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ON THE STRONG SUMMABILITY AND APPROXIMATION OF FOURIER SERIES

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In the present paper we give a survey of the results concerning the subject mentioned in the title and connected with the author's own work.

Let $f(x)$ be an integrable and 2π-periodic function and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1.1}$$

be its Fourier series. Denote by $s_n(x) = s_n(f; x)$ and $\sigma_n^\alpha(x) = \sigma_n^\alpha(f; x)$ the nth partial sum and the (C, α)-mean of (1.1); furthermore, denote by $\tilde{f}(x)$, $\tilde{s}_n(x)$, and $\tilde{\sigma}_n^\alpha(x)$ the conjugate functions of $f(x)$, $s_n(x)$, and $\sigma_n^\alpha(x)$, respectively.

First we mention some classical results.

Fejér (1904) proved that if $f(x)$ is a continuous function, then

$$\sigma_n(x) \equiv \sigma_n^1(x) \to f(x) \tag{1.2}$$

uniformly; more precisely, he proved that (1.2) holds at any point of continuity.

Lebesgue (1905) verified that if $f(x)$ is an integrable function, then (1.2) holds almost everywhere.

Privalov (1919) proved that if $f(x)$ is integrable, then

$$\tilde{\sigma}_n(x) - \tilde{f}(x, 1/n) \to 0 \quad \text{as} \quad n \to \infty \tag{1.3}$$

almost everywhere, where

$$\tilde{f}(x, \varepsilon) = -\frac{1}{\pi} \int_{\varepsilon}^{\pi} \frac{f(x+t) - f(x-t)}{2\,\mathrm{tg}(t/2)}\, dt.$$

Having the following result of Marcinkiewicz: if $f(x)$ is integrable, then there exists a $\lim\limits_{\varepsilon\to 0} \tilde{f}(x, \varepsilon) = \tilde{f}(x)$ almost everywhere, we can write (1.3)

$$\tilde{\sigma}_n(x) - \tilde{f}(x) \to 0 \text{ a.e.} \tag{1.4}$$

In 1912 Bernstein proved that if $f(x) \in \mathrm{Lip}\,\alpha$, then

(1.5) $$\sigma_n(x)-f(x) = O(n^{-\alpha}) \quad \text{for} \quad 0<\alpha<1,$$

and

(1.6) $$\sigma_n(x)-f(x) = O(n^{-1}\log n) \quad \text{for} \quad \alpha = 1.$$

M. Riesz (1923) generalized these results by proving that the means $\sigma_n(x)$ can be replaced by $\sigma_n^{\gamma}(x)$ with any positive γ.

Since $f(x) \in \mathrm{Lip}\,\alpha$ implies $\tilde f(x) \in \mathrm{Lip}\,\alpha$ if $\alpha < 1$ but does not imply $\tilde f(x) \in \mathrm{Lip}\,1$ if $\alpha = 1$ by the results of Bernstein, we have

$$\tilde\sigma_n(x)-\tilde f(x) = O(n^{-\alpha}) \quad \text{for} \quad 0<\alpha<1.$$

Therefore, the theorem of Alexits [1] (see also Zygmund [28] and Zamansky [26]) stating that $f(x) \in \mathrm{Lip}\,1$ if and only if

(1.7) $$\tilde\sigma_n(x)-\tilde f(x) = O(1/n),$$

seems to be a very surprising and interesting result.

In connection with strong summability we only recall the following results:

Marcinkiewicz [21] proved that for any integrable function $f(x)$ we have means

(1.8) $$\frac{1}{n+1}\sum_{\nu=0}^{n} |s_\nu(x)-f(x)|^p \to 0 \text{ a.e.}$$

with $p = 2$; and (1.8) for any positive p was verified by Zygmund [27]. Zygmund also proved that (1.8) holds at any point of continuity of $f(x)$.

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In [12] we generalized (1.8) by proving that if $f(x)$ is integrable, then we have

(2.1) $$\frac{1}{A_n^{\gamma}}\sum_{k=0}^{n} A_{n-k}^{\gamma-1}|s_k(x)-f(x)|^p \to 0 \text{ a.e.}$$

for any positive p and γ, where $A_n = \binom{n+\gamma}{n}$.

In connection with (2.1) we can mention the following open problem: Can we replace the partial sums $s_k(x)$ in (2.1) by (C, β)-means with negative β?

