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CONSTRUCTION OF DIFFUSION PROCESSES WITH WENTZELL'S BOUNDARY CONDITIONS BY MEANS OF POISSON POINT PROCESSES OF BROWNIAN EXCURSIONS

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1. Introduction

It is well known that, under certain regularity conditions, a diffusion process on a manifold with boundary is determined by a second order differential operator of elliptic type (possibly degenerate) plus a Wentzell's boundary condition (cf. Wentzell [11]). The problem of constructing the diffusion from a given pair of such analytic data has been discussed so far by many authors. K. Sato and T. Ueno [7] laid a fundamental route of construction in an analytical way and following it, J. M. Bony, Ph. Courrege and P. Priouret [1] succeeded in constructing diffusions in very general cases. In a probabilistic way, N. Ikeda [4] applied Itô's stochastic differential equations to this problem and S. Watanabe [8] extended his idea so as to cover more general cases.

In this paper, we will propose still another approach to this problem by constructing directly the *excursions* of the diffusion. Our plan is as follows: we prepare two kinds of Poisson point processes on function spaces which we call *Poisson point processes of the Brownian excursions of the first and second kinds*. Each point of these point processes represents a Brownian excursion and by solving a stochastic differential equation based on this excursion (which is an absorbing barrier Brownian motion with an infinite entrance law) we can associate an excursion of the diffusion to be constructed to each of Brownian excursions. Here, we make use of the space-time relation of the Brownian excursion to produce the frequency of excursions in proportion to the coefficient of the corresponding terms in the given boundary condition.

The path functions of the diffusion to be constructed will be defined by gluing the excursion thus constructed and, in doing this, we need the so-called *process on the boundary*. This is constructed by solving a stochastic differential equation of jump type based on the Poisson point processes of Brownian excursions. Usually stochastic differential equations of jump type, as discussed in e.g. K. Itô [5] and

I. I. Gihman and A. V. Skorohod [2], are based on Poisson point processes on Euclidean spaces but we need here essentially an equation based on Poisson point processes on function spaces.

The Markovian property and the structure of the diffusion thus constructed will be studied in another publication. In comparing our method with above mentioned former results, it has several advantages. In particular, it can cover the case that coefficient of reflection may degenerate on some part of the boundary and the case that jumps take place from the boundary to the interior.

2. Point processes and Poisson point processes

Here, we will summarize the theory of point processes as will be necessary below. For details, we refer to, e.g., Grigelionis [3], Itô [6] and Watanabe [9].

Let $(X, \mathscr{B}(X))$ be a measurable space. By a *point function p on X*, we mean a map $p\colon D_p \subset (0,\infty) \to X$, where the domain D_p is a countable subset of $(0,\infty)$. p defines a counting measure $N_p(dt, dx)$ on $(0,\infty)\times X$ by

(2.1) $$N_p((0,t]\times U) = \#\{s \in D_p;\ s \leqslant t, p(s) \in U\}, \quad t > 0,\ U \in \mathscr{B}(X).$$

A point process is obtained by randomizing the notion of point functions. Let Π_X be the set of all point functions on X and $\mathscr{B}(\Pi_X)$ be the smallest σ-field on Π_X with respect to which all $p \to N_p((0,t]\times U)$, $t > 0$, $U \in \mathscr{B}(X)$, are measurable. A *point process p on X* is, by definition, a $(\Pi_X, \mathscr{B}(\Pi_X))$-valued random variable, i.e. a measurable map $p\colon \Omega \to \Pi_X$ defined on a probability space $(\Omega, \mathscr{F}, P)$.

Let $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$ be a probability space with right-continuous increasing family $(\mathscr{F}_t)_{t\geqslant 0}$ of sub σ-fields of $\mathscr{F}$. From now on, we consider all point processes to be defined on the quadruplet $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$. A point process p on X is called *$(\mathscr{F}_t)$-adapted*, if $\{N_p((0,t]\times U)\}_{U\in\mathscr{B}(X)}$ is $\mathscr{F}_t$-measurable for all $t > 0$.

DEFINITION 2.1. A point process p on X is called *of the class* (QL) if

(i) it is $(\mathscr{F}_t)$-adapted,

(ii) it has a continuous compensating measure $\phi_p(dt, dx)$.

We refer to [9] for the precise meaning of (ii). Let $n(dx)$ be a σ-finite measure on $(X, \mathscr{B}(X))$.

DEFINITION 2.2. A point process p on X is called an *$(\mathscr{F}_t)$-stationary Poisson point process on X with the characteristic measure $n(dx)$* if it is of the class (QL) with

(2.2) $$\phi_p(dt, dx) = dt\, n(dx).$$

By Itô's formula (cf. [3], [9]), we can prove easily that such p satisfies, for every $t > s \geqslant 0$, $\lambda_i > 0$, $U_i \in \mathscr{B}(X)$, $i = 1, 2, \ldots, n$ such that $n(U_i) < \infty$ and are disjoint,

(2.3) $$E\big(e^{-\sum_{i=1}^{n}\lambda_i N_p((s,t]\times U_i)}/\mathscr{F}_s\big) = \exp\Big\{(t-s)\sum_{i=1}^{n} n(U_i)(e^{-\lambda_i}-1)\Big\}$$

and this justifies to call p as a Poisson point process.

