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ON THE APPLICATION OF THE ORTHONORMAL FRANKLIN SYSTEM TO THE APPROXIMATION OF ANALYTIC FUNCTIONS

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1. Introduction

Let $A = A(D)$ be the Banach space of analytic functions in the unit disc $D = \{z\colon |z| < 1\}$ and continuous in $\bar{D}$ with the norm

$$||f|| = \max_{|z|=1} |f(z)|.$$

Let $\Delta = \{-\pi = t_0 < t_1 < \ldots < t_n = \pi\}$ be a partition of the interval $I = [-\pi, \pi]$. The function $s_\Delta \in C(I)$ is called a *periodic spline of degree* 1 *with respect to the partition* Δ if it is piecewise linear with the knots at the points of the partition Δ and $s_\Delta(-\pi) = s(\pi)$. A function S_Δ defined with the aid of the Schwarz formula

$$(1)\qquad S_{\Delta}(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} s_\Delta(t)\frac{e^{it}+z}{e^{it}-z}\,dt+iA, \quad \text{where } A = \text{const}, \ |z| < 1,$$

will be called an *analytic spline of degree* 1 *with respect to the function* s_Δ. Since the function s_Δ satisfies the Lipschitz condition, we can define the function S_Δ on the unit circle $\Gamma = \{z\colon |z| = 1\}$ setting $S_\Delta(e^{i\phi}) = \lim_{r\to 1-} S_\Delta(re^{i\phi})$. The function S_Δ defined in this way belongs to the space A and

$$(2)\qquad S_\Delta(e^{i\phi}) = s_\Delta(\phi)+\frac{i}{2\pi}\int_{-\pi}^{\pi} s_\Delta(\phi-t)\,\mathrm{ctg}\frac{t}{2}\,dt+iA,$$

where the above integral is interpreted as the Cauchy Principal Values [8], [15].

The sequence $\{f_n\}_{n=0}^{\infty}$ of functions of the Banach space A is called a *basis* whenever each function $f \in A$ has a unique expansion

$$f = \sum_{n=0}^{\infty} a_n f_n$$

convergent in the norm.

The *modulus of continuity* of the function f is defined as follows:

(3) $$\omega(f, h) = \sup\{|f(t_2)-f(t_1)|,\ 0 < |t_2-t_1| \leqslant h,\ t_1, t_2 \in D_f\}.$$

The question of existence of a basis in the space A was raised by Banach in [3]. Bočkarev [4], [5] exhibited an orthonormal basis $\{G_n\}_{n=0}^{\infty}$ in A. He also proved that there exists a constant $B > 0$ such that for $f \in A$, $f(e^{it}) = u(t)+iv(t)$, $\|f-S_{n,f}\| \leqslant B[\omega(u, 1/n)+\omega(v, 1/n)]$, where $S_{n,f}$ is the nth Fourier sum of f with respect to the system $\{G_n\}_{n=0}^{\infty}$.

We can consider the space A as a space over R or C. The *Bočkarev system* is a basis in the complex space A. The purpose of this paper is to give a construction of a basis in the real space A and a construction of another basis in the complex space A and to estimate the error of the approximation of analytic functions by analytic splines of degree 1.

2. The orthonormal Franklin systems

Let $\{\Delta_n\}_{n=1}^{\infty}$ be a given sequence of partitions of $[0, 1]$, $\Delta_n = \{0 = t_{n,0} < t_{n,1} < \ldots \ldots < t_{n,n} = 1\}$ with $\Delta_n \subset \Delta_{n+1}$, i.e. each point of Δ_n is a point of Δ_{n+1}. Let $m_n = \min\limits_{1\leqslant i\leqslant n} (t_{n,i}-t_{n,i-1})$ and $M_n = \max\limits_{1\leqslant i\leqslant n} (t_{n,i}-t_{n,i-1})$.