As regards strong approximation, Alexits and Králik [4] proved that if $f(x) \in \mathrm{Lip}\,\alpha$, $0<\alpha<1$, then

(2.2) $$h_n(x) = \frac{1}{n+1}\sum_{\nu=0}^{n} |s_\nu(x)-f(x)|^{-p} = O(n^{-\alpha}),$$

and consequently

$$\tilde h_n(x) = \frac{1}{n+1}\sum_{\nu=0}^{n} |\tilde s_\nu(x)-\tilde f(x)| = O(n^{-\alpha}).$$

This shows that strong approximation for the class $\mathrm{Lip}\,\alpha$, $0<\alpha<1$ has the same approximation order as ordinary approximation. In connection with this it is natural to ask: Do we have, for all $f(x) \in \mathrm{Lip}\,1$,

(2.3) $$h_n(x) = O\left(\frac{\log n}{n}\right)$$

and

(2.4) $$\tilde h_n(x) = O\left(\frac{1}{n}\right)?$$

The answer is affirmative with regard to (2.3) and negative with regard to (2.4). (See Alexits and Leindler [6].) From this point of view the class $\mathrm{Lip}\,1$ with respect to strong approximation seems to have a certain extra property. But later we shall see that this is not the case.

In connection with strong approximation we proved ([9], Satz I) a very general theorem whose two special cases will be presented here.

THEOREM 1. *If $f(x)$ has a continuous r-th derivative and $f^{(r)}(x) \in \mathrm{Lip}\,\alpha$, $0<\alpha\leqslant 1$, then for any $p>0$*

$$\left\{\frac{1}{n}\sum_{k=n+1}^{2n} |s_k(x)-f(x)|^p\right\}^{1/p} = O(n^{-r-\alpha}),$$

and if $\beta > (r+\alpha)p$, then

$$h_n(f,p,\beta;x) = \left\{\frac{1}{(n+1)^{\beta}}\sum_{k=0}^{n} (k+1)^{\beta-1}|s_k(x)-f(x)|^p\right\}^{1/p} = (n^{-r-\alpha})$$

uniformly. The same estimations also hold for $\tilde f(x)$.

THEOREM 2. *If $f^{(r)}(x) \in \mathrm{Lip}\,\alpha$, $0<\alpha\leqslant 1$, $p>0$ and $(r+\alpha)p<1$, then for arbitrary $\gamma>0$*

(2.5) $$\sigma_n^{\gamma}|f,p;x| = \left\{\frac{1}{A_n^{\gamma}}\sum_{k=0}^{n} A_{n-1}^{\gamma-1}|s_k(x)-f(x)|^p\right\}^{1/p} = O(n^{-r-\alpha}).$$

This also holds with $\tilde f(x)$.

Comparing the results obtained with that of Jackson, we can see that the order of strong approximation is equal to that of the best approximation obtained by trigonometric polynomials of order at most n.

Earlier it was proved by Alexits [2] that if $f(x) \in \mathrm{Lip}\,\alpha$ and $\alpha<\gamma$, then

$$\frac{1}{A_n^{\gamma}}\sum_{k=0}^{n} A_{n-k}^{\gamma-1}|s_k(x)-f(x)| = O(n^{-\alpha}).$$

The following theorems show that the conditions $\beta>(r+\alpha)p$ and $(r+\alpha)p<1$ are essential with respect to the order of approximation.

THEOREM 3 ([9]). *If* $f^{(r)}(x) \in \operatorname{Lip}\alpha$, $0 < \alpha \leqslant 1$, $p > 0$ *and* $\beta = (r+\alpha)p$, *then we have only*

$$h_n(f, p, \beta; x) = O\left(\frac{(\log n)^{1/p}}{n^{r+\alpha}}\right)$$

and

$$h_n(\tilde{f}, p, \beta; x) = O\left(\frac{(\log n)^{1/p}}{n^{r+\alpha}}\right).$$

Furthermore, there exist $f_1(x)$ and $f_2(x)$ such that $f_i^{(r)}(x) \in \operatorname{Lip}\alpha$ $(i = 1, 2)$, but

$$h_n(f_1, p, \beta; 0) \geqslant c\left(\frac{(\log n)^{1/p}}{n^{r+\alpha}}\right) \tag{2.6}$$

and

$$h_n(\tilde{f}_2, p, \beta; 0) \geqslant c\left(\frac{(\log n)^{1/p}}{n^{r+\alpha}}\right)$$

with a positive $c = c(p, \beta)$.

