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SOLVING THE ABEL INTEGRAL EQUATION BY INTERPOLATION METHODS

In the paper we give a comparative study of methods for solving the Abel integral equation, based on interpolation on equally spaced data. The formulae quoted in various papers appear to be variants of one of two basic algebraic equations. A technique based on linear interpolation of the data function is modified. A data random error propagation formula for the algebraic form of the Abel integral equation is derived. The sensibility of different methods to random error propagation is compared. Systematic errors introduced by each of the methods considered are compared using various test functions. The influence of the number of data points on the two properties described above is examined. Suggestions concerning the choice of a method, appropriate for the experimental function processed, are made.

1. INTRODUCTION

In all side-on observations of non-homogeneous axially symmetric transparent media, the radial distribution of the physical property investigated is determined by the *Abel integral equation*

$$f(y) = 2\int_{y}^{R} \frac{g(r)\,r\,dr}{(r^2-y^2)^{1/2}} \tag{1}$$

or by the *Abel inverse integral equation*

$$g(r) = -\frac{1}{\pi}\int_{r}^{R} \frac{df(y)}{dy}\,\frac{dy}{(y^2-r^2)^{1/2}}, \tag{2}$$

where $f(y)$ is the lateral observation data function, $g(r)$ — the radial distribution function to be determined, and R — the overall radius of the medium column. None of these equations can be solved analytically for a function of any kind. An analytic solution can, however, be found if the unknown function $g(r)$ in equation (1) or the experimental data function $f(y)$ in equation (2) is, for example, a polynomial.

The solution of the Abel integral equation (1) based on the polynomial interpolation of the function $g(r)$ over the annular zones of equal width has been the subject of many studies. Mach [8] assumes that the radial distribution function to be determined is of the step type, its value in each zone being equal to the value at the internal zone boundary. A modification of this method has been developed by Czernichowski [4]. The value of $g(r)$ in each zone is assumed to be equal to its value in the center of this zone. In the comparison given in the sequel, this method is not considered; the set of data points necessary for the calculations differs from the one used in all other methods. The second modification of the Mach method is given by Pikalov and Preobrazhensky [13]: the function value in each zone is equal to the arithmetic mean of its values at the two boundaries of the zone, but no formula is given. Pearce's approach [11] is more sophisticated. His method is based on the assumption that the value of $f(y)$ in each lateral segment consists of partial sums, each of them being a product of constant values of $g(r)$ in individual zones multiplied by the common area of the lateral segment and the concentric zone considered. A linear variation of $g(r)$ in zones of equal width was previously proposed by van Voorhis (see [8] and [12]) and linear in r^2 variation of the same function was employed by Frie [5].

The solution of the Abel inverse integral equation (2) based on the polynomial interpolation of the function $f(y)$ in segments of equal width was also dealt with by many authors. Weyl [12] and Nestor and Olsen [10] assumed the linear variation of the data function in y^2. However, the final formulae for the radial distribution function differed from each other. Weyl's results are smaller by a factor of two than those of Nestor and Olsen. To check the results of both methods, appropriate calculations have been performed in paper [15], proving the correctness of Nestor's and Olsen's calculations. Bockasten [1] increased the degree of the approximation polynomial up to three.

Cremers and Birkebak [3] employed a method referred to as developed by Ladenburg et al. [8] which, however, was neither published by the authors nor mentioned elsewhere. In that method, equation (1) is solved under the assumption of linear variation of the data function $f(y)$ over segments of equal width. This approach leads to a very complicated formula which, moreover, does not include the axis point. Then extrapolation is necessary to evaluate the solution for $r = 0$. In the comparison of different interpolation methods of solving the Abel integral equation which is given in the sequel, this method is not considered, since it was not possible to calculate the coefficients of the complete Abel matrix.

Gorenflo ([6], p. 8) solved equation (2) under the same assumption as in [3]. As previously, extrapolation is necessary to evaluate the solution at the near-axis region, and the value of f_0 is not employed. The Gorenflo method is modified in Section 3 of this paper.

It is not easy to compare the properties of different interpolation methods of solving the Abel integral equation found in the literature because of various fields of application (e.g., spectroscopy or interferometry) and different notation of zone numbers. In this paper a uniform notation of zone numbers ($r_0 = 0$, $r_N = R$) is applied, and a uniform notation for the data function $f(y)$ and the radial distribution function $g(r)$ is adopted. This notation is applicable directly in spectroscopy; for interferometry, the factor $2\pi/\varrho_e\lambda$, where $\varrho_e = e^2/m_e c^2$ is the classical electron radius, and λ is the wavelength of traversing light, must be taken into account.

2. ABEL MATRICES

The set of N experimental data and the corresponding set of radial distribution function values to be determined can be expressed as the components of two N-dimensional vectors $\boldsymbol{f}$ and $\boldsymbol{g}$:

$$f_i = f(y_i), \quad g_i = g(r_i) \quad (i = 0, 1, \ldots, N-1).$$

For equidistant ($w = R/N$) data the integral equations (1) and (2) can be approximated by the algebraic expressions

$$f_i = w \sum_{k=i}^{N-1} a_{i,k} g_k, \tag{3}$$

$$g_i = \frac{1}{w} \sum_{k=i}^{N-1} b_{i,k} f_k, \tag{4}$$

which can also be presented in the matrix forms

$$\boldsymbol{f} = wA\boldsymbol{g}, \tag{5}$$

$$\boldsymbol{g} = \frac{1}{w} B\boldsymbol{f}. \tag{6}$$

The coefficients $a_{i,k}$ and $b_{i,k}$ are the elements of the Abel matrices A and B, respectively. Expressions (3)-(6) seem to be more appropriate than the formula given by Kock and Richter [7], since a variation in the number of equidistant data changes the matrix range without influencing the values of its elements. The components of the vector $\boldsymbol{g}$ are

