D, this successor belonging to Y, and that each node of the set Y has exactly one predecessor in D, this predecessor belonging to X.

(b) There exists a set of indices Q generating partitions of X and Y such that

$$\begin{gathered} \bigcup_{k\in Q} X^k = X, \quad \bigcup_{k \in Q} Y^k = Y, \\ X^k \cap X^l = \emptyset,\ Y^k \cap Y^l = \emptyset\ (k, l \in Q, k \neq l), \quad \Gamma(X^k) = Y^k\ (k \in Q). \end{gathered} \tag{13}$$

It follows from condition (12) that $\Gamma^{-1}(Y^k) = X^k$ for $k \in Q$.

PROPERTY 3. The set V of disjunctive arcs with zero length is

$$V = \bigcup_{k\in Q} V^k, \tag{14}$$

$$V = \{\langle y, x\rangle \in Y^k \times X^k \mid [\mathrm{War}(y, x) = 0 \vee \mathrm{War}(x, y) = 0]\}, \tag{15}$$

where $\mathrm{War}(x, y)$ is the statement function with arguments $x, y \in A$, defined by

(16) $\mathrm{War}(x, y) =$ (there exists a path from the node x to y in the graph D).

Now, similarly as in [4], we introduce some notions and prove a number of properties of the disjunctive graph.

It follows from Property 3 and from condition (12) that for each arc $\langle y, x\rangle \in V$ there exists exactly one arc $\langle u, v\rangle \in V$ such that the following condition is satisfied:

$$(u = \Gamma x) \wedge (v = \Gamma^{-1} y). \tag{17}$$

Two disjunctive arcs $[\langle y, x\rangle, \langle u, v\rangle]$ which satisfy condition (17) are called a *disjunctive pair*. Any arc from a disjunctive pair is called the *complement* of the other arc of this pair. Replacement of an arc by its complement is called *complementing*.

The subset of the set V containing at least one arc from each disjunctive pair is called a *selection*. A representation containing exactly one arc from each pair is called a *complete selection*.

Let $R_s = \{S_1, \ldots, S_p\}$ be the family of all selections (not necessarily complete), and let

$$R_s' = \{S_r \in R_s \mid S_r \text{ is a complete selection}\} \tag{18}$$

be the family of complete selections. In this paper we consider only complete selections. Each selection $S_r \in R_s'$ generates a conjunctive graph

$$D_r = \langle A, U \cup S_r\rangle = \langle A, U_r\rangle. \tag{19}$$

The longest path from the node 0 to the node z (if the graph D_r has no circuits) is called the *critical path* in D_r.

Let

$$R_D = \{D_1, \ldots, D_u\} \tag{20}$$

be the family of graphs of form (19) and let

$$R_D' = \{D_r \in R_D \mid D_r \text{ has no circuits}\} \tag{21}$$

be the family of graphs without circuits.

The critical path of the graph $D_0 \in R_D'$ is called the *minimaximal path* in $\bar{D} = \langle A, U; V\rangle$, and the associated selection S_0 is called the *optimal selection* if

$$L_0 = \min_{D_r \in R_D'} L_r, \tag{22}$$

where L_r is the length of the critical path in D_r.

Similarly as in [2], we can prove that Problem P is equivalent to finding the set of arcs $S_0 \in R_s'$ of the disjunctive graph $\bar{D} = \langle A, U; V\rangle$. The set S_0 is such that the graph $D_0 = \langle A, U \cup S_0\rangle$ has no circuits, and the critical path C_0 in D_0 is minimaximal in $\bar{D}$. The length L_0 of the path C_0 is the optimal value of t_z.

It follows from the above that the algorithm of finding the minimaximal path L_0 in $\bar{D}$ is equivalent to solving Problem P. For this purpose we prove some theorems.

LEMMA 1. *Let C_r be the set of arcs of the critical path in $D_r \in R'_D$. If the arc $\langle y, x\rangle \in S_r \cap C_r$, then $\langle y, x\rangle$ is the only path between the nodes y and x.*

Proof. Denote by $d'(y, x)$ any path from the node y to x which has no arc $\langle y, x\rangle$. We have to prove that $d'(y, x)$ does not exist in D_r for $\langle y, x\rangle \in S_r \cap C_r$. We infer from Property 3 that for each $\langle u, v\rangle \in S_r$ a path $d'(u, v)$ does not exist in D; so if $d'(y, x)$ exists, it will exist in D_r. That means that there exists at least one arc $\langle y_i, y_j\rangle$ such that

$$\langle y_i, y_j\rangle \in S_r \cap d'(y, x).$$

There is exactly one of two possibilities: (a) the path $d'(y, x)$ starts with the arc $\langle y_i, y_j\rangle \in S_r$, or (b) the path $d'(y, x)$ starts with the arc $\langle y, y_l\rangle \notin S_r$.