THEOREM 4 ([10]). *If* $f^{(r)}(x) \in \operatorname{Lip}\alpha$, $0 < \alpha \leqslant 1$, $p, \gamma > 0$ *and* $(r+\alpha)p = 1$, *then we have only*

$$\sigma_n^\gamma|f, p; x| = O\left(\frac{(\log n)^{1/p}}{n^{r+\alpha}}\right)$$

and

$$\sigma_n^\gamma|\tilde{f}, p; x| = O\left(\frac{(\log n)^{1/p}}{n^{r+\alpha}}\right).$$

Furthermore, there exist functions with $f_i^{(r)}(x) \in \operatorname{Lip}\alpha$ $(i = 1, 2)$ *such that*

$$\sigma_n^\gamma|f_1, p; 0| \geqslant d\frac{(\log n)^{1/p}}{n^{r+\alpha}} \quad \text{and} \quad \sigma_n^\gamma|\tilde{f}_2, p; 0| \geqslant d\frac{(\log n)^{1/p}}{n^{r+\alpha}}$$

with $d = d(p, \gamma) > 0$.

These theorems show very clearly that with respect to strong approximation the class Lip 1 has no extra property, namely any other class $\operatorname{Lip}\alpha$ has the same properties if $\beta = (r+\alpha)p$ or $(r+\alpha)p = 1$.

The counterexamples $f_1(x)$ and $f_2(x)$ of the previous theorems are the following functions:

If r is an odd integer, then

$$f_1(x) = \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu+1}}{2^{\nu\alpha}} \sum_{l=2^{\nu-1}+1}^{2^\nu} \left(\frac{\cos(5\cdot 2^\nu - l)x}{(5\cdot 2^\nu - l)^r l} - \frac{\cos(5\cdot 2^\nu + l)x}{(5\cdot 2^\nu + l)^r l}\right).$$

$$f_2(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k^{r+1+\alpha}},$$

and if r is even, then

$$f_1(x) = \sum_{k=1}^{\infty} \frac{\cos kx}{k^{r+1+\alpha}},$$

$$f_2(x) = \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu+1}}{2^{\nu\alpha}} \sum_{l=2^{\nu-1}+1}^{2^\nu} \left(\frac{\sin(5\cdot 2^\nu - l)x}{(5\cdot 2^\nu - l)^r l} - \frac{\sin(5\cdot 2^\nu + l)x}{(5\cdot 2^\nu + l)^r l}\right).$$

To prove e.g. that the rth derivative of the first function with $\alpha = 1$ belongs to the class Lip 1 was a difficult task.

As we have mentioned, it is well known that if $\alpha < 1$, then the condition $f(x) \in \operatorname{Lip}\alpha$ implies that $\tilde{f}(x) \in \operatorname{Lip}\alpha$, but this is not the case if $\alpha = 1$. Thus, in connection with the estimation (2.6), it is natural to ask whether the conditions $f^{(r)}(x) \in \operatorname{Lip} 1$ and $\tilde{f}^{(r)}(x) \in \operatorname{Lip} 1$ imply that

$$h_n(f, p, (r+1)p; x) = O(n^{-r-1}),$$

or equivalently, that the partial sums of the series

$$\sum_{k=1}^{\infty} k^{(r+1)p-1}|s_k(x) - f(x)|^p$$

are uniformly bounded. The answer is negative; namely we have

THEOREM 5 ([16]). *There exists a function* $f_0(x)$ *such that* $f_0^{(r)}(x)$ *and* $\tilde{f}_0^{(r)}(x)$ *belong to the class*, Lip 1; *moreover for any positive* p

$$\sum_{k=1}^{m} k^{(r+1)p-1}|s_k(f_0, 0) - f_0(0)|^p \geqslant K(r, p)\log m,$$

where $K(r, p)$ *is a positive constant depending on* r *and* p.