TABLE 1. Solving the Abel integral equation

Method no.	$c_{i,k}$	The numerical			
		k	$i = 0$	$i = 1$	$i = 2$
A1	Step interpolation I (Mach): $2\,[(k+1)^2 - i^2]^{1/2}$	0	2.000000		
		1	2.000000	3.464102	
		2	2.000000	2.192753	4.472136
		3	2.000000	2.089112	2.456067
		4	2.000000	2.051992	2.236948
		5	2.000000	2.034201	2.148557
		6	2.000000	2.024247	2.102699
		7	2.000000	2.018101	2.075526
		8	2.000000	2.014036	2.057995
		9	2.000000	2.011205	2.045989
A2	Step interpolation II (Pikalov and Preobrazhensky): $[(k+1)^2 - i^2]^{1/2} + (k^2 - i^2)^{1/2}$	0	1.000000		
		1	2.000000	1.732051	
		2	2.000000	2.828427	2.236068
		3	2.000000	2.140933	3.464102
		4	2.000000	2.070552	2.346508
		5	2.000000	2.043096	2.192753
		6	2.000000	2.029224	2.125628
		7	2.000000	2.021174	2.089112
		8	2.000000	2.016069	2.066760
		9	2.000000	2.012620	2.051992
A3	Step interpolation III (Pearce): $(k+1)^2\left(\arccos\frac{i}{k+1} - \arccos\frac{i+1}{k+1}\right) - i\,[(k+1)^2 - i^2]^{1/2} + (i+1)\,[(k+1)^2 - (i+1)^2]^{1/2}$	0	1.570796		
		1	2.255650	2.456739	
		2	2.060513	2.695986	3.097482
		3	2.028908	2.237191	3.102981
		4	2.017060	2.130229	2.433992
		5	2.011283	2.083482	2.255721
		6	2.008023	2.058381	2.171890
		7	2.006001	2.043227	2.124453
		8	2.004659	2.033341	2.094645
		9	2.003722	2.026518	2.074567
A4	Linear in r interpolation (van Voorhis): 1 for $i+k = 0$, $(k+1)\,[(k+1)^2 - i^2]^{1/2} - k\,(k^2 - i^2)^{1/2} - i^2 \ln \frac{k+1+[(k+1)^2 - i^2]^{1/2}}{k+(k^2-i^2)^{1/2}}$ for $i+k > 0$	0	1.000000		
		1	2.000000	2.147144	
		2	2.000000	2.428247	2.858509
		3	2.000000	2.130572	2.871556
		4	2.000000	2.068007	2.327041
		5	2.000000	2.042155	2.187349
		6	2.000000	2.028793	2.123446
		7	2.000000	2.020948	2.088045
		8	2.000000	2.015939	2.066172
		9	2.000000	2.012540	2.051640
A5	Linear in r^2 interpolation (Frie): $\frac{4}{3}\,\frac{[(k+1)^2 - i^2]^{3/2} - (k^2 - i^2)^{3/2}}{2k+1}$	0	1.333333		
		1	1.777778	2.309401	
		2	1.955556	2.338936	2.981424
		3	1.980952	2.107345	2.806934
		4	1.989418	2.056282	2.310141
		5	1.993266	2.034980	2.178500
		6	1.995338	2.023924	2.117856
		7	1.996581	2.017420	2.084151
		8	1.997386	2.013261	2.063288
		9	1.997936	2.010437	2.049411

by means of A-type interpolation methods

values of $a_{i,k}$

$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$	$i = 8$	$i = 9$
5.291503						
2.708497	6.000000					
2.392305	2.944272	6.633250				
2.256806	2.544853	3.164709	7.211103			
2.183286	2.367281	2.692037	3.371903	7.745967		
2.138166	2.268109	2.476634	2.833403	3.567742	8.246211	
2.108221	2.205787	2.353879	2.583592	2.969148	3.753789	8.717798
2.645751						
4.000000	3.000000					
2.550401	4.472136	3.316625				
2.324555	2.744563	4.898979	3.605551			
2.220046	2.456067	2.928373	5.291503	3.872983		
2.160726	2.317695	2.584335	3.102653	5.656854	4.123106	
2.123194	2.236948	2.415256	2.708497	3.268445	6.000000	4.358899
3.626494						
3.468359	4.087528					
2.625874	3.800845	4.501556				
2.385251	2.808785	4.107453	4.880569			
2.267169	2.513115	2.982572	4.393218	5.232189		
2.198125	2.363905	2.637501	3.147952	4.661793	5.561617	
2.153578	2.274679	2.459882	2.757899	3.305791	4.915903	5.872591
3.424716						
3.263057	3.909645					
2.524031	3.614747	4.340686				
2.316823	2.712543	3.936116	4.732607			
2.216782	2.446391	2.891496	4.233650	5.094454		
2.159070	2.313501	2.572981	3.061463	4.511881	5.432244	
2.122252	2.234771	2.410250	2.695654	3.223343	4.774093	5.750219
3.527668						
3.210071	4.000000					
2.510275	3.568814	4.422166				
2.309534	2.700708	3.895027	4.807402			
2.212114	2.440094	2.880977	4.196146	5.163978		
2.155776	2.309442	2.567375	3.051912	4.477166	5.497474	
2.119783	2.231886	2.406625	2.690564	3.214541	4.741628	5.811865