THEOREM 2.1. *Given a σ-finite measure $n(dx)$ on $(X, \mathscr{B}(X))$, we can always construct an $(\mathscr{F}_t)$-stationary Poisson point process on X with the characteristic measure n on some space $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$.*

A way of construction is as follows: let $U_n \in \mathscr{B}(X)$, $n = 1, 2, \ldots$, be such that they are disjoint, $n(U_n) < \infty$ and $X = \bigcup_n U_n$. Let $\{\tau_k^{(n)}\}_{n,k=1,2,\ldots}$ be non-negative random variables such that $P(\tau_k^{(n)} > t) = e^{-tn(U_n)}$ and $\{\xi_k^{(n)}\}_{k=1,2,\ldots}$ be, for each $n = 1, 2, \ldots$, U_n-valued random variables such that $P(\xi_k^{(n)} \in dx) = \dfrac{n(dx)}{n(U_n)}$. Furthermore, $\{\tau_k^{(n)}, \xi_k^{(n)}\}_{n,k=1,2,\ldots}$ are all independent. Preparing such random variables on a probability space $(\Omega, \mathscr{F}, P)$, we set

$$D_p = \bigcup_{n=1}^{\infty} \{\tau_1^{(n)}, \tau_1^{(n)}+\tau_2^{(n)}, \ldots, \tau_1^{(n)}+\tau_2^{(n)}+\ldots+\tau_k^{(n)}, \ldots\}$$

and

$$p(\tau_1^{(n)}+\tau_2^{(n)}+\ldots+\tau_k^{(n)}) = \xi_k^{(n)}, \quad n, k = 1, 2, \ldots$$

Thus we have defined a point process p on X and we set

$$\mathscr{F}_t = \bigcap_{\varepsilon>0} \sigma\{N_p((0,s]\times U)\colon s < t+\varepsilon,\ U \in \mathscr{B}(X)\}.$$

Then, this p is an $(\mathscr{F}_t)$-stationary Poisson point process with the characteristic measure n.

Given a point process p of the class (QL) and an $(\mathscr{F}_t)$-predictable process $f(t, x, \omega)$, we can define the stochastic integrals

$$\int_0^t\int_X f(s, x, \cdot)N_p(ds, dx) \quad \text{and} \quad \int_0^t\int_X f(s, x, \cdot)[N_p(ds, dx) - \phi_p(ds, dx)]$$

if f satisfies suitable integrability condition (cf. [9] for details). Note that the former is just an absolutely convergent sum $\sum_{\substack{s\in D_p\\ s\leqslant t}} f(s, p(s), \cdot)$ whereas the latter is a limit of a compensated sum defined as a martingale stochastic integral.

Finally, we discuss on stochastic differential equations involving a Poisson point process. Suppose we are given, on a space $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$, an r-dimensional $\mathscr{F}_t$-Wiener process $(B^1(t), B^{(2)}(t), \ldots, B^{(r)}(t))$ and an $\mathscr{F}_t$-stationary Poisson point process on $(E, \mathscr{B}(E))$ with the characteristic measure $n(du)$. Let $f(t, x, u) = (f^i(t, x, u))\colon [0,\infty)\times \boldsymbol{R}^n\times E \to \boldsymbol{R}^n$ be Borel measurable such that there exists $E_0 \in \mathscr{B}(E)$ such that, for a constant $K > 0$,

(2.4) $$\int_{E_0} |f(t, x, u) - f(t, x, u)|^2 n(du) \leqslant K|x-y|^2, \quad x, y \in \boldsymbol{R}^n,\ t > 0,$$

(2.5) $$\int_{E_0} |f|^2(t, x, u)n(du) \leqslant K(1+|x|^2), \quad t > 0,\ x \in \boldsymbol{R}^n,$$

(2.6) $$n(E\setminus E_0) < \infty.$$

Let

$$\sigma(t, x) = (\sigma_k^i(t, x))\colon [0,\infty)\times \boldsymbol{R}^n \to \boldsymbol{R}^n \otimes \boldsymbol{R}^r$$

and

$$b(t,x) = (b^i(t,x)) \colon [0,\infty)\times \mathbf{R}^n \to \mathbf{R}^n$$

be Borel measurable such that, for a constant $K > 0$,

$$\|\sigma(t,x)\|^2 + |b(t,x)|^2 \leqslant K(1+|x|^2), \quad t>0,\ x\in \mathbf{R}^n, \tag{2.7}$$

$$\|\sigma(t,x)-\sigma(t,y)\|^2 + |b(t,x)-b(t,y)|^2 \leqslant K|x-y|^2, \quad t>0,\ x,y\in \mathbf{R}^n. \tag{2.8}$$

Then the following stochastic differential equation

$$\begin{aligned} X^i(t) = X^i(0) &+ \sum_{k=1}^{r}\int_0^t \sigma_k^i(s,X(s))\,dB^k(s) + \int_0^t b^i(s,X(s))\,ds + \\ &+ \int_0^t\int_{E_0} f^i(s,X(s-),u)\,[N_p(ds,du)-ds\,n(du)] + \\ &+ \int_0^t\int_{E_0^c} f^i(s,X(s-),u)\,N_p(ds,du) \end{aligned} \tag{2.9}$$

has the unique strong solution (cf. [2], [5]).