(a) If the path $d'(y, x)$ starts with the arc $\langle y_i, y_j\rangle \in S_r$, there exists a successive arc $\langle y_j, y_e\rangle$, belonging to the path $d'(y, x)$, such that $\langle y_j, y_e\rangle \notin S_r$ (we know from Property 3 that two successive arcs belonging to S_r cannot exist). It follows from Property 3 that $y_j \in X$. In turn, from Property 2 it follows that $y_e \in Y$ and the length of the arc $c(y_j, y_e) > 0$. Thus the length of the path $d'(y, x)$ is greater than zero and is greater than the length of the arc $\langle y, x\rangle$ (as the length of each arc $\langle y, x\rangle \in S_r$ is equal to zero — which follows from Property 3), which contradicts that the arc $\langle y, x\rangle$ is the longest path from the node y to x.

(b) If the path $d'(y, x)$ starts with the arc $\langle y, y_l\rangle \notin S_r$, there exists a successive arc $\langle y_l, y_p\rangle$ belonging to the path $d'(y, x)$. It follows from Property 2 that $y_l \in X$, $y_p \in Y$ and the length of the arc $c(y_l, y_p) > 0$, which leads us to a contradiction like in case (a).

THEOREM 1. *Let C_r be the set of arcs of the critical path in $D_r \in R_D$. Any graph D_s obtained by complementing any arc $\langle y, x\rangle \in S_r \cap C_r$ has no circuits.*

Proof. Let the arc $\langle u, v\rangle$ be the complement of the arc $\langle y, x\rangle \in S_r \cap C_r$. Let $d(v, u)$ be any arbitrary path from the node v to the node u in D_r. It is easily seen that it suffices to prove that the path $d(v, u)$ does not exist in D_s. We infer from condition (17) that in D_r the node v is followed by the only node y and before the node u the only node x exists. According to Lemma 1 the only path from the node y to x in D_r is the arc $\langle y, x\rangle$. It follows from this that

$$d(v, u) = \langle\langle v, y\rangle, \langle y, x\rangle, \langle x, u\rangle\rangle$$

is the only path from the node v to u in D_r. As complementing the arc

$\langle y, x\rangle$ in D_r is made by rejecting this arc, and so winding up the only path from the node v to u, so in D_s a path from the node v to u does not exist.

For any $D_r \in R'_D$ and any node $x \in A$ we give a number of denotations.

The longest path from the node 0 to any $x \in A-\{0\}$ is

$$(23) \qquad L_r(0, x) = \max_{y \in \Gamma^{-1}x} [L_r(0, y)+c(y, x)].$$

The longest path from any node $x \in A-\{z\}$ to the node z is

$$(24) \qquad L_r(x, z) = \max_{y \in \Gamma x} [L_r(y, z)+c(x, y)].$$

Let x_i and x_j be nodes for which the right-hand sides of (23) and (24) take their maximal values. For $\Gamma^{-1}x-\{x_i\} \neq \emptyset$, let

$$(25) \qquad L'_r(0, x) = \max_{y \in \Gamma^{-1}x-\{x_i\}} [L_r(0, y)+c(y, x)],$$

and for $\Gamma x-\{x_j\} \neq \emptyset$, let

$$(26) \qquad L'_r(x, z) = \max_{y \in \Gamma x-\{x_j\}} [L_r(y, z)+c(x, y)].$$

$L'_r(0, x)$ is the longest path from the node 0 to x which does not contain the arc $\langle x_i, x\rangle$, and $L'_r(x, z)$ is the longest path from the node x to z which does not contain the arc $\langle x, x_j\rangle$.

Let

$$(27) \qquad \begin{aligned} a_r(0, x) &= L_r(0, x)-L'_r(0, x) \quad \text{for } x \in X, \\ \beta_r(y, z) &= L_r(y, z)-L'_r(y, z) \quad \text{for } y \in Y. \end{aligned}$$

Now we write the formula

$$(28) \qquad \Delta_r[(y, x), (u, v)]$$
$$= \max[-a_r(0, x),\ -\beta_r(y, z),\ c(x, u)+c(v, y)-a_r(0, x)-\beta_r(y, z)].$$

Similarly as in [4] we can prove that $\Delta_r[(y, x), (u, v)]$ exists for all $\langle y, x\rangle \in S_r$ and the arc $\langle u, v\rangle$ is the complement of the arc $\langle y, x\rangle$.

THEOREM 2. *Let $D_r \in R'_D$ and let D_r be the graph obtained from D_r by complementing one arc $\langle y, x\rangle \in S_r \cap C_r$, where C_r is the longest path in D_r. Then*

$$L_s(0, z) \geq L_r(0, z)+\Delta_r[(y, x), (u, v)].$$

Proof. Note that for each $\langle y, x\rangle \in C_r \cap S_r$

$$L_r(0, z) + \max[-\alpha_r(0, x),\ -\beta_r(y, z),\ c(x, u) + c(v, y) - \\ -\alpha_r(0, x) - \beta_r(y, z)]$$

$$= L_r(0, z) + \max[-L_r(0, x) + L'_r(0, x),\ -L_r(y, z) + \\ + L'_r(y, z),\ c(x, u) + c(v, y) - L_r(0, x) + L'_r(0, x) - \\ - L_r(y, z) + L'_r(y, z)]$$

$$= \max[L_r(x, z) + L'_r(0, x),\ L_r(0, y) + L'_r(y, z), \\ L'_r(0, x) + c(x, u) + c(v, y) + L'_r(y, z)]$$

$$= \max[L_r(x, z) + L'_r(0, x),\ L_r(0, v) + L_r(v, z), \\ l^d_s(0, u) + l^d_s(v, z)].$$