TABLE 2. Solving the Abel inverse integral equation

Meth- od no.	$d_{i,k}$	The numerical			
		k	$i = 0$	$i = 1$	$i = 2$
B1	Linear in y interpolation (Gorenflo modified): $\frac{2}{\pi}$ for $i+k=0$, $\frac{1}{\pi}\ln\frac{k+1+[(k+1)^2-i^2]^{1/2}}{k+(k^2-i)^{1/2}}$ for $i+k>0$	0	.636620		
		1	−.415984	.419201	
		2	−.091572	−.277302	.306349
		3	−.037492	−.046187	−.193497
		4	−.020543	−.022821	−.033325
		5	−.012994	−.013858	−.017155
		6	−.008967	−.009368	−.010774
		7	−.006563	−.006775	−.007484
		8	−.005013	−.005135	−.005533
		9	−.003954	−.004030	−.004271
B2	Linear in y^2 interpolation (Weyl, and Nestor and Olsen): $\frac{2}{\pi}\frac{[(k+1)^2-i^2]^{1/2}-(k^2-i^2)^{1/2}}{2k+1}$	0	.636620		
		1	−.424413	.367553	
		2	−.084883	−.227958	.284705
		3	−.036378	−.044597	−.173021
		4	−.020210	−.022423	−.032568
		5	−.012861	−.013710	−.016942
		6	−.008904	−.009300	−.010688
		7	−.006529	−.006739	−.007441
		8	−.004993	−.005114	−.005510
		9	−.003942	−.004017	−.004257
B3	Third degree polynomial interpolation (Bockasten): $g_i = \frac{1}{R}\sum_{k=i-1}^{N-1} Nb_{i,k} f_k$ ($Nb_{i,k}$ tabulated in paper [1])	0	.762597	.046342	
		1	−.580096	.360630	.032395
		2	−.058470	−.295128	.265385
		3	−.033947	−.018240	−.205837
		4	−.019704	−.021489	−.013873
		5	−.012688	−.013465	−.016250
		6	−.008828	−.009204	−.010503
		7	−.006491	−.006693	−.007368
		8	−.004825	−.004941	−.005318
		9	−.004488	−.004571	−.004835

calculated from (4) directly or from (3) by the use of the recurrence formula

$$g_i = \frac{1}{a_{i,i}}\left(\frac{f_i}{w} - \sum_{k=i+1}^{N-1} a_{i,k} g_k\right) \quad (i = N-1,\ N-2,\ \ldots,\ 0). \tag{7}$$

Each non-diagonal element of the matrix A or B can be presented as the difference between the neighbour elements of the matrix C or D, respectively,

$$a_{i,k} = \begin{cases} c_{i,k} & \text{for } k = i, \\ c_{i,k} - c_{i,k-1} & \text{for } k > i, \end{cases} \tag{8}$$

by means of B-type interpolation methods

values of $b_{i,k}$

$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$
.253173						
−.156646	.220636					
−.027025	−.134922	.198104				
−.014103	−.023202	−.120229	.181315			
−.008985	−.012172	−.020594	−.109457	.168181		
−.006325	−.007808	−.010826	−.018681	−.101131	.157542	
−.004733	−.005536	−.006970	−.009826	−.017205	−.094448	.148697
.240620						
−.144826	.212207					
−.026567	−.127007	.191948				
−.013968	−.022887	−.114459	.176567			
−.008928	−.012076	−.020362	−.105013	.164375		
−.006296	−.007767	−.010754	−.018501	−.097572	.154403	
−.004716	−.005514	−.006938	−.009770	−.017060	−.091515	.146051
.026318						
.219858	.022729					
−.166607	.191842	.020293				
−.011232	−.143490	.172381	.018502			
−.013382	−.009563	−.127859	.157851	.017114		
−.008769	−.011555	−.008415	−.116401	.146469	.015998	
−.006062	−.007429	−.010041	−.007262	−.107072	.138186	.025141
−.005337	−.006199	−.007699	−.010490	−.008647	−.103729	.098416

(9)
$$b_{i,k} = \begin{cases} d_{i,k} & \text{for } k = i, \\ d_{i,k} - d_{i,k-1} & \text{for } k > i, \end{cases}$$

where the values of elements $c_{i,k}$ and $d_{i,k}$ depend on the type of interpolation used.

The formulae for the elements of $c_{i,k}$ and $d_{i,k}$ are taken directly from the van Voorhis method and the method presented in the next section of this paper. The formulae of other authors have been simplified, and/or the notation has been unified. In Pearce's method, the elements $c_{i,k}$ can easily be derived from the elements $a_{i,k}$. For the Mach method and the Pikalov and Preobrazhensky method, the elements $c_{i,k}$ have been calculated in paper [15]. The formulae for the elements $c_{i,k}$ and $d_{i,k}$ are confronted in Tables 1 and 2, respectively. The numerical values of $a_{i,k}$ and $b_{i,k}$ have been computed for $N = 20$, using (8) and (9), and the first parts of the obtained matrices (for $N = 10$) have been printed in the same tables. The Bockasten method has been treated separately and has not been incorporated in (9).