The special case $p = 1$, $r = 0$ and $0 < \alpha < 1$ of Theorem 2 was generalized by Sunouchi [22]; he replaced the partial sums in (2.5) by (C, β)-means, where β can also be negative. We continued the generalization and proved

THEOREM 6 ([15]). *Suppose that* $f(x) \in \operatorname{Lip}\alpha$ *for some* $0 < \alpha < 1$, *that* $\beta > -1/2$ *and that the positive number* p *satisfies the inequality* $p\beta > -1$. *Suppose further that* $\gamma > \max(0, -p\beta)$. *Then we have*

$$\left\{\frac{1}{A_n^\gamma}\sum_{k=0}^{n} A_{n-k}^{\gamma-1}|\sigma_k^\beta(x) - f(x)|^p\right\}^{1/p} = \begin{cases} O(n^{-\alpha}) & \text{if } p\alpha < 1, \\ O\left(\frac{(\log n)^{1/p}}{n}\right) & \text{if } p\alpha = 1, \\ O(n^{-1/p}) & \text{if } p\alpha > 1. \end{cases}$$

A similar theorem in the case $f^{(r)}(x) \in \operatorname{Lip}\alpha$ is not yet proved. The case $f(x) \in \operatorname{Lip} 1$ has not been investigated, either.

In [17] we verified that these estimations are the best possible if $\beta \geqslant 0$. To verify this for $-1/2 < \beta < 0$ is also an open problem. We proved ([15], [17]) similar estimations for the following strong means:

$$\left\{\frac{1}{(n+1)^{\delta}}\sum_{k=0}^{n}(k+1)^{\delta-1}|\sigma_k^{\beta-1}(x)-f(x)|^p\right\}^{1/p}.$$

In [10] we investigated, among other things, the strong Riesz-means

$$\left\{\frac{1}{\lambda(n+1)}\sum_{k=0}^{n}\big(\lambda(k+1)-\lambda(k)\big)\,|s_k(x)-f(x)|^p\right\}^{1/p},$$

and gave conditions on $\{\lambda(k)\}$ which imply that these means have the order of the best approximation obtained by trigonometric polynomials of order n.

We also defined generalized strong de la Vallée Poussin means

$$V_n(f,\lambda,p;x)=\left\{\frac{1}{\lambda_n}\sum_{\nu=n-\lambda_n}^{n-1}|s_\nu(x)-f(x)|^p\right\}^{1/p},$$

where $\lambda=\{\lambda_n\}$ is a nondecreasing sequence of integers such that $\lambda_1=1$ and $\lambda_{n+1}-\lambda_n\leqslant 1$.

It is easy to see that these means include the strong $(C,1)$-means ($\lambda_n=n$, $V_{n+1}=\sigma_n$) and the proper strong de la Vallée Poussin means ($\lambda_n=[n/2]$, $[x]$ denotes the integral part of x), as special cases.

In [12] we proved, among other things, the following three theorems:

THEOREM 7. *Suppose that $n=O(\lambda_n)$. Then, for any continuous $f(x)$ and $p>0$, we have*

$$\left\{\frac{1}{\lambda_n}\sum_{k=n-\lambda_n}^{n-1}|s_k(x)-f(x)|^p\right\}^{1/p}=O(E_{n-\lambda_n}),$$

where $E_n=E_n(f)$ denotes the best approximation of $f(x)$ by trigonometric polynomials of order at most n.

See also Alexits–Králik [5] (Satz 1).

THEOREM 8. *Suppose that $f^{(r)}(x)\in\operatorname{Lip}\alpha$, $0<\alpha\leqslant 1$, and that $n=O(\lambda_n)$. Then for any $p>0$*

$$V_n(f,\lambda,p;x)=\begin{cases}O\left(\dfrac{1}{n^{r+\alpha}}\right) & \text{for}\quad (r+\alpha)p<1,\\[2ex] O\left(\dfrac{1}{n^{r+\alpha}}\left(1+\log\dfrac{n}{n-\lambda_n+1}\right)^{1/p}\right) & \text{for}\quad (r+\alpha)p=1.\end{cases}$$

The same estimate is also valid for the conjugate function $\tilde{f}(x)$.