Define the sequence of functions $\{\Psi_n\}_{n=0}^{\infty}$ as follows: $\Psi_0 = 1$, $\Psi_1 = t$ and for $n \geqslant 2$ Ψ_n is a spline of degree 1 with respect to Δ_n equal to one on $\Delta_n \diagdown \Delta_{n-1}$ and equal to zero on Δ_{n-1}. Applying the Schmidt procedure of orthonormalization, we obtain an orthonormal system in the space $L_2\,[0, 1]$ [7]. In this space the following scalar product is given:

$$(f, g) = \int_0^1 f(t)g(t)\,dt.$$

This system can also be obtained with the aid of cubic splines. Let g_n $(n > 0)$ be a cubic spline equal to one on $\Delta_n \diagdown \Delta_{n-1}$ and equal to zero on Δ_{n-1} with $g_n'(0) = g_n'(1) = 0$. Then the system $\{1, g_1'', g_2'', \ldots\}$ is orthogonal ([2], p. 100) and each function Ψ_n $(n \geqslant 0)$ is a linear combination of the functions $1, g_1'', \ldots, g_n''$. Hence the system $\{f_n\}_{n=0}^{\infty}$, where $f_n = \frac{g_n}{\|g_n\|_2}$, $\|g\|_2 = \left(\int_0^1 [g(t)]^2 dt\right)^{1/2}$, is an orthonormal Franklin system for this sequence of partitions.

If we reject the function Ψ_1 from the system $\{\Psi_n\}_{n=0}^{\infty}$ and put $\hat{\Psi}_1 = \Psi_0$, $\hat{\Psi}_n = \Psi_n$ for $n \geqslant 2$, then by applying the Schmidt procedure of orthonormalization, we obtain the orthonormal system in the space $C_p[0, 1]$ of continuous and periodic functions. This system can also be obtained with the aid of cubic splines. Let $\hat{g}_n$ $(n \geqslant 2)$ be a periodic cubic spline equal to one on $\Delta_n \diagdown \Delta_{n-1}$ and equal to zero on Δ_{n-1}. Then the system $\{1, \hat{g}_2'', \hat{g}_3'', \ldots\}$ is orthogonal and each function $\hat{\Psi}_n$ $(n \geqslant 1)$ is a linear combination of the functions $1, \hat{g}_2'', \ldots, \hat{g}_n''$. Hence the system $\{\hat{f}_n\}_{n=1}^{\infty}$, where $\hat{f}_n = \frac{\hat{g}_n''}{\|\hat{g}_n''\|_2}$, is an orthonormal periodic Franklin system for this sequence of partitions.

Further we need the following theorems:

THEOREM 1. *If $f \in C[0, 1]$ or $f \in C_p\,[0, 1]$ and $s_{n,f}$ is the n-th Fourier sum of f with respect to the system $\{f_n\}_{n=0}^{\infty}$ or $\{\hat{f}_n\}_{n=1}^{\infty}$, respectively, then*

(4) $$\|s_{n,f}\| \leqslant 3\|f\|.$$

Proof. For $f \in C[0, 1]$ the proof is in [6] and for $f \in C_p[0, 1]$ it is analogous.

The next theorem is analogous to the theorems on interpolation by cubic splines [2] and it will be given in detail considering its importance in the construction of a basis in the space A.

THEOREM 2. *Let $\{\Delta_n\}_{n=1}^{\infty}$ be a given sequence of partitions of $[0, 1]$. If*

(a) $f \in C[0, 1]$, $F(t) = \int_0^t\int_0^x f(y)\,dy\,dx$, $S_{n,f}$ is the cubic spline of interpolation to $F(t)$ on Δ_n satisfying the conditions $S_{n,f}'(0) = F'(0)$ and $S_{n,f}'(1) = F'(1)$ or

(b) $f \in C_p[0, 1]$, $S_{n,f}$ is the cubic spline of interpolation to $F(t)$ on Δ_n satisfying the conditions $S_{n,f}'(1)-S_{n,f}'(0) = F'(1)-f'(0)$ and $S_{n,f}''(0) = S_{n,f}''(1)$,

then

(1) $S_{n,f}'' = s_{n,f}$, where $s_{n,f}$ is the n-th Fourier sum of f with respect to the system $\{f_n\}_{n=0}^{\infty}$ or $\{\hat{f}_n\}_{n=1}^{\infty}$.