It can be easily observed that $L_r(x, z) + L'_r(0, x)$ and $L_r(0, y) + L'_r(y, z)$ are lengths of paths in D_r. It follows from the definitions of $L'_r(0, u)$, $L'(y, z)$, $L(0, x)$, $L(0, y)$ that $\langle y, x\rangle$ does not belong to the paths with these lengths. Summarizing, any path with length $L_r(x, z) + L'_r(0, x)$ or $L_r(0, y) + L'_r(y, z)$ does not contain the arc $\langle y, x\rangle$, so these paths are in D_s. On the contrary, the path with length $l^d_s(0, u) + l^d_s(v, z)$ is the longest path in D_s passing across the arc $\langle u, v\rangle$ which is the complement of the arc $\langle y, x\rangle \in C_r$. Since the longest path in D_s cannot be shorter than any other path, we have

$$L_s(0, z) \geqslant L_r(0, z) + \Delta_r[(y, x), (u, v)].$$

Therefore, by complementing the arc $\langle y, x\rangle \in C_r$, the lower bound of the longest path C_s in D_s is $L_r(0, z) + \Delta_r[(y, x), (u, v)]$. It should be mentioned that in [4] the lower bound is calculated for three given paths. Therefore, the lower bound used in this paper is stronger and allows of a better estimation of the longest path in D_s.

The subset $V^a \subset V$ of disjunctive arcs is called *full* if it contains arcs together with their complements. $\bar{D}^a = \langle A, U; V^a\rangle$ is called the *d-partial disjunctive graph* of the graph $\bar{D} = \langle A, U; V\rangle$. Let L^a_0 be the length of a minimaximal path of the graph $\bar{D}^a$. Let

$$R'_{D^a} = \{D^a_r = \langle A, U \cup S^a_r\rangle\} \tag{29}$$

be the family of graphs without circuits for the d-partial disjunctive graph $\bar{D}^a = \langle A, U; V^a\rangle$. Such a family exists, for the set V^a is full. It can be easily seen that the following theorem holds:

THEOREM 3. *Let $\bar{D}^a = \langle A, U; V^a\rangle$ be the d-partial disjunctive graph of $\bar{D} = \langle A, U; V\rangle$. Then*

$$L_0 \geqslant L^a_0, \tag{30}$$

where L_0 and L^a_0 are the lengths of the minimaximal paths of the graphs $\bar{D}$ and $\bar{D}^a$, respectively.

We determine a subset of nodes $A^a \subset A$ for any subset V^a of disjunctive arcs by

$$(31) \qquad A^a = \{x \in A \mid \langle y, x\rangle \in V^a\}$$

which is called the *full subset of nodes*. It follows from Property 2 that for each subset $A^a \subset X$ there exists a subset $B^a \subset Y$ such that $\Gamma(A^a) = B^a$ and $\Gamma^{-1}(B^a) = A^a$ in D.

The d-partial graph $\bar{D}^a = \langle A, U; V^a\rangle$ of the graph $\bar{D}$ is called a *disjunctive d-connected graph* if one of the following conditions is satisfied for each unordered pair of $x_j, x_i \in A^a$:

(a) there exists a path between nodes x_j and x_i in D,

(b) there exists an arc $\langle y, x\rangle \in V^a$ such that $y = \Gamma x_j$ and $x = x_i$ (or $y = \Gamma x_i$ and $x = x_j$).

Remark 1. It can be easily seen that if the d-partial graph $\bar{D}^a = \langle A, U; V^a\rangle$ is a d-connected graph, then for each unordered pair $y_j, y_i \in B^a$ one of the following conditions is satisfied:

(a) there exists a path between nodes y_j and y_i in D,

(b) there exists an arc $\langle y, x\rangle \in V^a$ such that $y = y_j$ and $x = \Gamma^{-1}y_i$ (or $y = y_i$ and $x = \Gamma^{-1}y_j$).

Since each node $x \in A^a$ has only one successor $\Gamma x \in B^a$ and each node $y \in B^a$ has only one predecessor $\Gamma^{-1}y \in A^a$ (this follows from Property 2), it is sufficient to put $y_j = \Gamma x_j$ and $y_i = \Gamma x_i$. Then the above-mentioned property follows from the definition of d-connectivity (see Fig. 5).