3. Conditions on analytic data

Let $D = \mathbf{R}_+^n = \{x = (x_1,x_2,\ldots,x_n);\ x_n \geqslant 0\}$ be the upper half space of $\mathbf{R}^n$, $\mathring{D} = \{x\in D;\ x_n > 0\}$ be its interior and $\partial D = \{x\in D;\ x_n = 0\}$ be its boundary. Suppose we are given the following analytic data:

(i) *a second order differential operator A on D of elliptic type* (possibly degenerate)

$$Af(x) = \frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(x)D_iD_jf(x) + \sum_{i=1}^{n} b^i(x)D_if(x) - c(x)f(x) \quad \left(D_i = \frac{\partial}{\partial x_i}\right) \tag{3.1}$$

where $a(x) = (a^{ij}(x))$ is symmetric and non-negative definite and $c(x)\geqslant 0$,

(ii) *a Wentzell's boundary condition L*, which is a map from a smooth function on D to a function on ∂D given as

$$\begin{aligned} Lf(x) = \frac{1}{2}\sum_{i,j=1}^{n-1}\alpha^{ij}(x)D_iD_jf(x) &+ \sum_{i=1}^{n-1}\beta^i(x)D_if(x) - \\ &-\gamma(x)f(x) + \mu(x)D_nf(x) - \varrho(x)Af(x) + \\ &+ \int_{u\in\mathbf{R}\setminus\{0\}}\Big[f(x+g(x,u)) - f(x) - I_{\{|u|\leqslant 1\}}\sum_{i=1}^{n-1} g^i(x,u)D_if(x)\Big]\frac{du}{|u|^2}, \end{aligned} \tag{3.2}$$

where $\alpha(x) = (\alpha^{ij}(x))$ is symmetric and non-negative definite $\gamma(x)\geqslant 0$, $\mu(x)\geqslant 0$, $\varrho(x)\geqslant 0$ ($x\in\partial D$) and $g(x,u) = \{g^i(x,u)\}_{i=1}^n$ $x\in\partial D$, $u\in\mathbf{R}$, satisfies $g(x,0)=0$, $g^n(x,u)\geqslant 0$ and

$$\int_{0<|u|\leqslant 1}\Big\{g^n(x,u) + \sum_{i=1}^{n-1} g^i(x,u)^2\Big\}\frac{du}{|u|^2} < \infty.$$

We assume $a^{nn}(x) > 0$ everywhere. Then, by a time change (or by a transformation of coordinates) we may, without loss of generality, assume that $a^{nn}(x)\equiv 1$. We may also assume that $b^n(x)\equiv 0$, $c(x)\equiv 0$ and $\gamma(x)\equiv 0$ since, if a diffusion is constructed in such a case, then by a well-known probabilistic method of transformations of drift and killing, a general case is easily constructed. Thus, we assume

(A.I) We have

$$a^{nn}(x)\equiv 1, \quad b^n(x)\equiv 0, \quad c(x)\equiv 0 \quad \text{and} \quad \gamma(x)\equiv 0. \tag{3.3}$$

Next, we assume the following regularity conditions on coefficients.

(A.II)$_1$ There exists $\sigma(x) = (\sigma_k^i(x))_{i,k=1,2,\ldots,n}$, $x\in D$, such that $\sigma(x)$ is bounded and Lipschitz continuous, $\sigma_n^n(x)\equiv 1$, $\sigma_k^n(x)\equiv 0$, $k = 1,2,\ldots,n-1$, and $a^{ij}(x) = \sum_{k=1}^{n}\sigma_k^i(x)\sigma_k^j(x)$.

N.B. If $a(x)$ is of the class C^2 on some neighborhood of D, then such $\sigma(x)$ exists.

(A.II)$_2$ $b(x) = (b^i(x))\colon x\in D\to\mathbf{R}^n$ is bounded and Lipschitz continuous.

(A.II)$_3$ There exists $\tau(x) = (\tau_k^i(x))_{i,k=1,2,\ldots,n-1}$, $x\in\partial D$ such that it is bounded and Lipschitz continuous and

$$\alpha^{ij}(x) = \sum_{k=1}^{n-1}\tau_k^i(x)\tau_k^j(x).$$

N.B. If $\alpha(x)$ is of the class C^2 on ∂D, then such $\tau(x)$ exists.

(A.II)$_4$ $\beta(x) = (\beta^i(x))\colon x\in\partial D\to\mathbf{R}^{n-1}$ is bounded and Lipschitz continuous.

(A.II)$_5$ $\mu(x)\colon x\in\partial D\to[0,\infty)$ is bounded and Lipschitz continuous.

(A.II)$_6$ $\varrho(x)\colon x\in\partial D\to[0,\infty)$ is bounded and Borel measurable.

(A.II)$_7$ $g(x,u) = (g^i(x,u))\colon (x,u)\in\partial D\times\mathbf{R}\to D$ satisfies the following;

(i) it is Borel measurable,

(ii) $g(x,0) = 0$

(iii) there exists bounded function $h(u)$ defined on $[-1,1]$ such that $h(0)=0$,

$$\int_{|u|\leqslant 1} h(u)\frac{du}{|u|^2} < \infty \tag{3.4}$$

and, for every $x,y\in\partial D$ and $u\in[-1,1]$,

$$0\leqslant g^n(x,u)\leqslant h(u), \tag{3.5}$$

$$|g^n(x,u)-g^n(y,u)|\leqslant h(u)|x-y|, \tag{3.6}$$

$$\sum_{i=1}^{n-1} |g^i(x,u)|^2 \leqslant h(u), \tag{3.7}$$

$$\sum_{n=1}^{n-1} |g^i(x,u)-g^i(y,u)|^2 \leqslant h(u)|x-y|^2. \tag{3.8}$$

Finally, we assume the *following transversality condition.*

(A.III)$_1$ For every $x \in \partial D$, one of the following occurs;

$$\mu(x) > 0 \quad \text{or} \quad \int_{-1}^{1} I_{\{g^n(x,u)>0\}} \frac{du}{|u|^2} = \infty \quad \text{or} \quad \varrho(x) > 0. \tag{3.9}$$

(A.III)$_2$ There exists a constant $K > 0$ such that, for every $x \in \partial D$,

$$\mu^2(x) + \int_R [g^n(x,u)^2 \wedge 1] \frac{du}{|u|^2} + \varrho(x) \geqslant K. \tag{3.10}$$

4. Construction of excursions of the first kind

Let

(4.1) $W = C([0,\infty) \to D) :=$ the set of all continuous paths $w\colon [0,\infty) \to D$.