(2) $$\begin{aligned} \|f-S_{n,f}''\| &\leqslant 6\omega(f, M_n/2),\\ \|F'-S_{n,f}'\| &\leqslant 6M_n\omega(f, M_n/2),\\ \|F-S_{n,f}\| &\leqslant 3M_n^2\omega(f, M_n/2). \end{aligned}$$

Proof. From the fundamental identity for cubic splines [2] we can deduce that $S_{n,f}'' = s_{n,f}$.

It follows from (1) and Theorem 1 that

$$\|S_{n,f}''\| \leqslant 3\|f\|.$$

Let ϕ be a spline of degree 1 of interpolation to f on Δ_n. Then we have the following estimation [10], [14];

$$\|f-\phi\| \leqslant \tfrac{3}{2}\omega(f, M_n/2).$$

Hence

(5) $$\|f-S_{n,f}''\| \leqslant \|f-\phi\|+\|\phi-S_{n,f}''\| = \|f-\phi\|+\|s_{n,f-\phi}\| \leqslant 6\omega(f, M_n/2).$$

Let $t \in [t_{i-1}, t_i]$. In consequence of the interpolation property of $S_{n,f}$, an application of Rolle's theorem yields the fact that there exists a point $\xi \in (t_{i-1}, t_i)$ for which $F'(\xi) = S_{n,f}'(\xi)$. Thus on this interval

$$|F'(t)-S_{n,f}'(t)| = \left|\int_\xi^t [f(x)-s_{n,f}(x)]dx\right| \leqslant M_n\|f-s_{n,f}\|$$

and a second integration yields the property

$$|F(t)-S_{n,f}(t)| \leqslant \frac{M_n^2}{2}\,\|f-s_{n,f}\|,$$

which completes the proof.

3. Construction of a basis

Let $\{\Delta_n\}_{n=1}^{\infty}$ be a given sequence of partitions of $[-\pi,\pi]$, $\Delta_n=\{-\pi=t_{n,0}<t_{n,1}<\ldots<t_{n,n}=\pi\}$ with $\Delta_n\subset\Delta_{n+1}$ and $K_n=M_n/m_n$.

THEOREM 3. *Let $f\in A$, $f(e^{it})=u(t)+iv(t)$ and let $s_{n,u}$ be the n-th Fourier sum of u with respect to the system $\{\hat{f}_n\}_{n=1}^{\infty}$ and $S_{n,f}$ the analytic spline of degree 1 with respect to the function $s_{n,u}$ such that $\operatorname{Im}S_{n,f}(0)=\operatorname{Im}f(0)$. Then*

$$\|f-S_{n,f}\| \leqslant \frac{2}{\pi}(3\pi+14+12K_n)\,\omega\left(u,\frac{M_n}{2}\right)+\frac{1}{\pi}(\pi+m_n)\,\omega(f,r_n), \tag{6}$$

where $r_n=|e^{im_n}-1|<m_n$.

Proof. Let $z=e^{i\phi}$. Since the addition of a constant to the function f changes neither modulus of continuity of f nor the difference $f(z)-S_{n,f}(z)$, we can assume that $f(z)=0$. From (2) we obtain

$$|f(z)-S_{n,f}(z)| \leqslant |u(\phi)-s_{n,u}(\phi)|+\frac{1}{2\pi}\left|\int_{-\pi}^{\pi}[u(\phi-t)-s_{n,u}(\phi-t)]\operatorname{ctg}\frac{t}{2}\,dt\right|. \tag{7}$$

It follows from Theorem 2 that

$$|u(\phi)-s_{n,u}(\phi)| \leqslant 6\omega(u,M_n/2). \tag{8}$$

To estimate the integral write it as follows

$$\frac{1}{2\pi}\int_{\pi-}^{\pi}[u(\phi-t)-s_{n,u}(\phi-t)]\operatorname{ctg}\frac{t}{2}\,dt \tag{9}$$

$$=\frac{1}{2\pi}\int_{m_n\leqslant|t|\leqslant\pi}[u(\phi-t)-s_{n,u}(\phi-t)]\operatorname{ctg}\frac{t}{2}\,dt+$$

$$+\frac{1}{2\pi}\int_{|t|\leqslant m_n}u(\phi-t)\operatorname{ctg}\frac{t}{2}\,dt-\frac{1}{2\pi}\int_{|t|\leqslant m_n}s_{n,u}(\phi-t)\operatorname{ctg}\frac{t}{2}\,dt$$

$$=I_1+I_2+I_3.$$

Introduce the following notation: $u_1(t)=\int_{-\pi}^{t}\int_{-\pi}^{x}u(y)\,dy\,dx$, $S(t)$ is a cubic spline of interpolation to $u_1(t)$ on Δ_n satisfying the conditions $S'(\pi)-S'(-\pi)=u_1'(\pi)-u_1'(-\pi)$ and $S''(-\pi)=S''(\pi)$, $r(t)=u(t)-s_{n,u}(t)$ and $R(t)=u_1'(t)-S'(t)$.