Furthermore, if $(r+\alpha)p=1$, $0<\alpha\leqslant 1$, then there exist functions $f_1(x)$ and $f_2(x)$ such that their r-th derivatives exist and belong to $\operatorname{Lip}\alpha$; moreover,

$$\overline{\lim_{n\to\infty}}\,V_n(f_1,\lambda,p;0)\geqslant\frac{c}{n^{r+\alpha}}\left(1+\log\frac{n}{n-\lambda_n+1}\right)^{1/p}$$

and

$$\overline{\lim_{n\to\infty}}\,V_n(\tilde{f}_2,\lambda,p;0)\geqslant\frac{c}{n^{r+\alpha}}\left(1+\log\frac{n}{n-\lambda_n+1}\right)^{1/p},$$

where $c>0$ is independent of n.

THEOREM 9. *If $f(x)$ is integrable, then for any positive p and δ*

$$V_n(f,\lambda,p;x)=o_x\left(\left(\frac{n}{\lambda_n}\right)^{\delta}\right)\ a.e.$$

Very recently we proved similar theorems without the restriction $n=O(\lambda_n)$. For example, we have ([19])

THEOREM 10. *If $f^{(r)}(x)\in\operatorname{Lip}\alpha$ $(0<\alpha\leqslant 1)$, then for any $p>0$*

$$V_n(f,\lambda,p;x)=\begin{cases}O\left(\left(\dfrac{n}{\lambda_n}\right)^{1/p}\dfrac{1}{n^{r+\alpha}}\right) & \text{if}\quad (r+\alpha)p<1,\\[2ex] O\left(\dfrac{1}{\lambda_n^{r+\alpha}}\left(1+\log\dfrac{n}{n-\lambda_n+l}\right)^{1/p}\right) & \text{if}\quad (r+\alpha)p=1,\\[2ex] O(\lambda_n^{-1/p}) & \text{if}\quad (r+\alpha)p>1.\end{cases}$$

The same estimate holds for $V_n(\tilde{f},\lambda,p,x)$.

These estimations, in general, are not the best possible. To give the best possible estimations is also an open problem.

We can also show that the partial sums in the means $V_n(f,\lambda,p;x)$ can be replaced by (C,β)-means of negative order. The restrictions on β are $\beta>-1/2$ and $p\beta>-1$ (see [19], Theorem 4).

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One other important problem of the theory of strong approximation by Fourier series is to deduce structural properties of the function $f(x)$ from the estimations given for the strong means.

By Theorem 3, in the special case $r=0$, $\beta=1$ and $\alpha=1/p$, $p>1$; for the whole class $\operatorname{Lip}(1/p)$ we have only the estimation

$$h_n(f,p,1;x)=\left\{\frac{1}{n}\sum_{k=1}^{n}|s_k(x)-f(x)|^p\right\}^{1/p}=O\left(\left(\frac{\log n}{n}\right)^{1/p}\right).$$

Freud [7] raised the following question: If we know that a function $f(x)$ has the property

$$h_n(f,p,1;x)=O(n^{-1/p}),\quad p>1,$$

for all x, or equivalently

(3.1) $$\sum_{k=1}^{\infty}|s_k(x)-f(x)|^p\leqslant K,\quad p>1,$$

what can we say about the function $f(x)$?

He proved that (3.1) implies

$$\lim_{h\to+0}\,(f(x+h)-f(x))h^{-1/p}=0 \tag{3.2}$$

almost everywhere.

In his paper the following problem was raised: Is the assertion (3.2) true for all x?

We answered this problem negatively ([13]); that is, we gave a function such that the estimation (3.2) is not fulfilled in $x=0$, but (3.1) holds. Our counter-example is

$$f_0(x)=\sum_{n=1}^{\infty}\frac{\sin nx}{n^{1+1/p}}.$$

The case $p=1$ was investigated in the paper of Leindler and Nikisin [20].

THEOREM 11. *If*

$$\sum_{n=0}^{\infty}|s_n(x)-f(x)|\leqslant K$$

for all x, then

$$f(x+h)-f(x)=O_x(h)$$

for almost all x, and

$$|f(x+h)-f(x)|\leqslant K_1 h\log(1/h)$$

for all x.

