ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XV, 4 (1977)

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PROPERTIES OF CERTAIN IMPROPER INTEGRALS

The functions F and G determined by integrals (1) and (2), respectively, are discussed in the paper. Series expansions and asymptotic formulae for these functions are derived. The application of both integrals is illustrated by examples.

1. Fundamental equations. In the sequel, we consider the functions

$$F(x, y) = \int_0^\infty \frac{e^{-ux}\cos uy}{u+i}\, du, \tag{1}$$

$$G(x, y) = \int_0^\infty \frac{e^{-ux}\sin uy}{u+i}\, du \tag{2}$$

with real y and $x > 0$. The function F is even, whereas G is odd with respect to y, i.e. $F(x, -y) = F(x, y)$ and $G(x, -y) = -G(x, y)$. Each of the functions F and G satisfies the two-dimensional Laplace equation.

Similar functions involving $a = \exp(i\pi/4)$ instead of i in the denominator of each integrand in (1) and (2) are discussed in [2].

2. Series expansions. Substituting

$$\cos uy = \frac{1}{2}(e^{iuy} + e^{-iuy})$$

into (1) we obtain

$$F(x, y) = \frac{1}{2}\left[\int_0^\infty \frac{e^{-uz}\, du}{u+i} + \int_0^\infty \frac{e^{-uz^*}\, du}{u+i}\right], \tag{3}$$

where

$$z = x+iy = re^{i\theta}, \quad z^* = x-iy = re^{-i\theta} \tag{4}$$

and

$$r = \sqrt{x^2+y^2}, \quad \theta = \operatorname{arctg}\frac{y}{x}. \tag{5}$$

The integrals appearing in (3) can be expressed in terms of the integral exponential function, giving

$$F(x,y) = \frac{1}{2}[-e^{iz}\mathrm{Ei}(-iz) - e^{iz^*}\mathrm{Ei}(-iz^*)], \tag{6}$$

according to formula No. 3.352.4 in [1].

The exponential function as well as the integral exponential function which appear in (6) are complex and can be represented in the forms

$$e^{iz} = A_1 + iA_2 \quad \text{and} \quad -\mathrm{Ei}(-iz) = B_1 + iB_2$$

with A_1, A_2, B_1, B_2 determined later. Hence

$$-e^{iz}\mathrm{Ei}(-iz) = (A_1+iA_2)(B_1+iB_2). \tag{7}$$

Substituting z from (4) into the expression

$$e^{iz} = \sum_{k=0}^{\infty} \frac{i^k z^k}{k!}$$

we obtain

$$A_1 = \sum_{k=0}^{\infty} \frac{i^k r^k \cos k\theta}{k!} \quad \text{and} \quad A_2 = \sum_{k=1}^{\infty} \frac{i^k r^k \sin k\theta}{k!} \tag{8}$$

if the Euler identity is applied. For $i^{k+4} = i$, equations (8) take the forms

$$A_1 = 1 - c_2 + ic_1 \quad \text{and} \quad A_2 = -s_2 + is_1, \tag{9}$$

where, for $m = 1, 2$,

$$c_m = \sum_{k=0}^{\infty} (-1)^k \frac{r^{m+2k}\cos(m+2k)\theta}{(m+2k)!},$$

$$s_m = \sum_{k=0}^{\infty} (-1)^k \frac{r^{m+2k}\sin(m+2k)\theta}{(m+2k)!}.$$

The functions B_1 and B_2 associated with $-\mathrm{Ei}(-iz)$ are derived in a similar way if the series (see [3])

$$\mathrm{Ei}(z) = C + \ln(-z) + \sum_{k=1}^{\infty} \frac{z^k}{k\cdot k!}$$

is used, where $C = 0.5772\ldots$ is Euler's constant. We obtain

$$B_1 = (-C - \ln r + c_2') + i\left(-\frac{\pi}{2} + c_1'\right),$$

$$B_2 = (-\theta + s_2') + is_1',$$

where, for $m = 1, 2$,

$$c_m' = \sum_{k=0}^{\infty} (-1)^k \frac{r^{m+2k}\cos(m+2k)\theta}{(m+2k)\cdot(m+2k)!},$$

$$s_m' = \sum_{k=0}^{\infty} (-1)^k \frac{r^{m+2k}\sin(m+2k)\theta}{(m+2k)\cdot(m+2k)!}.$$

The relationships for e^{iz^*} and $-\mathrm{Ei}(-iz^*)$ can be derived if θ is replaced by $-\theta$, giving

$$-e^{iz^*}\mathrm{Ei}(-iz^*) = (A_1 - iA_2)(B_1 - iB_2), \tag{10}$$

since A_1 and B_1 are even, whereas A_2 and B_2 are odd with respect to θ. Inserting (7) and (10) into (6), we obtain

$$F(x, y) = A_1B_1 - A_2B_2.$$

Now we consider the function G. Substituting

$$\sin uy = \frac{1}{2i}(e^{iuy} - e^{-iuy})$$

into (2) we have

$$G(x, y) = \frac{1}{2i}[e^{iz}\mathrm{Ei}(-az) - e^{iz^*}\mathrm{Ei}(-iz^*)]. \tag{11}$$

Inserting (7) and (10) into (11), we obtain

$$G(x, y) = -(A_1B_2 + A_2B_1).$$

For small r we have

$$F(x, y) = -C + \ln\frac{1}{r} - i\frac{\pi}{2} + O(r), \qquad G(x, y) = \theta + O(r),$$

where r and θ are defined by (5).