From Theorem 2 we obtain

$$|r(t)|\leqslant 6\omega(u,M_n/2),$$
$$|R(t)|\leqslant 6M_n\omega(u,M_n/2).$$

Hence

$$\left|\int_{m_n}^{\pi}r(\phi-t)\operatorname{ctg}\frac{t}{2}\,dt\right|=\left|R(\phi-t)\operatorname{ctg}\frac{t}{2}\Big|_{m_n}^{\pi}+\int_{m_n}^{\pi}\frac{R(\phi-t)}{2\sin^2\frac{t}{2}}\,dt\right|$$

$$\leqslant\|R\|\operatorname{ctg}\frac{m_m}{2}+\|R\|\int_{m_n}^{\pi}\frac{dt}{2\sin^2\frac{t}{2}}$$

$$=2\|R\|\operatorname{ctg}\frac{m_n}{2}\leqslant\frac{4\|R\|}{m_n}\leqslant 24K_n\omega\left(u,\frac{M_n}{2}\right).$$

Analogously,

$$\left|\int_{-\pi}^{-m_n}r(\phi-t)\operatorname{ctg}\frac{t}{2}\,dt\right|\leqslant 24K_n\omega\left(u,\frac{M_n}{2}\right).$$

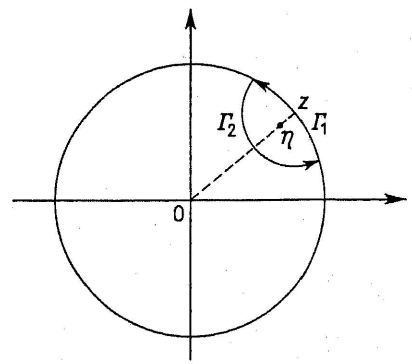
Then

$$|I_1|\leqslant\frac{24}{\pi}K_n\left(u,\frac{M_n}{2}\right). \tag{10}$$

Write the second integral in the following form:

$$I_2=\frac{1}{2\pi}\int_{-m_n}^{m_n}u(\phi-t)\operatorname{ctg}\frac{t}{2}\,dt=-\frac{i}{2\pi}\int_{-m_n}^{m_n}u(\phi-t)\frac{e^{it}+1}{e^{it}-1}\,dt$$

$$=\frac{i}{2\pi}\int_{\phi-m_n}^{\phi+m_n}u(\tau)\frac{e^{i\tau}+e^{i\phi}}{e^{i\tau}-e^{i\phi}}\,d\tau=\frac{1}{2\pi}\operatorname{Re}\int_{\phi-m_n}^{\phi+m_n}if(e^{i\tau})\frac{e^{i\tau}+e^{i\phi}}{e^{i\tau}-e^{i\phi}}\,d\tau=\operatorname{Re}J.$$



Further we need the following notation: $\Gamma_1 = \{\xi\colon \xi = e^{i(\phi+t)},\ -m_n \leqslant t \leqslant m_n\}$, $\Gamma_2 = \{\xi\colon |\xi-z| = |e^{im_n}-1| = r_n\}\cap\bar{D}$, $0 < \varepsilon < r_n$, $\eta = (1-\varepsilon)z$. Hence

$$J = \frac{1}{2\pi}\int_{\Gamma_1}\frac{f(\xi)}{\xi}\cdot\frac{\xi+z}{\xi-z}\,d\xi = \frac{1}{\pi}\int_{\Gamma_1}\frac{f(\xi)}{\xi-z}\,d\xi - \frac{1}{2\pi}\int_{\Gamma_1}\frac{f(\xi)}{\xi}\,d\xi = J_1+J_2.$$