3. MODIFICATION OF THE SOLUTION OF THE ABEL INVERSE INTEGRAL EQUATION BASED ON LINEAR VARIATION OF THE DATA FUNCTION

The region of integration $y \in \langle 0, R \rangle$ in (2) is divided into N segments $i = 0, 1, \ldots, N-1$. In the i-th segment $(i = 1, 2, \ldots, N-1)$ a linear variation of the data function is assumed. To avoid singularity in the integral in equation (2), a linear in y^2 variation of the data function in the near axis segment $(i = 0)$ is assumed. Moreover, this assumption reflects better the physical reality, since in that case the derivative of the data function becomes continuous at the axis of symmetry. Calculations performed for segments of equal width $w = R/N$ in the Appendix allow us to replace equation (2) by the system of algebraic equations (4), where the coefficients $b_{i,k}$ are given by (9) and

$$d_{i,k} = \begin{cases} \dfrac{2}{\pi} & \text{for } i+k = 0, \\[2ex] \dfrac{1}{\pi} \ln \dfrac{k+1+[(k+1)^2 - i^2]^{1/2}}{k+(k^2-i^2)^{1/2}} & \text{for } i+k > 0. \end{cases} \tag{10}$$

4. COMPARISON OF THE QUALITY OF DIFFERENT METHODS

Various interpolation methods for solving the Abel integral equation have been compared by many authors, but no unified procedure has been established. The criteria applied in this paper seem to be simpler than in [13], and the presentation of results appears to be clearer than in [3].

A. Comparison of data random error amplification. The resultant function values g_i are influenced by the statistical random errors superimposed on the data points f_i. The standard deviation σ in the distribution of errors is supposed. Applying the law of random error propagation (see [9], p. 138) to the linear combinations of variables in (4) and (3) we obtain

$$\sigma_{g_i}^2 = \frac{\sigma^2}{w^2} \sum_{k=i}^{N-1} b_{i,k}^2$$

and

$$\sigma_{g_i}^2 = \frac{1}{a_{i,i}^2}\left(\frac{\sigma^2}{w^2} + \sum_{k=i+1}^{N-1} a_{i,k}^2 \sigma_{g_k}^2\right),$$

respectively. The random error amplification defined as the standard deviation ratio $\gamma_i = \sigma_{g_i}/\sigma$ can be calculated directly from the formula

$$\gamma_i = \frac{1}{w}\left(\sum_{k=i}^{N-1} b_{i,k}^2\right)^{1/2} \tag{11}$$

for the elements of the Abel matrix B, and from the recurrence formula

$$\gamma_i = \frac{1}{a_{i,i}}\left(\frac{1}{w^2} + \sum_{k=i+1}^{N-1} a_{i,k}^2 \gamma_k^2\right)^{1/2}$$

for the elements of the Abel matrix A. A formula analogous to (11) has been used by Bockasten [1].

The radial distribution of random error amplification in all the methods listed in Tables 1 and 2 has been computed for the number of zones $N = 10$, $N = 20$, and is confronted in Table 3. The results are also presented in a graphical form in Fig. 1.

B. Comparison of systematic error distribution. All interpolation methods for solving the Abel integral equation introduce the systematic error [13]

$$\eta_i = \frac{g_i - g_e(r_i)}{g_e(r_i)},$$

where $g_e(r_i)$ is the exact analytic solution of the Abel integral equation for the test curve, and g_i is the approximative numerical solution at $r = r_i$ given by the method examined. To compare the systematic error distributions, three test curves useful in practical applications have been used [2], [14] (Table 4 and Fig. 2). Two of them are similar to the functions applied by van Trigt in an unpublished paper, announced in [16].

An example of the computation of the values of g_i for one of the test curves by different interpolation methods is presented in a graphical form in Fig. 3. The systematic error distributions have been computed for any of the test curves ($N = 10$ and $N = 20$) and are compared in Figs. 4A and 4B.

TABLE 3. Radial distribution of random error amplification for some interpolation methods of solving the Abel integral equation

Number of zones	r	γ							
		A1	A2	A3	A4	A5	B1	B2	B3
10	.00	8.990	385.917	34.269	53.997	33.483	7.674	7.712	9.609
	.10	4.588	160.655	17.469	19.452	16.224	5.056	4.358	4.695
	.20	3.311	86.525	11.553	12.209	10.834	3.646	3.355	3.384
	.30	2.649	49.990	8.360	8.669	7.946	2.995	2.827	2.779
	.40	2.226	29.842	6.326	6.503	6.090	2.601	2.488	2.413
	.50	1.923	18.141	4.907	5.023	4.778	2.330	2.248	2.161
	.60	1.688	11.133	3.853	3.935	3.790	2.128	2.065	1.974
	.70	1.493	6.837	3.029	3.091	3.007	1.970	1.919	1.824
	.80	1.320	4.126	2.345	2.393	2.348	1.837	1.795	1.735
	.90	1.147	2.294	1.703	1.739	1.721	1.487	1.461	1.016
20	.00	23.402	47574.060	256.115	397.656	235.970	15.348	15.423	19.219
	.05	11.951	19805.449	130.668	143.328	114.416	10.113	8.717	9.391
	.10	8.643	10668.000	86.609	90.137	76.579	7.293	6.711	6.769
	.15	6.943	6165.465	62.938	64.258	56.419	5.992	5.655	5.560
	.20	5.871	3683.620	47.977	48.550	43.573	5.204	4.979	4.828
	.25	5.119	2243.862	37.663	37.941	34.611	4.663	4.498	4.325
	.30	4.555	1384.304	30.160	30.311	28.011	4.261	4.135	3.952
	.35	4.110	861.709	24.498	24.593	22.970	3.948	3.847	3.661
	.40	3.747	539.990	20.111	20.181	19.019	3.695	3.611	3.426
	.45	3.443	340.133	16.644	16.702	15.864	3.485	3.415	3.232
	.50	3.183	215.123	13.860	13.911	13.305	3.307	3.247	3.067
	.55	2.955	136.510	11.594	11.642	11.203	3.154	3.102	2.924
	.60	2.752	86.858	9.729	9.775	9.458	3.021	2.974	2.800
	.65	2.568	55.384	8.178	8.223	7.995	2.902	2.861	2.691
	.70	2.400	35.367	6.875	6.917	6.755	2.797	2.760	2.593
	.75	2.242	22.595	5.767	5.806	5.694	2.702	2.668	2.505
	.80	2.090	14.410	4.811	4.847	4.772	2.615	2.585	2.425
	.85	1.941	9.125	3.968	4.001	3.952	2.535	2.507	2.347
	.90	1.784	5.646	3.191	3.221	3.193	2.455	2.429	2.317
	.95	1.601	3.203	2.390	2.414	2.402	2.057	2.039	1.409