Furthermore, there exists a function $f_0(x)$ such that

$$\sum_{n=0}^{\infty}|s_n(f_0;x)-f_0(x)|\leqslant K$$

for all x, but

$$f_0\left(\frac{\pi}{2^n}\right)-f_0(0)\geqslant\frac{1}{8}\frac{\pi}{2^n}\log\frac{2^n}{\pi},\qquad n\geqslant 6.$$

This last result was generalized in [16] as follows:

THEOREM 12. *If r is a nonnegative integer and*

$$\sum_{n=1}^{\infty}n^r|s_n(x)-f(x)|\leqslant K, \tag{3.3}$$

for all x, then we have

$$f^{(r)}(x+h)-f^{(r)}(x)=O_x(h), \tag{3.4}$$

for almost all x; furthermore,

$$|f^{(r)}(x+h)-f^{(r)}(x)|\leqslant K_1 h\log(1/h),$$

for all x. Estimate (3.4) also holds with $\tilde f^{(r)}(x)$ instead of $f^{(r)}(x)$.

If r is even, then (3.3) implies that

$$\tilde f^{(r)}(x)\in \mathrm{Lip}\,1,$$

and if r is an odd integer, then it implies that

$$f^{(r)}(x)\in\mathrm{Lip}\,1.$$

Furthermore, there exists a function $f_0(x)$ having the following properties:

$$\sum_{n=1}n^r|s_n(f_0;x)-f_0(x)|\leqslant K$$

for all x, and if r is an even integer, then

$$\left|f_0^{(r)}\left(\frac{\pi}{2^m}\right)-f_0^{(r)}(0)\right|>\frac{1}{4}\frac{\pi}{2^m}\log\frac{2^m}{\pi}, \tag{3.5}$$

$$\left|\tilde f_0^{(r)}\left(\frac{\pi}{2^m}\right)-\tilde f_0^{(r)}(0)\right|>\frac{1}{2}\frac{\pi}{2^m} \tag{3.6}$$

for all $m\geqslant 10$; and if r is an odd integer, then $f_0^{(r)}(x)$ and $\tilde f_0^{(r)}(x)$ have to be interchanged in (3.5) and (3.6).

In [14] we generalized the result of Freud, namely we deduced structural properties from the condition

$$\sum_{n=0}^{\infty}\lambda_n|s_n(x)-f(x)|^p\leqslant K,\qquad p\geqslant 1,\ \lambda_n>0.$$

Let λ_k be positive numbers, let $\Lambda_n=\sum_{k=0}^{n}\lambda_k$, and let $\Lambda(x)$ be an increasing function between n and $n+1$, linear and such that $\Lambda(n)=\Lambda_n$.

We have the following theorem with respect to this problem.

THEOREM 13. *Let $p\geqslant 1$. Suppose that $n\Lambda_n^{-1/p}$ is increasing,*

$$\sum_{k=0}^{n}\Lambda_k^{-1/p}\leqslant Kn\Lambda_n^{-1/p}, \tag{3.7}$$

$$\sum_{k=n}^{\infty}k^{-1}\Lambda_k^{-1/p}\leqslant K\Lambda_n^{-1/p},$$

$$\Big\{\sum_{k=n}^{2n}\lambda_k^{1/(1-p)}\Big\}^{p-1}\leqslant Kn^p\Lambda_n^{-1},\quad \text{if}\quad p>1,$$

and

$$\Lambda_n\leqslant Kn\lambda_n,\quad \text{if}\quad p=1.$$

Then, if

$$\sum_{k=0}^{\infty}\lambda_n|s_n(x)-f(x)|^p\leqslant K$$

for all x, then

(3.8) $$|f(x+h)-f(x)| \leqslant K_1 \Lambda^{-1/p}(1/h)$$

for all x; furthermore for almost all x we have

(3.9) $$\lim_{h\to+0} \Lambda^{1/p}(1/h)\big(f(x+h)-f(x)\big) = 0.$$

We have proved the estimate (3.8) is the best possible.

It is easy to verify that, for example, in the cases

$$\lambda_k = k^{\beta-1}, \quad 0<\beta<p,$$

$$\lambda_k = \frac{\log k}{k^\beta}, \quad 0<\beta<1,$$

the conditions of Theorem 13 are satisfied. Thus Theorem 13 includes the result of Freud, but it does not include Theorem 11. It would be an interesting task to give a generalization which includes both results. A similar theorem with the rth derivatives of $f(x)$ has not been investigated.