From Sochocki's theorem and Cauchy's theorem [9], [11] we obtain

$$J_1 = \frac{1}{\pi}\int_{\Gamma_1}\frac{f(\xi)}{\xi-z}\,d\xi = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\Gamma_1}\frac{f(\xi)}{\xi-\eta}\,d\xi = \lim_{\varepsilon\to 0}J_{1,\varepsilon},$$

$$\frac{1}{\pi}\int_{\Gamma_1}\frac{f(\xi)}{\xi-\eta}\,d\xi = 2if(\eta) - \frac{1}{\pi}\int_{\Gamma_2}\frac{f(\xi)}{\xi-\eta}\,d\xi.$$

Hence

$$|J_{1,\varepsilon}| \leqslant \frac{1}{\pi}\int_{\Gamma_2}\frac{|f(\xi)-f(z)|}{|\xi-z|}\cdot\left|\frac{\xi-z}{\xi-\eta}\right||d\xi| + 2|f(\eta)|$$

$$\leqslant (f, r_n)\frac{r_n}{r_n-\varepsilon} + 2|f(\eta)| \to \omega(f, r_n) \quad \text{as} \quad \varepsilon\to 0.$$

Since

$$|J_2| \leqslant \frac{1}{2\pi}\int_{\phi-m_n}^{\phi+m_n}|f(e^{it})-f(z)|\,dt \leqslant \frac{m_n}{\pi}\omega(f, r_n),$$

we have

(11) $$|I_2| \leqslant \left(1+\frac{m_n}{\pi}\right)\omega(f, r_n).$$

To estimate the integral I_3 we shall estimate the derivative of the function $s_{n,u}$. To do this it suffices to estimate the divided difference of the function $s_{n,u}$ taken at the points t_{i-1}, t_i, $i = 1, 2, \ldots, n$.

$$\frac{s_{n,u}(t_i)-s_{n,u}(t_{i-1})}{t_i-t_{i-1}} = \frac{s_{n,u}(t_i)-u(t_i)}{t_i-t_{i-1}} + \frac{u(t_i)-u(t_{i-1})}{t_i-t_{i-1}} + \frac{u(t_{i-1})-s_{n,u}(t_{i-1})}{t_i-t_{i-1}}.$$

Hence from Theorem 2 and the property of the modulus of continuity [1], [13], we obtain

$$|s'_{n,u}(t)| \leqslant \frac{14}{m_n}\omega\left(u, \frac{M_n}{2}\right)$$

and an application of Lagrange's theorem yields

$$|s_{n,u}(\phi-t)-s_{n,u}(\phi)| \leqslant \frac{14\omega(u, M_n/2)}{m_n}|t|.$$

Hence

$$|I_3| \leqslant \frac{1}{2\pi}\int_{-m_n}^{m_n}||s'_{n,u}||\cdot\left|t\operatorname{ctg}\frac{t}{2}\right|dt \leqslant \frac{2}{\pi}||s'_{n,u}||m_n.$$

Then

(12) $$|I_3| \leqslant \frac{28}{\pi}\omega\left(u, \frac{M_n}{2}\right).$$

Hence from (7)–(12), together with an application of the principle of maximum for analytic functions, we obtain the theorem.

Remark. Since there exists a constant $C > 0$ such that for $f\in A$, $\omega(f, \delta) \leqslant C\tilde{\omega}(f, \delta)$, where $\tilde{\omega}(f, \delta) = \max_{\substack{|z_1|=|z_2|=1\\ |z_2-z_1|\leqslant\delta}}|f(z_2)-f(z_1)|$, [12], the inequality (6) can be written as follows:

$$||f-S_{n,f}|| \leqslant (A+8K_n)\omega(u, M_n/2)+B\omega(v, m_n),$$

where A and B are constants.(*)

Let $\{\hat{f}_n\}_{n=1}^{\infty}$ be an orthonormal periodic Franklin system in the interval $[-\pi, \pi]$ and let

(13) $$g_0 = i,\quad g_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}_n(t)\frac{e^{it}+z}{e^{it}-z}\,dt,\quad n = 1, 2, \ldots$$

From Theorem 3 we obtain

THEOREM 4. *If for a given sequence of partitions $\{\Delta_n\}_{n=1}^{\infty}$ $\overline{\lim_{n\to\infty}}\,K_n < \infty$, then the system $\{g_n\}_{n=0}^{\infty}$ is a basis in the real Banach space A.*

Proof. Define a scalar product in the real space A as follows:

$$f, g\in A,\quad f = u+iv,\quad g = u_1+iv_1,\quad (f, g) = \int_{-\pi}^{\pi}u(e^{it})u_1(e^{it})\,dt+v(0)v_1(0).$$

Now the system $\{g_n\}_{n=0}^{\infty}$ is orthonormal and from (6) we obtain the theorem.