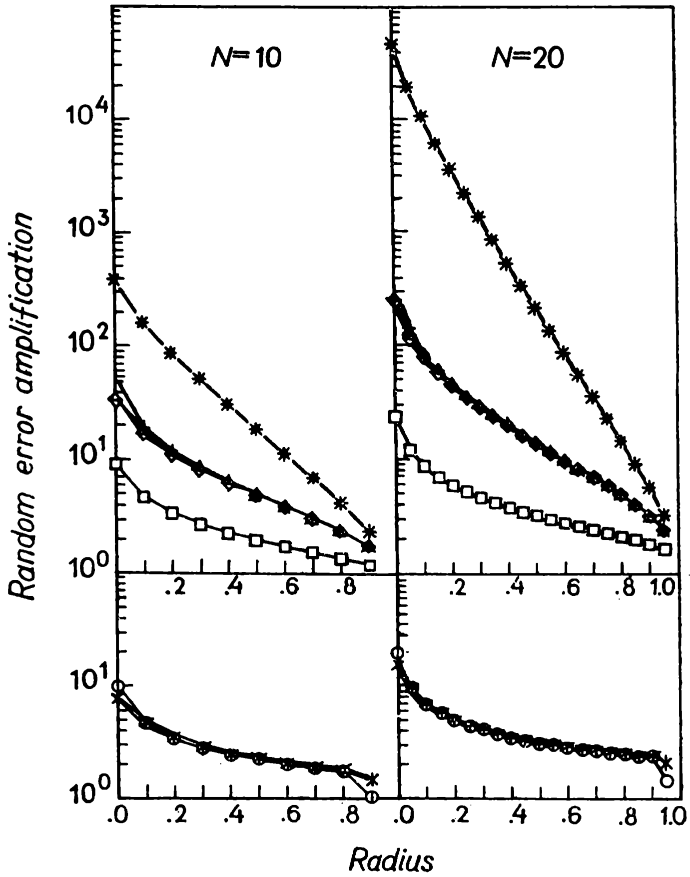

Fig. 1. Radial distribution of random error amplification for some interpolation methods for solving the Abel integral equation

□ method A1, * method A2, △ method A3, + method A4, ◇ method A5
× method B1, | method B2, ○ method B3

TABLE 4. Test functions used to examination of interpolation methods for solving the Abel integral equation (1)

No.	$f(y)$	$g(r)$	$\left.\dfrac{dg(r)}{dr}\right\|_{r=1}$
1	$\cos\dfrac{\pi y^2}{2}$	$\sin\left(\dfrac{\pi r^2}{2}\right) C[(1-r^2)^{1/2}] + \cos\left(\dfrac{\pi r^2}{2}\right) S[(1-r^2)^{1/2}]$	$-\infty$
2	$\frac{4}{3}(1-y^2)^{3/2}$	$1-r^2$	-2
3	$(1-y^2)^{1/2} + \cos(\pi y^2) C[(2-2y^2)^{1/2}]/\sqrt{2} - \sin(\pi y^2) S[(2-2y^2)^{1/2}]/\sqrt{2}$	$\cos^2\left(\dfrac{\pi r^2}{2}\right)$	0

(1) $C[h] = \int_0^h \cos\dfrac{\pi v^2}{2} dv$ and $S[h] = \int_0^h \sin\dfrac{\pi v^2}{2} dv$ (Fresnel integrals).

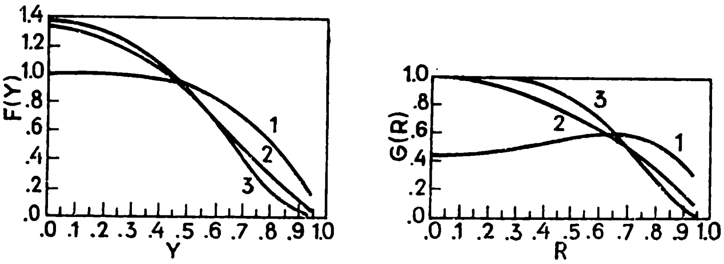

Fig. 2. Test function used to examine the interpolation methods for solving the Abel integral equation

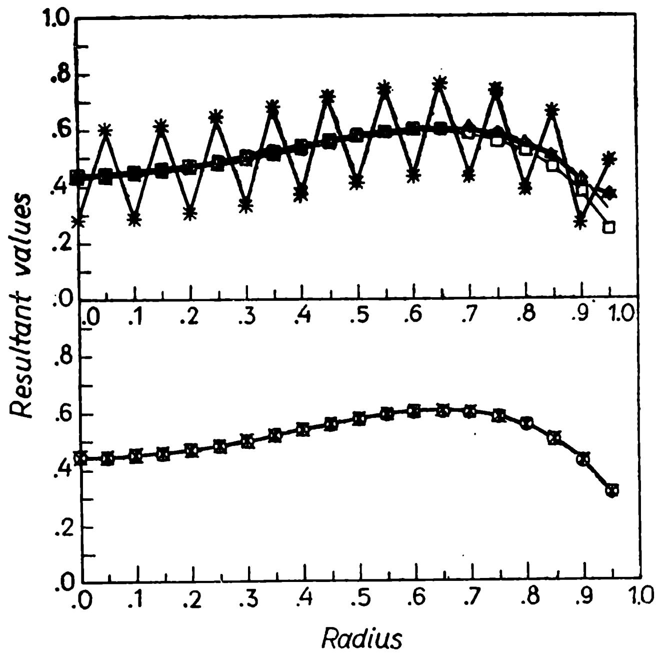

Fig. 3. An example of numerical solution of the Abel integral equation (test function no. 1). Points correspond to values calculated by means of various interpolation methods. Continuous lines correspond to the analytical solution