Theorem 11 can be interpreted in such a way that the condition

$$\sum_{n=0}^{\infty} |s_n(x)-f(x)| \leqslant K$$

does not imply that $f(x)\in \mathrm{Lip}\,1$. In connection with this I ([16], [18]) raised the following problem: Does the condition

(3.10) $$\sum_{n=0}^{\infty} |s_n(x)-f(x)|^p \leqslant K \quad \text{with} \quad 0<p<1$$

imply $f(x)\in\mathrm{Lip}\,1$?

The answer has recently been given in an affirmative form by Oskolkov [22] and by Szabados [24], independently. They have both proved a stronger statement. To formulate their result we denote by $\Omega(\delta)$ an arbitrary modulus of continuity, i.e. $\Omega(\delta)$ is positive, increasing, subaddative and $\lim_{\delta\to+0}\Omega(\delta)=0$. They have proved that if

$$\sum_{k=0}^{\infty} \Omega\big(|s_k(x)-f(x)|\big) \leqslant K$$

for all x and

(3.11) $$\int_0^1 \frac{dx}{\Omega(x)} < \infty,$$

then $f(x)\in\mathrm{Lip}\,1$.

Under a certain restriction on $\Omega(x)$ they have also proved the necessity of condition (3.11). As regards the additional conditions on $\Omega(x)$, Oskolkov and Szabados differ. Oskolkov claims the following condition:

$$\varlimsup_{x\to+0} \frac{\Omega(x/2)}{\Omega(x)} < 1.$$

It seems to be a difficult task to prove the necessity of (3.11) without an additional condition.

Szabados has also proved that if $0<p\leqslant 1$, $1/p = r+\alpha$, where $r=[1/p]$, then condition (3.10) implies that $f^{(r-1)}(x)$ is continuous and

$$\omega(f^{(r-1)};\delta) = \begin{cases} O\big(\delta\log(1/\delta)\big)^{1/p} & \text{if} \quad \alpha=0,\\ O(\delta) & \text{otherwise,}\end{cases}$$

where $\omega(f;\delta)$ denotes the modulus of continuity of $f(x)$.

In connection with this result I have the following conjecture: Condition (3.10) implies that

(1) $$f^{(r)}(x)\in\mathrm{Lip}\,\alpha, \quad \text{if} \quad \alpha>0,$$

(2) $$\omega(f^{(r-1)};\delta) = O\big(\delta\log(1/\delta)\big) \quad \text{if} \quad \alpha=0.$$

In this direction we could prove ([19]) that if $0<p<1$ and $1/p-r=\alpha>0$, then condition (3.10) implies that $f^{(r)}(x)$ is continuous and

$$\omega(f^{(r)};h) = O\Big(h^\alpha\big(\log(1/h)\big)^{1/p}\Big).$$

In a conversation Stečkin mentioned the following conjecture: The conditions

(3.12) $$\sum_{n=1}^{\infty} |s_n(x)-f(x)| \leqslant K$$

and

(3.13) $$\sum_{n=1}^{\infty} E_n(f) < \infty$$

are equivalent. Using Theorem 11, we can show that this conjecture is not true. Namely (3.12) does not imply that $f(x)\in\mathrm{Lip}\,1$, but (3.13) does, since

$$\omega\Big(f,\frac{1}{n}\Big) \leqslant \frac{1}{n}\sum_{k=1}^{n} E_k.$$

But, by Theorem 7, it is easy to verify that (3.13) implies (3.12), namely

$$\sum_{k=2^m}^{2^{m+1}} |s_k(x)-f(x)| \leqslant K 2^m E_{2^m}.$$

4

Finally we mention one more field of problems in connection with strong summability.

We have mentioned the following result of Zygmund [27]: At any point of continuity of $f(x)$

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}|s_k(x)-f(x)|^p = 0 \tag{4.1}$$

holds for any $p > 0$.

In connection with the result Turán [25] raised the following problem: Can one replace the exponent p in (4.1) by a sequence $\{\lambda_n\}$ increasing to infinity and such that

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}|s_k(x)-f(x)|^{\lambda_n} = 0$$

holds?

Turán gave a negative answer to this question, proving that for any sequence $\lambda_n \nearrow +\infty$ a continuous function can be given having the property

$$\overline{\lim_{n\to\infty}}\frac{1}{n+1}\sum_{k=0}^{n}|s_k(x_0)-f(x_0)|^{\lambda_n} = +\infty$$

at a suitable point x_0.