By the relationships derived, the numerical values of the functions F and G can be computed using digital computers. Tables of the functions iF and iG are given in [5].

3. Asymptotic formulae. The asymptotic formulae of the functions F and G are derived from the asymptotic formulae for the integral exponential function (see [3])

$$\mathrm{Ei}(z) \sim \frac{e^z}{z}\sum_{k=0}^{\infty}\frac{k!}{z^k}.$$

Then

$$-e^{iz}\mathrm{Ei}(-iz) \sim -\sum_{k=0}^{\infty}\frac{k!\,i^{k+1}}{z^{k+1}},$$

whence

$$-e^{iz}\mathrm{Ei}(-iz) \sim \sum_{m=0}^{\infty}(-1)^m\frac{(2m+1)!}{z^{2m+2}} - i\sum_{m=0}^{\infty}(-1)^m\frac{(2m)!}{z^{2m+1}}.$$

Substituting $z = re^{i\theta}$ into the last equation we have

$$(12)\qquad -e^{-iz}\mathrm{Ei}(-iz) \sim \sum_{m=0}^{\infty}(-1)^m\frac{(2m+1)!}{r^{2m+2}}e^{-i2(m+1)\theta} - \\ -i\sum_{m=0}^{\infty}(-1)^m\frac{(2m)!}{r^{2m+1}}e^{-i(2m+1)\theta}$$

and, similarly,

$$(13)\qquad -e^{-iz^*}\mathrm{Ei}(-iz^*) \sim \sum_{m=0}^{\infty}(-1)^m\frac{(2m+1)!}{r^{2m+2}}e^{i2(m+1)\theta} - \\ -i\sum_{m=0}^{\infty}(-1)^m\frac{(2m)!}{r^{2m+1}}e^{i(2m+1)\theta},$$

since $z^* = re^{-i\theta}$. Inserting (12) and (13) into (6) and (11), we obtain

$$F(x,y) \sim \sum_{m=0}^{\infty}(-1)^m\frac{(2m+1)!\cos 2(m+1)\theta}{r^{2m+2}} - \\ -i\sum_{m=0}^{\infty}(-1)^m\frac{(2m)!\cos(2m+1)\theta}{r^{2m+1}},$$

$$G(x,y) \sim \sum_{m=0}^{\infty}(-1)^m\frac{(2m+1)!\sin 2(m+1)\theta}{r^{2m+2}} - \\ -i\sum_{m=0}^{\infty}(-1)^m\frac{(2m)!\sin(2m+1)\theta}{r^{2m+1}}.$$

4. Applications. The functions discussed appear in problems concerning the electromagnetic field in a very thin metallic plate which is treated as a conducting plane, due to alternating currents in parallel conductors placed in a non-conducting medium above the plate (see [4]-[6]). We give the examples which illustrate the application of the functions.

1. The unit-length external impedance of the conductor k placed above the plate is

$$Z_{ek} = \frac{i\omega\mu_0}{2\pi}\left[F(2ch_k, 0) + \ln\frac{2h_k}{r_k}\right],$$

and the unit-length mutual impedance of the conductors k and m is

$$Z_{km} = \frac{i\omega\mu_0}{2\pi}\left\{F[c(h_k+h_m), cb_{km}] + \ln\sqrt{\frac{(h_k+h_m)^2+b_{km}^2}{(h_k-h_m)^2+b_{km}^2}}\right\},$$

where ω denotes the angular frequency, μ_0 — the permeability of vacuum, h_k — the distance from the conductor k to the plate, r_k — the radius of the conductor k, b_{km} — the horizontal distance between the conductors k and m, $c = \frac{1}{2}\omega\mu_0\gamma d$, γ being the conductivity of the plate, and d — the thickness of the plate.

2. The electric intensity at the plate due to the current flowing through the conductor k (see [5] and [6]) is

$$E = -\frac{i\omega\mu_0 I}{2\pi}F(ch_k, cy),$$

where I is the complex current, and y is the horizontal distance from the conductor k to the point under consideration.

The density of the active power on the plate is described by

$$T(y) = \gamma d|E|^2 = \gamma d\left(\frac{\omega\mu_0}{2\pi}\right)^2 |I|^2 |F(ch_k, cy)|^2.$$

3. The function G occurs in relationships for the normal component of the magnetic intensity (see [5] and [6]).

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Received on 2. 1. 1976

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WŁASNOŚCI PEWNYCH CAŁEK NIEWŁAŚCIWYCH

STRESZCZENIE

W pracy rozpatrzono funkcje F i G określone wzorami (1) i (2) oraz wyznaczono szeregi i wzory asymptotyczne. Zastosowanie obu funkcji zilustrowano przykładami.