□ method A1, * method A2, △ method A3, + method A4, ◇ method A5
× method B1, | method B2, ○ method B3

C. Comparison of standard deviations. To facilitate the comparison of the accuracy of different interpolation methods, a standard deviation [3] has been calculated for each set of results

$$S = \left(\frac{1}{N}\sum_{i=0}^{N-1} W_i[g_i - g_e(r_i)]^2\right)^{1/2}.$$

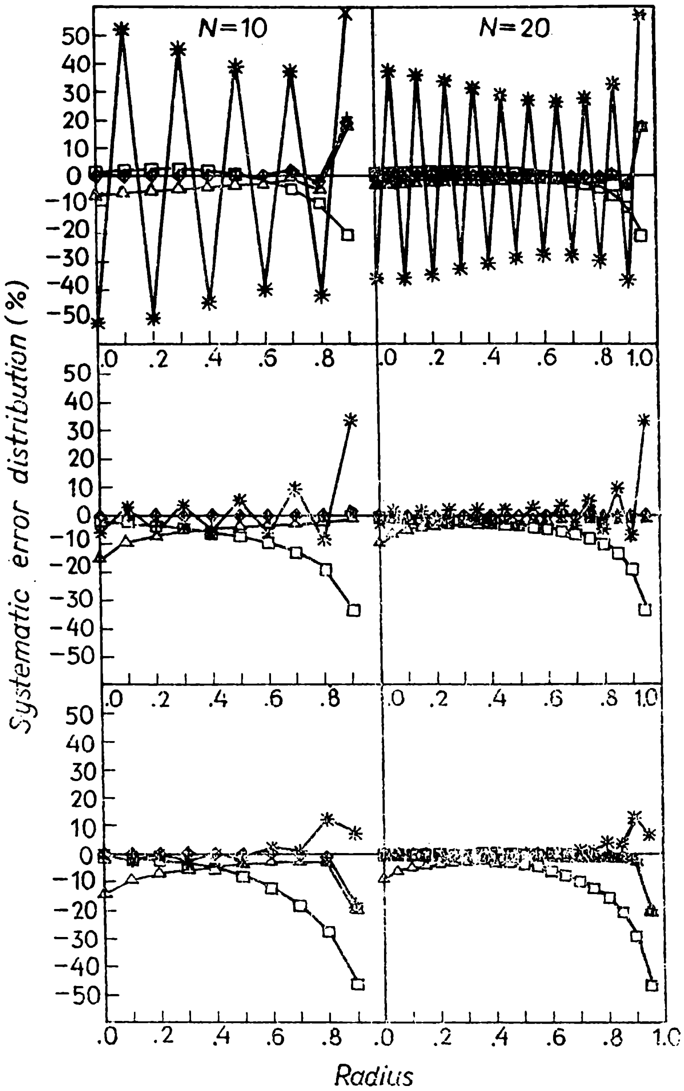

Fig. 4A. Systematic error radial distribution of the Abel integral equation solution by means of some A-type interpolation methods. Test functions from Table 4 (Fig. 2). The upper part corresponds to the test function no. 1, the middle part to no. 2, and the bottom part to no. 3

□ method A1, * method A2, △ method A3, + method A4, ◇ method A5

Equal weights $W_i \equiv 1$ are assumed. Other values of W_i can be used if a specific region of radial distances is to be stressed. The computed standard deviations for different methods, different test functions and different numbers of zones are compared in Table 5. The systematic error introduced by each of the methods has been compared by arranging the values of the standard deviations from Table 5. The results are given in Table 6.

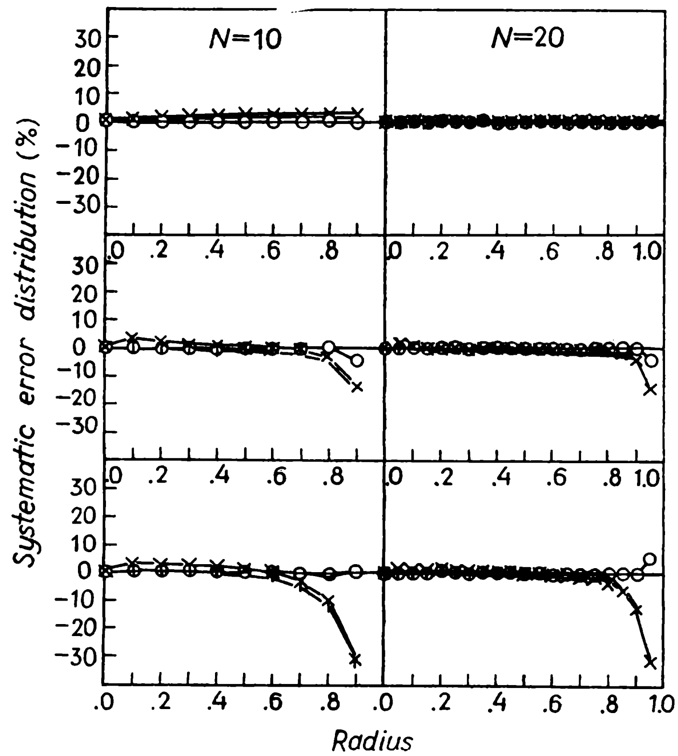

Fig. 4B. Systematic error radial distribution of the Abel integral equation solution by means of some B-type interpolation methods. Test functions from Table 4 (Fig. 2). The upper part corresponds to the test function no. 1, the middle part to no. 2, and the bottom part to no. 3