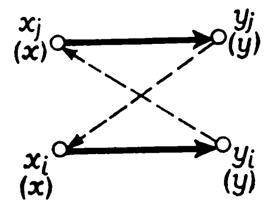

Fig. 5

The d-partial graph $\bar{D}^a = \langle A, U; V^a\rangle$ of the graph $\bar{D}$ is called a *symmetrical graph* to the node z or, shortly, *z-symmetrical* if, for each pair of nodes $x_j, x_i \in A^a \cup \{z\}$ for which the longest path $C(x_j, x_i)$ exists in D and this path does not contain other nodes of the set $A^a \cup \{z\}$, we have in D

$$(32) \qquad L(x_j, x_i) = c(x_j, \Gamma x_i), \quad \text{i.e.,} \quad L(\Gamma x_j, x_i) = 0.$$

It should be noted that since z is the end node of all paths in the graph, the first node x_j of each pair of nodes is $x_j \neq z$, $x_j \in A^a$.

The d-partial graph $\bar{D}^a = \langle A, U; V^a \rangle$ of the graph D is called the *symmetrical graph* to the node 0 or, shortly, 0-*symmetrical* if, for each pair of nodes $x_j, x_i \in B^a \cup \{0\}$ for which the longest path $C(x_j, x_i)$ exists in D and this path does not contain other nodes of the set $B^a \cup \{0\}$, we have in D

$$L(x_j, x_i) = c(\Gamma^{-1}x_i, x_i), \quad \text{i.e.,} \quad L(x_j, \Gamma^{-1}x_i) = 0. \tag{33}$$

In order to illustrate the above notions let us consider the disjunctive graph shown in Fig. 4. Let us create the disjunctive graphs

$$\bar{D}^1 = \langle A, U; V^1 \rangle, \quad \bar{D}^2 = \langle A, U; V^2 \rangle,$$
$$\bar{D}^3 = \langle A, U; V^3 \rangle \quad \text{and} \quad \bar{D}^4 = \langle A, U; V^4 \rangle,$$

where $V^4 = V^1 \cup V^2$. It can be easily shown that each graph $\bar{D}^k$ for $k = 1, 2, 3, 4$ is a d-partial graph of the graph $\bar{D}$ and

$$\begin{aligned} A^1 &= \{x_1, x_4, x_6\}, & B^1 &= \{y_1, y_4, y_6\}, \\ A^2 &= \{x_2, x_5, x_7\}, & B^2 &= \{y_2, y_5, y_7\}, \\ A^3 &= \{x_3, x_8\}, & B^3 &= \{y_3, y_8\}, \\ A^4 &= A^1 \cup A^2, & B^4 &= B^1 \cup B^2. \end{aligned}$$

Each graph $\bar{D}^k$ for $k = 1, 2, 3$ is d-connected. Consequently, for each pair of nodes $x_j, x_i \in A^k$ there exists an arc $\langle y, x \rangle \in V^k$ such that $y = \Gamma x_j$ and $x = x_i$. The graph $\bar{D}^4$ is not d-connected. It follows from this, for example, that for the pair $x_1, x_5 \in A^4$ a path between nodes x_1 and x_5 does not exist in D and $\langle y_1, x_5 \rangle \notin V^4$ ($y_1 = \Gamma x_1$ in D). The graph $\bar{D}^1$ is 0-symmetrical. Since the pair of nodes $x_1, z \in A^1 \cup \{z\}$, we have in D

$$L(x_1, z) = 1+2+3 > c(x_1, y_1) = 1,$$

so $\bar{D}^1$ is not a z-symmetrical graph. The graph $\bar{D}^3$ is z-symmetrical. Since the pair of nodes $y_3, 0 \in B^3 \cup \{0\}$, we have in D

$$L(0, y_3) = 1+2+3 > c(x_3, y_3) = 3,$$

so $\bar{D}^3$ is not a 0-symmetrical graph.

The graphs $\bar{D}^2$ and $\bar{D}^4$ are neither 0-symmetrical nor z-symmetrical.

LEMMA 2. *Let $\bar{D}^a = \langle A, U; V^a \rangle$ be the disjunctive d-partial and z-symmetrical graph. If for some pair of nodes $x_j, x_i \in A^a \cup \{z\}$ the longest path $C(x_j, x_i)$ exists in $D^a_r \in R'_{D^a}$ and this path does not contain other nodes of the set $A^a \cup \{z\}$, then the length of the path is*

$$L(x_j, x_i) = c(x_j, \Gamma x_j). \tag{34}$$

Proof. Since the graph D_r^a has no circuits, so if there exists a path between any nodes of this graph, then the longest path between them exists.

In order to prove the lemma note that if the path $C(x_j, x_i)$ exists in D_r^a, then: (a) the path exists in D, or (b) the path exists in D_r^a.