Remark. Let $f\in A$, $f(e^{it}) = u(t)+iv(t)$, $U(t) = \int_{-\pi}^{t}\int_{-\pi}^{x}[u(t)-2\pi\operatorname{Re}f(0)]\,dy\,dx$ and let $H(t)$ be a periodic cubic spline of interpolation to the function U at the points of the partition Δ_n. Reasoning as in the proof of Theorem 2, we can see that

(14) $$S_{n,f}(z) = f(0)+\frac{1}{2\pi}\int_{-\pi}^{\pi}H''(t)\frac{e^{it}+z}{e^{it}-z}\,dt.$$

(*) The constant C is equal to 3. This is proved by L. A. Rubel, A. L. Shields and B. A. Taylor in J. Approximation Theory 15 (1975), p. 23.

4. Orthonormal systems

Let $\{\Delta_n\}_{n=1}^{\infty}$ be a given sequence of partitions of $[0, 2\pi]$, $\Delta_n = \{0 = t_{n,0} < t_{n,1} < \ldots$ $\ldots < t_{n,n} = 2\pi\}$ with $\Delta_n \subset \Delta_{n+1}$ and let $\{f_n\}_{n=0}^{\infty}$ be an orthonormal Franklin system for this sequence of partitions.

Define the following system of functions $\{F_n\}_{n=0}^{\infty}$:

$$F_n(t) = \begin{cases} f_n(2t) & \text{for} \quad t \in [0, \pi] \\ f_n(-2t) & \text{for} \quad t \in [-\pi, 0] \end{cases} \qquad n = 0, 1, \ldots \tag{15}$$

Let

$$g_0(t) = \frac{1}{\sqrt{2\pi}}, \qquad g_n(t) = \frac{1}{\sqrt{2}}[F_n(t) + i\tilde{F}_n(t)], \tag{16}$$

where

$$\tilde{F}_n(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(t-x)\operatorname{ctg}\frac{x}{2}\,dx. \tag{17}$$

Since the functions satisfy the Lipschitz condition, the functions

$$G_n(z) = G_n(re^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g_n(x)\frac{1-r^2}{1-2r\cos(t-x)+r^2}\,dx$$

belong to the space A.

The functions $G_n(z)$ can also be written with the aid of the Schwarz formula [5]

$$\begin{aligned} G_0(z) &= \frac{1}{\sqrt{2\pi}}, \\ G_n(z) &= \frac{1}{2\sqrt{2\pi}}\int_{-\pi}^{\pi} F_n(t)\frac{e^{it}+z}{e^{it}-z}\,dt, \qquad n \geqslant 1. \end{aligned} \tag{18}$$

This system will be called the *Bočkarev system.*

Let $f, g \in A$. The scalar product is defined as follows:

$$(f, g) = \int_{-\pi}^{\pi} f(e^{it})\overline{g(e^{it})}\,dt.$$

The next theorem is a conclusion from Parseval's equality and it will be given in detail considering the simplicity of the proof.

THEOREM 5. *The system $\{G_n\}_{n=0}^{\infty}$ is orthonormal.*

Proof. From Cauchy's formula we obtain

$$\frac{1}{2\pi i}\int_{|\xi|=1}\frac{G_n(\xi)G_m(\xi)}{\xi}\,d\xi = G_n(0)G_m(0). \tag{19}$$