× method B1, | method B2, ○ method B3

TABLE 5. Standard deviations of the Abel integral equation solution by some interpolation methods

Number of zones	Test function no.	S							
		A1	A2	A3	A4	A5	B1	B2	B3
10	1	.0347	.2326	.0326	.0278	.0260	.0131	.0075	.0006
	2	.0510	.0459	.0642	.0018	.0000	.0166	.0118	.0028
	3	.0577	.0174	.0631	.0057	.0056	.0210	.0164	.0010
20	1	.0217	.1636	.0166	.0133	.0129	.0046	.0027	.0009
	2	.0264	.0220	.0340	.0004	.0000	.0073	.0047	.0012
	3	.0299	.0060	.0330	.0013	.0013	.0082	.0060	.0009

TABLE 6. Interpolation methods of the Abel integral equation solution arranged according to the increasing standard deviation values

Test function no.	Place							
	1	2	3	4	5	6	7	8
1	B3	B2	B1	A5	A4	A3	A1	A2
2	A5	A4	B3	B2	B1	A2	A1	A3
3	B3	A5	A4	B2	A2	B1	A1	A3

5. DISCUSSION

1° The random error amplification grows fast with the decreasing radius and also with the number of zones. It means that the conclusions given by Bockasten for his method can be extended to cover all the methods considered. A comparison of the random error amplification distributions in various methods permits us to divide them into three groups. All the methods of B-type (Gorenflo's, Nestor's and Olsen's, Bockasten's) and Mach's method belong to the first group having smallest random error amplification. Other A-type methods (Pearce's, van Voorhis', Frie's) belong to the second group. In the Pikalov and Preobrazhensky method, random error amplifications are considerably greater. The systematic error decreases with an increase of the number of zones for all $i < N-1$. In other words, if a larger N is chosen, a better accuracy can be achieved at all points except the exterior one, but, as has been noted above, the incertitude of the result increases for larger N.

2° The methods of A-type (except Mach's) seem to affect the character of the solution, since they introduce "oscillations" not existing in an analytical solution. This phenomenon appears in the cases where the variation of the function processed in the exterior zone does not agree with the type of interpolation. For illustration, a part of Fig. 3 is presented in Fig. 5. It is obvious that in the exterior zone ($r \in \langle R-w, R\rangle$) the Mach method would best approximate the test function no. 1, the van Voorhis or Frie method — the test function no. 2, and the Pikalov and Preobrazhensky method — the test function no. 3.

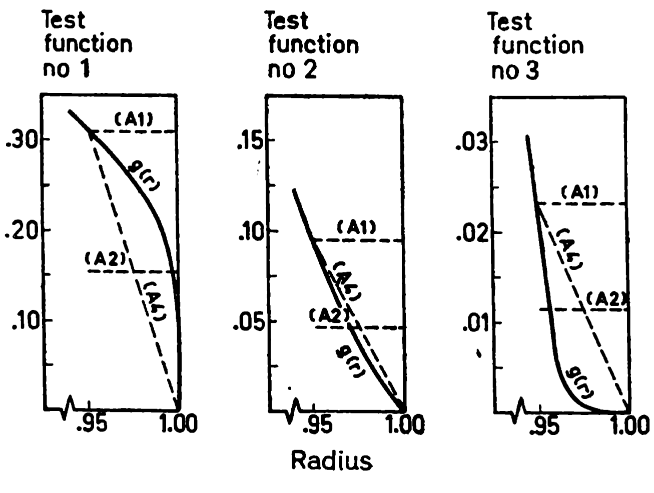

Fig. 5. Different kinds of interpolation in the exterior zone of the test function used. Continuous lines correspond to the analytical solution. Broken lines correspond to the interpolation used

Oscillations are generated, since a positive value of the systematic error appearing during computing the first equation of (7) becomes negative in the second step, positive in the third step, and so on.

"Oscillations" in the solution of the Abel inverse integral equation (all the B-type methods) have not been observed, since the result of solving any of equations (4) does not depend on the preceding results. Thus, the methods based on the interpolation of the data function $f(y)$ are safer for processing functions of unknown character.

6. CONCLUSIONS

The method presented in Section 3 belongs to the group of methods less sensitive to random error propagation. For all the test functions used, the systematic error introduced by this method appears to be smaller than in the Mach method and in the Pearce method; for some test functions it appears to be smaller than the error introduced by the Pikalov and Preobrazhensky method, the van Voorhis method, and the Frie method. The modified method is characterized by the Abel matrix B; it does not introduce "oscillations" which can occur by using one of methods characterized by the Abel matrix A. Formula (10) defining the elements of the matrix D is simple.

Taking into account all the above-mentioned properties, this method is to be preferred to all the A-type techniques. Nevertheless, the Bockasten method seems to be most worth-while among all techniques considered; the Frie method or van Voorhis method could be recommended only in the cases where the experimental curve is plotted with a good accuracy, and the variation of its slope in the last exterior zone is insignificant.

Acknowledgement. The author wishes to express his gratitude to Dr. M. Sadowski for his valuable comments on the manuscript.