With respect to this negative result Alexists [3] raised the following problem: For a given sequence $\{\lambda_n\}$, what kind of function have the property

$$\lim_{n\to\infty} h_n(f, \lambda; x) \equiv \lim_{n\to\infty}\left\{\frac{1}{n+1}\sum_{k=0}^{n}|s_k(x)-f(x)|^{\lambda_n}\right\}^{1/\lambda_n} = 0 \tag{4.2}$$

either at a fixed point $x = x_0$ or in an interval?

Králik [8] gave sufficient conditions implying the property (4.2). To explain his result we require the following definition: A positive differentiable function $F(x)$ $(x > 0)$ increasing to infinity is said to belong to the class Λ if the functions $x^{1/2\lambda_n}/F(x)$ are monotone increasing for $x > n^{1/2\lambda_n}$ in case of sufficiently large n.

The theorem of Králik states: If $f(x)$ belongs to $L^{\lambda_n/(\lambda_n-1)}[-\pi, \pi]$ for sufficiently large n and if at x_0

$$f(x_0+t)+f(x_0-t)-2f(x_0) = O\big((F(1/t))^{-1}\big) \quad (t \to 0) \tag{4.3}$$

is fulfilled for some $F(x) \in \Lambda$, and furthermore $\lambda_n = O\big(\min(\log n, F(n))\big)$, then (4.2) holds. If (4.3) holds uniformly in $[c, d]$, then (4.2) also holds uniformly in any interval internal to (c, d).

We proved ([11]), among other things that (4.2) also holds for integrable functions satisfying (4.3) with a function $F(x)$ belonging to a larger class Λ.

First we define a new class $\Lambda(\varrho)$ of functions in the following way: A positive unction $\varphi(x)$ $(x > 0)$ increasing to infinity is said to belong to the class $\Lambda(\varrho)$ if $\lambda_n = (\varphi(n))$ and there exists a positive number $\varrho < 1$ such that the functions $x^{\varrho/\lambda_n}(\varphi(x))^{-1}$ are monotone nondecreasing for sufficiently large n.

We consider a regular summation method T_n determined by a triangular matrix $\left\|\dfrac{\alpha_{nk}}{A_n}\right\|$ $(\alpha_{nk} \geqslant 0$ and $A_n = \sum_{k=0}^{n}\alpha_{nk})$, i.e. if s_k tends to s, then

$$T_n = \frac{1}{A_n}\sum_{k=0}^{n}\alpha_{nk}s_k \to s.$$

Now we are in a position to formulate our

THEOREM 14. *If there exists a $p > 1$ such that*

$$\Big\{\sum_{k=1}^{n}\alpha_{nk}^{p}\Big\}^{1/p} \leqslant Kn^{(1-p)/p}A_n \tag{4.4}$$

and if $f(x)$ is integrable, and furthermore at x_0

$$f(x_0+t)+f(x_0-t)-2f(x_0) = O\big((\varphi(1/t))^{-1}\big) \quad (t \to +0) \tag{4.5}$$

is satisfied for any $\varphi(x) \in \Lambda(\varrho)$, $\varrho < (p-1)/p$, then we have

$$\lim_{n\to\infty}\left\{\frac{1}{A_n}\sum_{k=0}^{n}\alpha_{nk}|s_k(x_0)-f(x_0)|^{\lambda_n}\right\}^{1/\lambda_n} = 0. \tag{4.6}$$

If (4.5) holds uniformly in $[c, d]$, then (4.6) is also valid uniformly in any interval internal to (c, d).

We mention that in the cases

$$\alpha_{nk} = A_{n-k}^{\alpha-1},\ 0 < \alpha \leqslant 1 \text{ and } 1 < p < 1/(1-\alpha),$$

$$\alpha_{nk} = A_{n-k}^{\alpha-1},\ \alpha > 1 \text{ and } 1 < p < \infty,$$

$$\alpha_{nk} = k^{\beta} \text{ and } p\beta > -1$$

condition (4.4) is satisfied, and thus Theorem 14 can be used for the most frequent means.

5

Besides the problems mentioned in connection with our theorems we can also raise the following ones:

1. Can the result given here be generalized to very strong approximation, that is, can the partial sums $s_k(x)$ in our theorems be replaced by $s_{\nu_k}(x)$ with any increasing sequence $\{\nu_k\}$?

2. What kind of additional conditions are needed for such a transition?

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