(a) In this case, since the graph $\bar{D}^a$ is z-symmetrical, by (32) we have

$$L(x_j, x_i) = c(x_j, \Gamma x_j).$$

(b) In this case, the path $C(x_j, x_i)$ must contain disjunctive arcs of the representation S_r^a. Since $x_j \in A^a$, x_j has one successor $\Gamma x_j \in B^a$. Now we have

$$C(x_j, x_i) = C(x_j, \Gamma x_j) \cup C(\Gamma x_j, x_i).$$

It is obvious that $C(x_j, \Gamma x_j)$ is a path in D, and so $C(\Gamma x_j, x_i)$ contains the disjunctive arcs. Since each disjunctive arc terminates in a node from the set A^a, $C(\Gamma x_j, x_i)$ must contain at least one such node. From the assumption of the lemma it follows that the path $C(x_j, x_i)$ does not contain other nodes (except x_j and x_i) of the set $A^a \cup \{z\}$ and, consequently, x_i is the only node belonging to the path $C(\Gamma x_j, x_i)$. Therefore, x_i must be the end of the disjunctive arc

$$\langle p, x_i \rangle \in S_r^a \cap C(\Gamma x_j, x_i).$$

$\langle p, x_i \rangle$ is the only arc of path $C(\Gamma x_j, x_i)$, i.e. $p = \Gamma x_j$. In order to prove this let us suppose that $p \neq \Gamma x_j$. Since $p \in B^a$, we have $\Gamma^{-1} p \in A^a$, and also $\Gamma^{-1} p \neq x_j$. Thus we obtain

$$C(\Gamma x_j, x_i) = C(\Gamma x_j, \Gamma^{-1} p) \cup (\Gamma^{-1} p, p) \cup \{\langle p, x_i \rangle\}.$$

From the above it follows that the path $C(\Gamma x_j, x_i)$ contains another node (apart from x_i) $\Gamma^{-1} p \in A^a \cup \{z\}$ which contradicts the assumption that the path $C(\Gamma x_j, x_i)$ contains only one node of the set $A^a \cup \{z\}$. Therefore, $C(\Gamma x_j, x_i) = \{\langle p, x_i \rangle\}$, i.e. $p = \Gamma x_j$ and

$$C(x_j, x_i) = C(x_j, \Gamma x_j) \cup \{\langle \Gamma x_j, x_i \rangle\},$$

and $\langle \Gamma x_j, x_i \rangle \in S_r^a$. Since $c(\Gamma x_j, x_i) = 0$, we have $L(x_j, x_i) = c(x_j, \Gamma x_j)$.

LEMMA 3. *Let $\bar{D}^a = \langle A, U; V^a \rangle$ be a disjunctive d-partial, d-connected and z-symmetrical graph and let C_r^a be the set of arcs of the critical path in $D_r^a \in R'_{D^a}$. If there exists an arc $\langle y, x \rangle \in C_r^a \cap S_r^a$, then the path C_r^a contains such a node $v \in A^a \cup \{z\}$ that the path $C(x, v)$ is of length*

$$L(x, v) = c(x, \Gamma x) \tag{35}$$

and there exists a path $d(\Gamma^{-1}y, v)$ which does not contain the arc $\langle y, x\rangle$ and the length of this path is

$$l^d(\Gamma^{-1}y, v) = c(\Gamma^{-1}y, y). \tag{36}$$

Proof. Since the graph D_r^a has no circuits, there exists a critical path and it can be written as

$$C_r^a = C(0, y) \cup \{\langle y, x\rangle\} \cup C(x, z). \tag{37}$$

Since $x \in A^a$, x has only one successor $\Gamma x \neq z$. Thus we have

$$C(x, z) = C(x, \Gamma x) \cup C(\Gamma x, z). \tag{38}$$

Since $\Gamma x \notin A^a \cup \{z\}$, any $u \in A^a \cup \{z\}$ belonging to $C(x, z)$ is contained in $C(\Gamma x, z)$. This path contains the node z, so it contains at least one node of the set $A^a \cup \{z\}$. Let u be such a node the earliest to Γx. Assume that u is the required node v. So we have to prove that the node satisfies the assumptions of Lemma 3.

The path $C(x, z)$ can be defined by

$$C(x, z) = C(x, \Gamma x) \cup C(\Gamma x, u) \cup C(u, z) = C(x, u) \cup C(u, z). \tag{39}$$

Let us consider the path $C(x, u)$. In order to prove the first part of the lemma, observe that we have: (a) the path $C(x, u)$ exists in D, or (b) the path $C(x, u)$ exists in D_r^a.

Observe that, in both cases, $x, u \in A^a \cup \{z\}$ and $C(x, u)$ is the longest path between nodes x and u such that it does not contain other nodes of the set $A^a \cup \{z\}$ (because u is the first node in the path $C(\Gamma x, u)$). In case (a), since the graph D_r^a is z-symmetrical, by (32) we have

$$L(x, u) = c(x, \Gamma x). \tag{40}$$

However, in case (b), by (34) we obtain also relation (40). This completes the proof of the first part of the lemma.