APPENDIX

Calculations for Section 3. The region of integration $y \in \langle 0, R\rangle$ in (2) is divided into N segments. In the i-th segment ($y \in \langle y_i, y_{i+1}\rangle$, $i = 0, 1, \ldots, N-1$), the data function $f(y)$ is interpolated by the analytical expression $f_i(y)$. Then equation (2) takes the form

$$g_i = g(r_i) = -\frac{1}{\pi}\sum_{k=i}^{N-1}\int_{r_k}^{r_{k+1}} \frac{df_k(y)}{dy}\frac{dy}{(y^2-r_i^2)^{1/2}}. \tag{12}$$

The linear interpolation

$$f_i(y) = f_i + \frac{f_{i+1} - f_i}{y_{i+1} - y_i}(y - y_i)$$

is assumed. To avoid singularity of integral (12) for $i = 0$, the interpolation given above is used only for $i = 1, 2, \ldots, N-1$. For such a function, equation (2) takes the form

$$g_i = -\frac{1}{\pi}\sum_{k=i}^{N-1} \frac{f_{k+1} - f_k}{y_{k+1} - y_k} I_{i,k},$$

where

$$I_{i,k} = \int_{r_k}^{r_{k+1}} \frac{dy}{(y^2 - r_i^2)^{1/2}} = \ln \frac{r_{k+1} + (r_{k+1}^2 - r_i^2)^{1/2}}{r_k + (r_k^2 - r_i^2)^{1/2}}.$$

Thus, since $y_i = r_i$, we have

$$g_i = \frac{1}{\pi}\left(\frac{f_i I_{i,i}}{r_{i+1} - r_i} + \sum_{k=i+1}^{N-1} \frac{f_k I_{i,k}}{r_{k+1} - r_k} - \sum_{k=i}^{N-1} \frac{f_{k+1} I_{i,k}}{r_{k+1} - r_k}\right).$$

In the second sum the index $k+1$ is replaced by k, and the index $N-1$, which becomes N, is replaced by $N-1$, since $f_N = 0$. Then

$$g_i = \frac{1}{\pi}\left[\frac{f_i I_{i,i}}{r_{i+1} - r_i} + \sum_{k=i+1}^{N-1} f_k\left(\frac{I_{i,k}}{r_{k+1} - r_k} - \frac{I_{i,k-1}}{r_k - r_{k-1}}\right)\right] = \sum_{k=i}^{N-1} B_{i,k} f_k,$$

where

$$B_{i,k} = \begin{cases} D_{i,k} & \text{for } k = i > 0, \\ D_{i,k} - D_{i,k-1} & \text{for } k > i > 0, \end{cases}$$

and

$$D_{i,k} = \frac{I_{i,k}}{\pi(r_{k+1} - r_k)} = \frac{1}{\pi(r_{k+1} - r_k)} \ln \frac{r_{k+1} + (r_{k+1}^2 - r_i^2)^{1/2}}{r_k + (r_k^2 - r_i^2)^{1/2}}.$$

For $i = 0$, a linear in y^2 variation of the data function in the first segment is assumed:

$$f_0(y) = f_0 + \frac{f_1 - f_0}{y_1^2} y^2.$$

For such a function, equation (12) takes the form

$$g_0 = -\frac{1}{\pi}\int_0^{r_1} \frac{df_0(y)}{dy}\frac{dy}{y} - \frac{1}{\pi}\sum_{k=1}^{N-1}\int_{r_k}^{r_{k+1}} \frac{df_k(y)}{dy}\frac{dy}{y}$$

$$= -\frac{2}{\pi}\frac{f_1 - f_0}{r_1} - \frac{1}{\pi}\sum_{k=1}^{N-1} \frac{f_{k+1} - f_k}{r_{k+1} - r_k} I_{0,k} = -\frac{1}{\pi}\sum_{k=0}^{N-1} \frac{f_{k+1} - f_k}{r_{k+1} - r_k} I_{0,k},$$

where $I_{0,0} = 2$.

Finally, for all $i = 0, 1, \ldots, N-1$ we get

$$g_i = \sum_{k=i}^{N-1} B_{i,k} f_k, \tag{13}$$

where

$$B_{i,k} = \begin{cases} D_{i,k} & \text{for } k = i, \\ D_{i,k} - D_{i,k-1} & \text{for } k > i, \end{cases}$$

and

$$D_{i,k} = \begin{cases} \dfrac{2}{\pi r_1} & \text{for } i+k = 0, \\ \dfrac{1}{\pi (r_{k+1} - r_k)} \ln \dfrac{r_{k+1} + (r_{k+1}^2 - r_i^2)^{1/2}}{r_k + (r_k^2 - r_i^2)^{1/2}} & \text{for } i+k > 0. \end{cases}$$

For segments of equal width $w = R/N$, equation (13) becomes (4), where $b_{i,k} = wB_{i,k}$, and $d_{i,k} = wD_{i,k}$ are given by (10).

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ROZWIĄZYWANIE RÓWNANIA CAŁKOWEGO ABELA METODAMI INTERPOLACYJNYMI

STRESZCZENIE

W pracy porównano metody rozwiązania równania całkowego Abela, oparte na interpolacji danych pomiarowych równomiernie rozmieszczonych. Okazało się, że wzory podawane w różnych pracach są wariantami jednego z dwóch podstawowych równań algebraicznych. Do znanych z literatury metod dodano zmodyfikowaną metodę, opartą na liniowej interpolacji funkcji danych pomiarowych. Określono propagację przypadkowych błędów pomiarowych dla równania całkowego Abela w postaci algebraicznej. Porównano wrażliwość różnych metod na propagację błędów przypadkowych. Porównano również błędy systematyczne, wprowadzane przez każdą z metod, używając w tym celu funkcji próbnych różnego rodzaju. Zbadano zależność obu tych własności od liczby punktów pomiarowych. Wysunięto sugestię, dotyczącą wyboru właściwej metody dla określonej funkcji danych eksperymentalnych.