On the other hand, for $n \geqslant 1$, $m \geqslant 0$,

$$\begin{aligned} \frac{2}{i}\int_{|\xi|=1}\frac{G_n(\xi)G_m(\xi)}{\xi}\,d\xi &= 2\int_{-\pi}^{\pi} G_n(e^{it})G_m(e^{it})\,dt \\ &= \int_{-\pi}^{\pi}[F_n(t)+i\tilde{F}_n(t)]\,[F_m(t)+i\tilde{F}_m(t)]\,dt \\ &= \int_{-\pi}^{\pi}[F_n(t)F_m(t)-\tilde{F}_n(t)\tilde{F}_m(t)]\,dt + \\ &\qquad + i\int_{-\pi}^{\pi}[F_n(t)\tilde{F}_m(t)+\tilde{F}_n(t)F_m(t)]\,dt \\ &= 0 \quad \text{because} \quad G_n(0) = 0. \end{aligned}$$

Then

$$\begin{aligned} \int_{-\pi}^{\pi} F_n(t)F_m(t)\,dt &= \int_{-\pi}^{\pi}\tilde{F}_n(t)\tilde{F}_m(t)\,dt, \\ \int_{-\pi}^{\pi} F_n(t)\tilde{F}_m(t)\,dt &= -\int_{-\pi}^{\pi}\tilde{F}_n(t)F_m(t)\,dt. \end{aligned} \tag{20}$$

Hence

$$\begin{aligned} (G_n, G_m) = \int_{-\pi}^{\pi} G_n(e^{it})\overline{G_m(e^{it})}\,dt &= \frac{1}{2}\int_{-\pi}^{\pi}[F_n(t)F_m(t)+\tilde{F}_n(t)\tilde{F}_m(t)]\,dt + \\ &\quad + \frac{i}{2}\int_{-\pi}^{\pi}[F_m(t)\tilde{F}_n(t)-F_n(t)\tilde{F}_m(t)]\,dt. \end{aligned}$$

Since the functions F_n are even and $\tilde{F}_n$ odd, we infer from (20) that $(G_n, G_m) = \delta_{n,m}$.

THEOREM 6. *Let $S_{n,f}$ be the n-th Fourier sum of a given function $f \in A$ with respect to the system $\{G_n\}_{n=0}^{\infty}$. Then*

$$\|f - S_{n,f}\| \leqslant \frac{2}{\pi}(3\pi+14+12K_n)\left[\omega\left(u, \frac{M_n}{4}\right)+\omega\left(v, \frac{M_n}{4}\right)\right] + \frac{1}{\pi}(\pi+m_n)\omega(f, r_n), \tag{21}$$

where $r_n = |e^{im}-1| < m_n$.

Proof. From (18),

$$S_{n,f}(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} s_{n,f}(t)\frac{e^{it}+z}{e^{it}-z}\,dt,$$

where $s_{n,f}$ is the nth Fourier sum of the function $f(e^{it})$ with respect to the system $\{F_n\}_{n=0}^{\infty}$. On the circle Γ,

$$S_{n,f}(e^{i\phi}) = s_{n,f}(\phi)+\frac{i}{2\pi}\int_{-\pi}^{\pi} s_{n,f}(\phi-t)\operatorname{ctg}\frac{t}{2}\,dt.$$

Let $f(e^{it}) = u(t)+iv(t)$. Divide the functions u and v into the even and odd parts, $u = u_1+u_2$, $v = v_1+v_2$, where $u_1(t) = \dfrac{u(t)+u(-t)}{2}$, $u_2(t) = \dfrac{u(t)-u(-t)}{2}$, and do the same with the functions v_1 and v_2.

Since the functions F_n are even and $\tilde{u}_1 = v_2$ and $\tilde{v}_1 = -u_2+\operatorname{Re}f(0)$, we have

$$f(e^{i\phi})-S_{n,f}(e^{i\phi}) = [u_1(\phi)-s_{n,u_1}(\phi)]+i[v_1(\phi)-s_{n,v_1}(\phi)]+$$
$$+\frac{i}{2\pi}\int_{-\pi}^{\pi}[u_1(\phi-t)-s_{n,u_1}(\phi-t)]\operatorname{ctg}\frac{t}{2}\,dt-$$
$$-\frac{1}{2\pi}\int_{-\pi}^{\pi}[v_1(\phi-t)-s_{n,v_1}(\phi-t)]\operatorname{ctg}\frac{t}{2}\,dt,$$

and reasoning as in the proof of Theorem 3 we obtain (21).