To prove the second part of the lemma, observe that $\Gamma^{-1}y \in A^a \cup \{z\}$. From the property of d-connectivity of the graph $\bar{D}^a = \langle A, U; V^a\rangle$ and from completeness of the representation S_r^a it follows that: (a) there exists a path between nodes $\Gamma^{-1}y$ and u in D, or (b) there exists a disjunctive arc $\langle y, u\rangle \in S_r^a$ (or a disjunctive arc $\langle \Gamma u, x\rangle \in S_r$).

(a) If there exists a path between nodes $\Gamma^{-1}y$ and u in D, then the path does not contain the disjunctive arc $\langle y, x\rangle$. Further, since the graph $\bar{D}^a = \langle A, U; V^a\rangle$ is z-symmetrical, by (32) we have

$$L(\Gamma^{-1}y, u) = c(\Gamma^{-1}y, y).$$

(b) In this case it follows from (37) and (38) that the critical path C_r^a contains the path

$$C(y, u) = \{\langle y, x\rangle\} \cup C(x, u).$$

Since the graph D_r^a has no circuits, it cannot contain the disjunctive arc $\langle \Gamma u, x\rangle$, however, it contains the complement of the arc $\langle y, u\rangle$, which implies that there exists a path between nodes $\Gamma^{-1}y$ and u:

$$d(\Gamma^{-1}y, u) = d(\Gamma^{-1}y, y) \cup \{\langle y, u\rangle\}.$$

Since $u \neq x$, the path $d(\Gamma^{-1}y, u)$ does not contain the disjunctive arc $\langle y, x\rangle$ and the length of the path is

$$l^d(\Gamma^{-1}y, u) = c(\Gamma^{-1}y, y).$$

Therefore, the node u is the required node v. The above considerations are illustrated in Fig. 6.

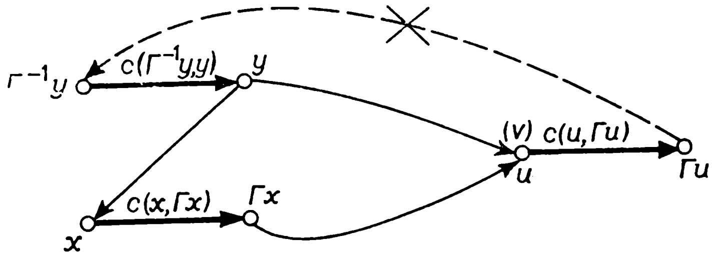

Fig. 6

LEMMA 4. *Let $\bar{D}^a = \langle A, U; V^a\rangle$ be a disjunctive d-partial, d-connected and z-symmetrical graph and let C_r^a be the set of arcs of the critical path in $D_r^a \in R'_{D^a}$. If there exists an arc $\langle y, x\rangle \in C_r^a \cap S_r^a$ belonging to the path C_r^a and if there exist nodes from A^a which belong to the path C_r^a and which precede the node $\Gamma^{-1}y \in A^a$, then*

$$L_s^a(0, z) \geqslant L_r^a(0, z). \tag{41}$$

Moreover, the graph D_s^a has been obtained from the graph D_r^a by complementing the arc $\langle y, x\rangle$.

Proof. By Theorem 1, complementing the disjunctive arc belonging to the critical path produces no circuit in the graph D_s^a.

It follows from Lemma 3 that for the arc $\langle y, x\rangle \in C_r^a \cap S_r^a$ there exists a node $v \in A^a \cup \{z\}$ belonging to the critical path and having the property

$$C_r^a = C(0, v) \cup C(v, z). \tag{42}$$

From the assumption of Lemma 4 it follows that there exists at least one node of the set A^a belonging to the path C_r^a and preceding the node $\Gamma^{-1}y$. Let u be the earliest such node (to $\Gamma^{-1}y$). Thus we have

$$C(0, v) = C(0, u) \cup C(u, \Gamma^{-1}y) \cup \{\langle \Gamma^{-1}y, y\rangle\} \cup \{\langle x, y\rangle\} \cup C(x, v). \tag{43}$$

Since $u, \Gamma^{-1}y \in A^a$ and the path $C(u, \Gamma^{-1}y)$ does not contain other nodes of the set A^a, by (34) we have $L(u, \Gamma^{-1}y) = c(u, \Gamma u)$. However, it follows from Lemma 3 that the path $C(x, v)$ has the length $c(x, \Gamma x)$. Therefore, the longest length of the path is

$$L(0, v) = L(0, u) + c(u, \Gamma u) + c(\Gamma^{-1}y, y) + 0 + c(x, \Gamma x). \tag{44}$$

Let us consider now the path $d(0, v)$ obtained as a result of complementing the arc $\langle y, x\rangle$:

$$d(0, v) = C(0, u) \cup d(u, x) \cup \{\langle x, \Gamma x\rangle\} \cup \{\langle \Gamma x, \Gamma^{-1}y\rangle\} \cup d(\Gamma^{-1}y, v). \tag{45}$$

The path $d(0, v)$ (if it exists) does not contain the disjunctive arc $\langle y, x\rangle$, however, it contains the complement of the arc $\langle \Gamma x, \Gamma^{-1}y\rangle$. Therefore, $d(0, v)$ is a path in D_s^a.