We introduce the following subspaces of W;

$$\mathscr{W} = \{w \in W;\ w(0) \in \partial D,\ w(t) = w(t \wedge \sigma(w))\}, \tag{4.2}$$

$$\mathscr{W}_+ = \{w \in \mathscr{W};\ \sigma(w) > 0\}, \tag{4.3}$$

$$\mathscr{W}_0 = \{w \in \mathscr{W};\ \sigma(w) > 0,\ w(0) = 0 \in \partial D\}, \tag{4.4}$$

where

$$\sigma(w) = \inf\{t > 0;\ w(t) \in \partial D\}. \tag{4.5}$$

Clearly, $\mathscr{W} \supset \mathscr{W}_+ \supset \mathscr{W}_0$. Let $\mathscr{B}(W)$, $\mathscr{B}(\mathscr{W})$, ... be the σ-fields on W, $\mathscr{W}$, ... generated by Borel cylinder sets.

Let

$$K(t,x) = \prod_{i=1}^{n-1} \left((2\pi t)^{-1/2} \exp\left(-\frac{x_i^2}{2t}\right)\right) \cdot \left(\frac{2}{\pi t^3}\right)^{-1/2} x_n \exp\left(-\frac{x_n^2}{2t}\right) \tag{4.6}$$

and

$$p_{*}^0(t,x,y) = \prod_{i=1}^{n-1} \left((2\pi t)^{-1/2} \exp\left(-\frac{(x_i-y_i)^2}{2t}\right)\right) \cdot (2\pi t)^{-1/2} \times \tag{4.7}$$

$$\times \left(\exp\left(-\frac{(x_n^2-y_n^2)}{2t}\right) - \exp\left(-\frac{(x_n^2+y_n^2)}{2t}\right)\right), \quad t > 0,\ x, y \in D.$$

$p^0(t,x,y)$ is the transition density of Brownian motion on D with absorbing boundary ∂D and $K(t,x)$ is the density of an entrance law:

$$\int_D K(t,x) p^0(s,x,y)\,dx = K(t+s,y), \quad t, s > 0, \quad y \in D. \tag{4.8}$$

As is well known, there exists a σ-finite measure Q on $(\mathscr{W}_0, \mathscr{B}(\mathscr{W}_0))$ such that

$$Q\{w;\ w(t_1) \in E_1,\ w(t_2) \in E_2, \ldots, w(t_m) \in E_m,\ \sigma(w) > t_m\} \tag{4.9}$$

$$= \int_{E_1} dx_1 \int_{E_2} dx_2 \ldots \int_{E_m} dx_m K(t_1,x_1) \prod_{i=1}^{m-1} p^0(t_{i+1}-t_i, x_i, x_{i+1}),$$

$$0 < t_1 < t_2 < \ldots < t_m,\ E_i \in \mathscr{B}(D).$$

Q is constructed, for example, in the following way. Consider a stochastic differential equation on R^n:

$$de_t^i = dB_t^i, \quad i = 1, 2, \ldots, n-1,$$

$$de_t^n = dB_t^n + \frac{1 - \dfrac{(e_t^n)^2}{T-t}}{e_t^n}\,dt, \tag{4.10}$$

$$e_0^i = 0, \quad i = 1, 2, \ldots, n,$$

where $(B_t^1, B_t^2, \ldots, B_t^n)$ is an n-dimensional Wiener process. Then, there exists a unique solution e_t for $t \in [0,T)$ such that $e_t^n > 0$, $t \in (0,T)$ and $\lim_{t\uparrow T} e_t^n = 0$. In fact, the uniqueness of the solution in $[0,T)$ of

$$df_t^n = 2(f_t^n \vee 0)^{1/2} dB_t^n + \left[3 - \frac{2f_t^n}{T-t}\right] dt, \quad f_0^n = 0$$

is well known and $e_t^n = (f_t^n)^{1/2}$. By setting $e_t^i = e_T^i = B_T^i$, $i = 1, 2, \ldots, n-1$ and $e_t^n = 0$ for $t \geqslant T$, $t \to e_t$ is a $\mathscr{W}_0$-valued random variable such that $\sigma(e) \equiv T$ and we denote the probability law as P_T. Then

$$Q(B) = \int_0^\infty P_T(B \cap \{w;\ \sigma(w) = T\}) \frac{dT}{\sqrt{2\pi T^3}}. \tag{4.11}$$

By Theorem 2.1, there exists a stationary Poisson point process p_1 on $\mathscr{W}_0$ with the characteristic measure Q.