COROLLARY. *There exists a constant A such that for $f\in A$,*

$$\|f-S_{n,f}\| \leqslant (A+8K_n)[\omega(u, M_n/2)+\omega(v, M_n/2)].$$

This follows from Theorem 5 and Theorem 6.

THEOREM 7. *If for a given sequence of partitions $\{\Delta_n\}_{n=1}^{\infty}$ $\varlimsup_{n\to\infty} K_n < \infty$, then the system $\{G_n\}_{n=0}^{\infty}$ is a basis in the complex Banach space A.*

Remark 1. Let $F(t) = \int_0^t\int_0^x [u_1(y/2)+iv_1(y/2)]\,dy\,dx$ and let $H(t)$ be a cubic spline of interpolation to the function F at the points of the partition Δ_n satisfying the conditions $H'(0) = F'(0)$ and $H'(2\pi) = F'(2\pi)$. Reasoning as in the proof of Theorem 2 and from (15) we can see that for $0 \leqslant t \leqslant \pi$, $s_{n,f}(t) = H''(2t)$, and since $s_{n,f}$ is an odd function, we have for $-\pi \leqslant t \leqslant 0$, $s_{n,f}(t) = H''(-2t)$ and

(22) $$S_{n,f}(z) = \frac{1}{2\pi}\int_0^{\pi} H''(2t)\left[\frac{e^{it}+z}{e^{it}-z}+\frac{e^{-it}+z}{e^{-it}-z}\right]dt.$$

Remark 2. Let $\{\Delta_n\}_{n=1}^{\infty}$ be a given sequence of partitions of $[0, 2\pi]$, $\Delta_n = \{0 = t_{n,0} < t_{n,1} < \ldots < t_{n,n} = 2\pi\}$ with $\Delta_n \subset \Delta_{n+1}$ and let $\{\hat{f}_n\}_{n=1}^{\infty}$ be an orthonormal periodic system with respect to $\{\Delta_n\}_{n=1}^{\infty}$ with $\hat{f}_n(0) = 0$ $(n = 1, 2, \ldots)$ obtained as the Franklin system.

Define the following system of functions $\{\hat{F}_n\}_{n=2}^{\infty}$:

$$\hat{F}_n(t) = \begin{cases}\hat{f}_n(2t) & \text{for} \quad t\in[0,\pi],\\ -\hat{f}_n(-2t) & \text{for} \quad t\in[-\pi, 0].\end{cases}$$

Let

$$\hat{G}_1(z) = \frac{1}{\sqrt{2\pi}},$$

$$\hat{G}_n(z) = \frac{1}{2\sqrt{2\pi}}\int_{-\pi}^{\pi}\hat{F}_n(t)\frac{e^{it}+z}{e^{it}-z}\,dt, \quad n \geqslant 2.$$

For the system $\{\hat{G}_n\}_{n=1}^{\infty}$ theorems 5–7 are true. The proofs are analogous.

Remark 3. Let $\{\Delta_n\}_{n=1}^{\infty}$ be a given sequence of partitions of $[-\pi, \pi]$, $\Delta_n = \{-\pi = t_{n,0} < t_{n,1} < \ldots < t_{n,n} = \pi\}$ with $\Delta_n \subset \Delta_{n+1}$ and let $\{\hat{f}_n\}_{n=1}^{\infty}$ be an orthonormal periodic Franklin system with respect to $\{\Delta_n\}_{n=1}^{\infty}$.

Instead of Schwarz's formula use Cauchy's formula and define the following system $\{\hat{g}_n\}_{n=1}^{\infty}$:

(23) $$\hat{g}_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\hat{f}_n(t)e^{it}dt}{e^{it}-z}, \quad n = 1, 2, \ldots$$

As for the system $\{g_n\}_{n=1}^{\infty}$, we can see that if $\varlimsup_{n\to\infty} K_n < \infty$, then each function $f\in A$ has a uniform convergent expansion $f = \sum_{n=1}^{\infty} a_n\hat{g}_n$. However, this expansion is not unique because each function $\hat{g}_n$ $(n \geqslant 2)$ has at least two different expansions. Then the system $\{\hat{g}_n\}_{n=1}^{\infty}$ is not a basis in the space A.

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