Now we prove that (a) the path $d(0, v)$ exists, and (b) the length of the path is equal to the length of the path $L(0, v)$.

(a) In order to prove that the path $d(0, v)$ exists, it should be proved that the second and the last components of (45) exist. Since $u, x \in A^a$, repeating for these nodes similar considerations as for the nodes $\Gamma^{-1}y$, u in the second part of the proof of Lemma 3 for case (b) we can prove that the path $d(u, x)$ exists and that the length of the path is $l^d(u, x) = c(u, \Gamma u)$. Further, it follows from Lemma 3 that the path $d(\Gamma^{-1}y, v)$ exists and the length of the path is $c(\Gamma^{-1}y, y)$.

(b) In view of the previous considerations we have

$$l^d(0, v) = L(0, u) + c(u, \Gamma u) + c(x, \Gamma x) + 0 + c(\Gamma^{-1}y, y), \tag{46}$$

i.e. by (44) and (45) we obtain

$$L(0, v) = l^d(0, v). \tag{47}$$

Further, since $d(0, z) = d(0, v) \cup C(v, z)$ is a certain path from the node 0 to z in the graph D_s^a and the longest path is not shorter, we have

$$L_s^a(0, z) \geqslant l^d(0, z).$$

Therefore, taking into consideration (42) and (47) we obtain

$$L_s^a(0, v) \geqslant l^d(0, v) + L(v, z) = L(0, v) + L(v, z) = L_r^a(0, z). \tag{48}$$

The above considerations are illustrated in Fig. 7.

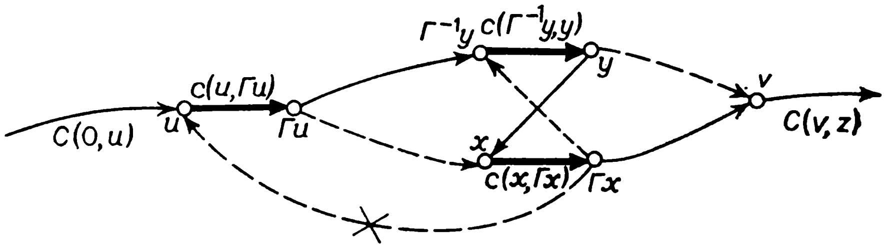

Fig. 7

CONCLUSION 1. *Let $\bar{D}^a = \langle A, U; V^a\rangle$ be a disjunctive d-partial, d-connected and z-symmetrical graph and let C_r^a be the set of arcs of the critical path in $D_r^a \epsilon R'_{D^a}$. If there exist two disjunctive arcs $\langle y_j, x_i\rangle$, $\langle y_k, x_l\rangle \epsilon C_r^a \cap S_r^a$ such that the arc $\langle y_j, x_i\rangle$ precedes $\langle y_k, x_l\rangle$, then*

$$(49) \qquad L_s^a(0, z) \geqslant L_r^a(0, z).$$

Moreover, the graph D_s^a has been obtained from the graph D_r^a by complementing the arc $\langle y_k, x_l\rangle$.

Proof. Since each disjunctive arc is terminated by a node of the set A^a, so $x_i \epsilon A^a$. From the assumptions of Conclusion 1 it follows that x_i precedes $\Gamma^{-1}y_k \epsilon A^a$ and, consequently, by Lemma 4 we obtain inequality (49).

THEOREM 4. *Let $\bar{D}^a = \langle A, U; V^a\rangle$ be a disjunctive d-partial, d-connected and z-symmetrical graph. The set S_0^{az} defined by*

$$(50) \qquad S_0^{az} = \{\langle y, x\rangle \epsilon B^a \times A^a \mid [\mathrm{War}(y, x) = 0 \vee \mathrm{War}(x, y) = 0] \wedge \wedge [L(0, \Gamma^{-1}y) \leqslant L(0, x)]\}$$

is the optimal solution (optimal selection) of the disjunctive graph $\bar{D}^a$ with minimaximal path L_0^{az}, where $L(0, x)$, $x \epsilon A$, is the maximal path from the node 0 to the node x of the graph D.

The proof of this theorem will be given after stating the algorithm for solution of our sequencing problem in part two of the paper.

For the disjunctive graph $\bar{D}^a = \langle A, U; V^a\rangle$ which is d-connected and 0-symmetrical we can prove similar properties as in the case of d-connection and z-symmetry.