DEFINITION 4.1. p_1 is called a *Poisson point process of Brownian excursions of the first kind.*

Let $(\mathscr{W}_0, \mathscr{B}(\mathscr{W}_0), Q)$ be the above σ-finite measure space and $\mathscr{B}_t(\mathscr{W}_0)$ be the sub-σ-field of $\mathscr{B}(\mathscr{W}_0)$ generated by cylinder sets up to time t. Let us denote by E^Q the integration by the measure Q. It is easy to see, for $0 < t_1 < t_2$ and $i, j = 1, 2, \ldots, n$,

$$E^Q\big(w_i(t_2) - w_i(t_1)/\mathscr{B}_{t_1}(\mathscr{W}_0)\big) = 0 \text{ a.s.}, \tag{4.12}$$

$$E^Q\big([w_i(t_2)-w_i(t_1)][w_j(t_2)-w_j(t_1)]/\mathscr{B}_{t_1}(\mathscr{W}_0)\big) \tag{4.13}$$

$$= \delta_{ij} E^Q\big([t_2 \wedge \sigma(w) - t_1 \wedge \sigma(w)]/\mathscr{B}_{t_1}(\mathscr{W}_0)\big) \text{ a.s.}$$

For a $\mathscr{B}_t(\mathscr{W}_0)$-adapted measurable process $f(t,w)$ such that for every t,

$$E^Q\Big[\int_0^{t\wedge\sigma(w)} f^2(s,w)\,ds\Big] < \infty, \tag{4.14}$$

(hence, in particular, for any bounded $f(t, w)$) we can define the stochastic integral

$$I_i(f)(t) = \int_0^t f(s, w)\, dw_i(s), \quad i = 1, 2, \ldots, n$$

in exactly the same way as ordinary Itô's integral and, for almost all $w(Q)$, $t \in (0, \infty) \to I_i(f)(t)$ is continuous, $I_i(f)(0+) = 0$ and $I_i(f)(t) = I_i(f)(t \wedge \sigma(w))$.

Let $\sigma(x)$: $x \in D \to \sigma(x) \in \boldsymbol{R}^n \otimes \boldsymbol{R}^n$ and $b(x)$: $x \in D \to b(x) \in \boldsymbol{R}^n$ be given as in § 3, and let $c \geqslant 0$ be given. Consider the following stochastic differential equation for the process $X(t)$ on D:

$$\text{(4.15)} \quad \begin{aligned} X_i(t) &= x_i + c \sum_{k=1}^{n} \int_0^t \sigma_k^i(X(s))\, dw_k(s) + c^2 \int_0^{t \wedge \sigma(w)} b^i(X(s))\, ds, \\ &\qquad\qquad i = 1, 2, \ldots, n-1, \\ X_n(t) &= c w_n(t). \end{aligned}$$

THEOREM 4.1. *For given $x = (x_1, x_2, \ldots, x_{n-1}, 0) \in \partial D$ and $c \geqslant 0$, there exists a unique solution $X^{x,c}(t)$ of* (4.15). *This solution defines a map $(x, c, w) \in \partial D \times [0, \infty) \times \mathscr{W}_0 \to X^{x,c} \in \mathscr{W}$ which is measurable $\mathscr{B}(\partial D) \times \mathscr{B}[0, \infty) \times \overline{\mathscr{B}(\mathscr{W}_0)}^Q / \mathscr{B}(\mathscr{W})$.* (1)

Furthermore, for every $T > 0$ and $C > 0$, there exists a constant $K = K(T, C)$ such that

$$\text{(4.16)} \quad E_Q\{|(X^{x,c}(t) - x) - (X^{y,c'}(t) - y)|^2\} \leqslant K\{|c - c'|^2 + |x - y|^2\},$$
$$c, c' \in [0, C], \ x, y \in \partial D, \ t \in [0, T].$$

Proof. Let $\sigma_c(x) = c\sigma(x)$ and $b_c(x) = c^2 b(x)$. Then, for a fixed $C > 0$, there exists a constant $K > 0$ such that for all $c, c' \in [0, C]$ and $x, y \in D$,

$$\begin{aligned} \|\sigma_c(x)\|^2 + |b_c(x)|^2 &\leqslant K, \\ \|\sigma_c(x) - \sigma_{c'}(x)\|^2 + |b_c(x) - b_{c'}(x)|^2 &\leqslant K|c - c'|^2, \\ \|\sigma_c(x) - \sigma_c(y)\|^2 + |b_c(x) - b_c(y)|^2 &\leqslant K|x - y|^2. \end{aligned}$$

Now

$$\begin{aligned} I(t) :&= (X^{x,c}(t) - x) - (X^{y,c'}(t) - y) \\ &= \int_0^{t\wedge\sigma} [\sigma_c(X^{x,c}(s)) - \sigma_{c'}(X^{y,c'}(s))]\, dw(s) + \\ &\quad + \int_0^{t\wedge\sigma} [b_c(X^{x,c}(s)) - b_{c'}(X^{y,c'}(s))]\, ds \\ &= \int_0^{t\wedge\sigma} [\sigma_c(X^{x,c}(s)) - \sigma_c(X^{y,c'}(s))]\, dw(s) + \\ &\quad + \int_0^{t\wedge\sigma} [\sigma_c(X^{y,c'}(s)) - \sigma_{c'}(X^{y,c'}(s))]\, dw(s) + \\ &\quad + \int_0^{t\wedge\sigma} [b_c(X^{x,c}(s)) - b_c(X^{y,c'}(s))]\, ds + \\ &\quad + \int_0^{t\wedge\sigma} [b_c(X^{y,c'}(s)) - b_{c'}(X^{y,c'}(s))]\, ds. \end{aligned}$$

(1) $\overline{\mathscr{B}(\mathscr{W}_0)}^Q$ is the completion of $\mathscr{B}(\mathscr{W}_0)$ by Q.

Hence, noting $E^Q(t \wedge \sigma) \leqslant E^Q(T \wedge \sigma)$, $t \in [0, T]$, we see, by a standard argument, that there exists a constant $K = K(T, C)$ such that, if $t \in [0, T]$ and $c, c' \in [0, C]$,

$$E^Q(|I|^2(t)) \leqslant K(|x - y|)^2 + |c - c'|^2 + \int_0^t E^Q[|I(s)|^2]\, ds.$$

The conclusion follows at once from this estimate.

DEFINITION 4.2. Let, for given $x \in \partial D$ and $c \geqslant 0$, $(Y^{x,c}(t))_{t \geqslant 0}$ be defined by

$$\text{(4.17)} \quad Y^{x,c}(t) = \begin{cases} X^{x,c}(t/c^2), & \text{if} \quad c > 0, \\ x, & \text{if} \quad c = 0. \end{cases}$$

Let $\mu(x)$ be given as in § 3.

DEFINITION 4.3. Let a map Φ: $(x, w) \in \partial D \times \mathscr{W}_0 \to \Phi(x, w) \in \mathscr{W}$ be defined by

$$\text{(4.18)} \quad \Phi(x, w)(t) = Y^{x, \mu(x)}(t).$$

$\Phi(x, w)$ is called the *excursion of the first kind* (of the diffusion to be constructed) *corresponding to a Brownian excursion w.*

Clearly,

$$\text{(4.19)} \quad \Phi(x, w)(0) = x$$

and

$$\text{(4.20)} \quad \sigma[\Phi(x, w)] = \mu^2(x)\sigma(w).$$

DEFINITION 4.4. Let a map φ: $(x, w) \in \partial D \times \mathscr{W}_0 \to \varphi(x, w) \in \partial D$ be defined by

$$\text{(4.21)} \quad \begin{aligned} \varphi(x, w) &= \Phi(x, w)(\sigma[(x, w)]) - x \\ &= X^{x, \mu(x)}(\sigma(w)) - x. \end{aligned}$$

From (4.16) and the Lipschitz continuity of μ, we have at once the following

THEOREM 4.2. *There exists a constant $K > 0$ such that*

$$\text{(4.22)} \quad E^Q\{|\varphi(x, w) - \varphi(y, w)|^2 \colon \sigma(w) \leqslant 1\} \leqslant K|x - y|^2, \quad x, y \in \partial D,$$

$$\text{(4.23)} \quad Q\{w;\ \sigma(w) > 1\} = \int_1^\infty (2\pi u^3)^{-1/2}\, du < \infty.$$

Let

$$\beta_1^i(x) = \mu^2(x)E^Q\Big[\int_0^{\sigma(w)} b^i(X^{x,\mu(x)}(s))\,ds:\ \sigma(w) \leqslant 1\Big] - \tag{4.24}$$

$$-\mu(x)E^Q\Big[\int_0^1 \sigma_n^i(X^{x,\mu(x)}(s))\,dw_n(s):\ \sigma(w) > 1\Big],$$

$$i = 1, 2, \ldots, n-1,\ x \in \partial D.$$

Then, using (4.16), we see easily the following

THEOREM 4.3. *$\beta_1(x) = (\beta_1^i(x))_{i=1}^{n-1}$ is bounded and Lipschitz continuous on ∂D.*

5. Construction of excursions of the second kind

We introduce the following subspaces of $W = C([0, \infty) \to D)$:

$$\mathscr{W}^* = \{w \in W;\ w(t) = w(t \wedge \sigma^*(w))\}, \tag{5.1}$$

$$\mathscr{W}_+^* = \{w \in \mathscr{W}^*;\ w(0) \in \mathring{D}\} \tag{5.2}$$

and

$$\mathscr{W}_1^* = \{w \in \mathscr{W}^*;\ w(0) = (0, 0, \ldots, 0, 1)\}, \tag{5.3}$$

where

$$\sigma^*(w) = \inf\{t \geqslant 0;\ w(t) \in \partial D\}. \tag{5.4}$$

Clearly, $\mathscr{W}^* \supset \mathscr{W}_+^* \supset \mathscr{W}_1^*$. Let P_1 be the probability measure on $(\mathscr{W}_1^*, \mathscr{B}(\mathscr{W}_1^*))$ which is the probability law of the Brownian motion on D with the absorbing boundary ∂D starting at $(0, 0, \ldots, 0, 1)$. On the product space $\mathscr{W}_1^* \times (\boldsymbol{R}\setminus\{0\})$, we define the following σ-finite measure $P_1(dw)\dfrac{du}{|u|^2}$ as a product measure.