LEMMA 2.1. *Let $\bar{D}^a = \langle A, U; V^a\rangle$ be a disjunctive d-partial and 0-symmetrical graph. If for some pair of nodes $x_j, x_i \epsilon B^a \cup \{0\}$ the longest path $C(x_j, x_i)$ exists in $D_r^a \epsilon R'_{Da}$ and this path does not contain other nodes of the set $B^a \cup \{0\}$, then the length of the path is*

$$(34') \qquad L(x_j, x_i) = c(\Gamma^{-1}x_i, x_i).$$

LEMMA 3.1. *Let $\bar{D}^a = \langle A, U; V^a\rangle$ be a disjunctive d-partial, d-connected and 0-symmetrical graph and let C_r^a be the set of arcs of the critical path in $D_r^a \epsilon R'_{Da}$. If there exists an arc $\langle y, x\rangle \epsilon C_r^a \cap S_r^a$, then the path C_r^a contains such a node $v \epsilon B^a \cup \{0\}$ that the path $C(v, y)$ has the length*

$$(35') \qquad L(v, y) = c(\Gamma^{-1}y, y),$$

and there exists a path $d(v, \Gamma x)$ which does not contain the arc $\langle y, x\rangle$ and the length of the path is (see Fig. 8)

$$(36') \qquad l^d(v, \Gamma x) = c(x, \Gamma x).$$

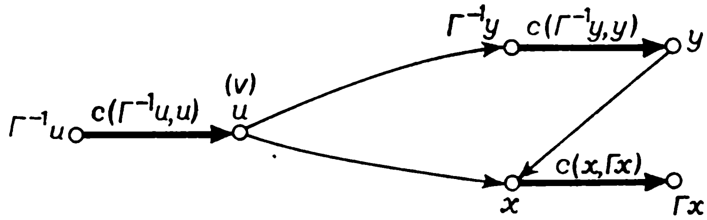

Fig. 8

LEMMA 4.1. *Let* $\bar{D}^a = \langle A, U; V^a\rangle$ *be a disjunctive d-partial, d-connected and* 0*-symmetrical graph and let* C_r^a *be the set of arcs of the critical path in* $D_r^a \in R'_{D^a}$. *If there exists an arc* $\langle y, x\rangle \in C_r^a \cap S_r^a$ *belonging to the path* C_r^a *and if there exist nodes from* B^a *belonging to the path* C_r^a *which follow the node* $\Gamma x \in B^a$, *then* (see Fig. 9)

$$L_s^a(0, z) \geqslant L_r^a(0, z). \tag{41'}$$

Moreover, the graph D_s^a *has been obtained from the graph* D_r^a *by complementing the arc* $\langle y, x\rangle$.

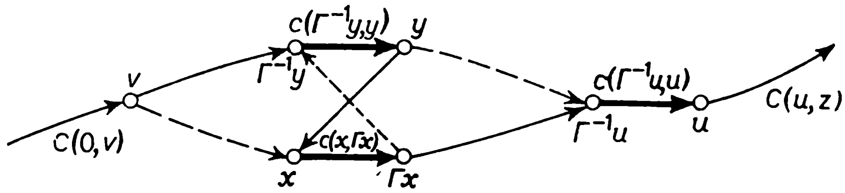

Fig. 9

THEOREM 4.1. *Let* $\bar{D}^a = \langle A, U; V^a\rangle$ *be a disjunctive d-partial, d-connected and* 0*-symmetrical graph. The set* S_0^{a0} *defined by*

$$S_0^{a0} = \{\langle y, x\rangle \in B^a \times A^a \mid [\mathrm{War}(y, x) = 0 \vee \mathrm{War}(x, y) = 0] \wedge \wedge [L(y, z) \geqslant L(\Gamma x, z)]\} \tag{50'}$$

is the optimal solution (optimal selection) of the disjunctive graph $\bar{D}^a$ *with minimaximal path* L_0^{a0}, *where* $L(y, z)$ $(y \in B)$ *is a maximal path from the node* y *to the node* z *of the graph* D.

Remark 2. Let us consider the disjunctive graph $\bar{D}^k = \langle A, U; V^k\rangle$ $(k \in Q)$, where V^k is the set of disjunctive arcs defined by (15) and, as can be easily seen, it is also the full subset of V. That is to say, D^k is the partial graph of the graph D. It follows from Property 3 that each graph D^k is d-connected, and $A^a = X^k$ and $B^a = Y^k$ are the full subsets. If we assume $\mathrm{War}(x, y) = 0$ for each $x, y \in X^k$ and $L(x, z) = c(x, z)$ in D for each $x \in X^k$, then D^k is the z-symmetrical graph. However, if $\mathrm{War}(x, y) = 0$ for each $x, y \in X^k$ and $L(0, y) = c(\Gamma^{-1}y, y)$ $(L(0, \Gamma^{-1}y) = 0)$ for each $x \in Y^k$ in D, then D^k is the 0-symmetrical graph.

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NOWE SFORMUŁOWANIE I ROZWIĄZANIE ZAGADNIENIA KOLEJNOŚCIOWEGO: MODEL MATEMATYCZNY

STRESZCZENIE

W pracy sformułowano ogólne zagadnienie kolejnościowe, prowadzące do nowej konstrukcji grafu dysjunktywnego. Uogólniono istniejące definicje i wprowadzono nowe pojęcia do teorii grafów dysjunktywnych. Pozwoliło to na skonstruowanie stosunkowo efektywnych algorytmów, umożliwiających rozwiązanie zagadnień o większych rozmiarach.