DEFINITION 5.1. A stationary Poisson point process p_2 on $\mathscr{W}_1^* \times (\boldsymbol{R}\setminus\{0\})$ with the characteristic measure $P_1(dw)\dfrac{du}{|u|^2}$ is called a *Poisson point process of Brownian excursions of the second kind.*

Let $x = (x_1, x_2, \ldots, x_n) \in D$. $w \in \mathscr{W}_1^*$ is a sample of a Wiener process with absorbing boundary starting at $(0, 0, \ldots, 0, 1)$ with respect to the probability P_1 and hence, we can consider the following stochastic differential equation for $\sigma(x)$ and $b(x)$ given in § 3:

$$\begin{aligned} X_i(t) &= x_i + x_n\sum_{k=1}^{n}\int_0^t \sigma_k^i(X(s))\,dw_k(s) + x_n^2\int_0^{t\wedge\sigma^*(w)} b^i(X(s))\,ds, \quad i = 1, 2, \ldots, n-1,\\ X_n(t) &= x_n w_n(t). \end{aligned} \tag{5.5}$$

Clearly, the solution exists uniquely which we denote as $X^x(t)$. Note that $X^x(t) = X^x(t\wedge\sigma^*(w))$.

THEOREM 5.1. *The solution $X^x(t)$ defines a map $(x, w) \in D \times \mathscr{W}_1^* \to X^x \in \mathscr{W}^*$ which is measurable $\mathscr{B}(D)\times\overline{\mathscr{B}(\mathscr{W}_1^*)}^{P_1}/\mathscr{B}(\mathscr{W}^*)$.* (2) *Let γ be the projection $\boldsymbol{R}^n \to \boldsymbol{R}^{n-1}$: $\gamma x = (x_1, x_2, \ldots, x_{n-1})$ for $x = (x_1, x_2, \ldots, x_{n-1}, x_n)$.*

Then, there exists a constant K such that, for every $t > 0$, $x, y \in D$,

$$\begin{aligned} &E^{P_1}[|\gamma(X^x(t)-x) - \gamma(X^y(t)-y)|^2]\\ &\quad = K[|\gamma x-\gamma y|^2\{x_n^2E^{P_1}(t\wedge\sigma^*)(1+tx_n^2)\} +\\ &\qquad |x_n-y_n|^2E^{P_1}(t\wedge\sigma^*)\{1+(x_n^2+y_n^2)t+x_n^4t^2\}]\exp K(tx_n^2+t^2x_n^4). \end{aligned} \tag{5.6}$$

Proof. Let $\xi_i(t) = X_i^x(t)-x_i$, $\eta_i(t) = X_i^y(t)-y_i$, $i = 1, 2, \ldots, n-1$. Then, $I(t) := \xi(t)-\eta(t)$

$$\begin{aligned} &= \int_0^{t\wedge\sigma^*}[x_n\sigma(X^x(s)) - y_n\sigma(X^y(s))]\,dw(s) + \int_0^{t\wedge\sigma^*}[x_n^2b(X^x(s)) - y_n^2b(X^y(s))]\,ds\\ &= x_n\int_0^{t\wedge\sigma^*}[\sigma(X^x(s))-\sigma(X^y(s))]\,dw(s) + (x_n-y_n)\int_0^{t\wedge\sigma^*}\sigma(X^y(s))\,dw(s) +\\ &\quad + x_n^2\int_0^{t\wedge\sigma^*}[b(X^x(s))-b(X^y(s))]\,ds + (x_n^2-y_n^2)\int_0^{t\wedge\sigma^*} b(X^y(s))\,ds. \end{aligned}$$

Therefore, for some constants K and K',

$$\begin{aligned} E^{P_1}(|I(t)|^2) &\leqslant K\Big\{\big(|x_n-y_n|^2E^{P_1}(t\wedge\sigma^*) + |x_n-y_n|^2(x_n+y_n)^2E^{P_1}((t\wedge\sigma^*)^2)\big) +\\ &\quad + x_n^2E^{P_1}\Big[\int_0^{t\wedge\sigma^*}\{|\gamma x-\gamma y|^2 + |x_n-y_n|^2|w_n(s)|^2 + |I(s)|^2\}\,ds\Big] +\\ &\quad + x_n^4E^{P_1}\Big[\Big(\int_0^{t\wedge\sigma^*}\{|\gamma x-\gamma y|^2 + |x_n-y_n|^2|w_n(s)|^2 + |I(s)|^2\}\,ds\Big)\,t\wedge\sigma^*\Big]\Big\}\\ &\leqslant K'\Big\{|\gamma x-\gamma y|^2[x_n^2E^{P_1}(t\wedge\sigma^*)\,(1+tx_n^2)] + |x_n-y_n|^2[E^{P_1}(t\wedge\sigma^*)\,(1+\\ &\quad + (x_n+y_n)^2t + x_n^4t^2)] + (x_n^2+x_n^4t)\int_0^t E^{P_1}(|I(s)|^2)\,ds\Big\}. \end{aligned}$$

(5.6) follows easily from this estimate. Here we used the following inequality:

$$E^{P_1}\Big[\int_0^{t\wedge\sigma^*}|w_n(s)|^2\,ds\Big] \leqslant \int_0^t E^{P_1}[w_n(s\wedge\sigma^*)^2]\,ds = \int_0^t E^{P_1}(s\wedge\sigma^*)\,ds \leqslant tE^{P_1}(t\wedge\sigma^*).$$

DEFINITION 5.2. Let, for a given $x \in D$, $(Y^x(t))_{t\geqslant 0}$ be defined by

$$Y^x(t) = \begin{cases} X^x\left(\dfrac{t}{x_n^2}\right), & x_n > 0\\ x, & x_n = 0. \end{cases} \tag{5.7}$$

Let $g(x, u) = (g^i(x, u))$ be given as in § 3.

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(2) $\overline{\mathscr{B}(\mathscr{W}_1^*)}^{P_1}$ is the completion of $\mathscr{B}(\mathscr{W}_1^*)$ with respect to P_1.

DEFINITION 5.3. Let a map $\Psi\colon (x, w, u) \in \partial D \times \mathscr{W}_1^* \times \boldsymbol{R} \to \Psi(x, w, u) \in \mathscr{W}^*$ be defined by

$$\Psi(x, w, u)(t) = Y^{x+g(x,u)}(t). \tag{5.8}$$

$\Psi(x, w, u)$ is called the *excursion of the second kind* (of the diffusion to be constructed) *corresponding to a Brownian excursion* (w, u) *of the second kind.*

Clearly,

$$\Psi(x, w, u)(0) = x+g(x, u), \tag{5.9}$$

$$\sigma^*[\Psi(x, w, u)] = g_n(x, u)^2\sigma^*(w). \tag{5.10}$$

DEFINITION 5.4. Let a map $\psi\colon (x, w, u) \in \partial D \times \mathscr{W}_1^* \times \boldsymbol{R} \to \psi(x, w, u) \in \partial D$ be defined by

$$\begin{aligned}\psi(x, w, u) &= \Psi(x, w, u)\big(\sigma^*[\Psi(x, w, u)]\big)-x\\ &= X^{x+g(x,u)}\big(\sigma^*(w)\big)-x.\end{aligned} \tag{5.11}$$

THEOREM 5.2. *There exists a constant $K > 0$ such that*

$$\int\limits_{0<|u|\leqslant 1} E^{P_1}\{|\psi(x, w, u)-\psi(y, w, u)|^2\colon \sigma^*(w) \leqslant h^{-2}(u)\}\frac{du}{|u|^2} \leqslant K|x-y|^2, \quad x, y \in \partial D, \tag{5.12}$$

$$\int\limits_{\mathscr{W}_1^*}\int\limits_{R\setminus\{0\}} I_{\{\sigma^*(w)\leqslant h^{-2}(u),\,|u|\leqslant 1\}}\, cP_1(dw)\frac{du}{|u|^2} < \infty. \tag{5.13}$$

Proof. By (5.6),

$$\begin{aligned}E^{P_1}\big(|\psi(x, w, u)-\psi(y, w, u)|^2\colon \sigma^*(w) \leqslant h^{-2}(u)\big) &\leqslant \sum_{i=1}^{n-1}|g^i(x, u)-g^i(y, u)|^2+\\ &+K\Big\{\Big(|x-y|^2+\sum_{i=1}^{n-1}|g^i(x, u)-g^i(y, u)|^2\Big)E^{P_1}\big(\sigma^*(w)\wedge h^{-2}(u)\big)g^n(x, u)^2\times\\ &\times\big(1+g^n(x, u)^2\cdot h^{-2}(u)\big)+|g^n(x, u)-g^n(y, u)|^2E^{P_1}\big(\sigma^*(w)\wedge h^{-2}(u)\big)\times\\ &\times\big(1+h^{-2}(u)\cdot g^n(x, u)^2+h^{-2}(u)\cdot g^n(y, u)^2+h^{-4}(u)\cdot g^n(x, u)^4\big)\Big\}\times\\ &\times\exp\{K\big(h^{-2}(u)\cdot g^n(x, u)^2+h^{-4}(u)\cdot g^n(x, u)^4\big)\}.\end{aligned}$$

If $|u| \leqslant 1$, we note the assumptions (3.5), (3.6), (3.7) and (3.8). Also, noting $P_1(\sigma^* > t) = \int_t^\infty (2\pi s^3)^{-1/2}e^{-1/2s}ds$, we see that $E^{P_1}\big(\sigma^*(w)\wedge h^{-2}(u)\big) = O\big(h^{-1}(u)\big)$. Hence,

$$\int_0^1\frac{du}{|u|^2}E^{P_1}\big(|\psi(x, w, u)-\psi(y, u, w)|^2\colon \sigma^*(w) \leqslant h^{-2}(u)\big) \leqslant K'|x-y|^2\int_0^1 h(u)\frac{du}{|u|^2}$$

and by (3.4)

$$\int_0^1 h(u)\frac{du}{|u|^2} < \infty.$$

As for (5.13), we have

$$\int\limits_{\mathscr{W}_1^*}\int\limits_{R\setminus\{0\}} I_{\{\sigma^*(w)\leqslant h^{-2}(u),\,|u|\leqslant 1\}}\,cP_1(dw)\frac{du}{|u|^2} \leqslant \int\limits_{|u|>1}\frac{du}{|u|^2}+\int\limits_{|u|\leqslant 1}P_1\{\sigma^*(w) > h^{-2}(u)\}\frac{du}{|u|^2} < \infty$$

since $P_1\{\sigma^*(w) > h^{-2}(u)\} = O\big(h(u)\big)$.

Let, for $x \in \partial D$ and $i = 1, 2, \ldots, n-1$,

$$\begin{aligned}\beta_2^i(x) = &-\int_0^1 g^i(x, u)P_1[\sigma^*(w) > h^{-2}(u)]\frac{du}{|u|^2}+\\ &+\int_0^1 g^n(x, u)^2E^{P_1}\Big[\int_0^{\sigma^*} b^i\big(X^{x+g(x,u)}(s)\big)\,ds\colon \sigma^*(w) \leqslant h(u)^{-2}\Big]\frac{du}{|u|^2}-\\ &-\int_0^1 g^n(x, u)E^{P_1}\Big[\int_0^{h^{-2}} \sigma_n^i\big(X^{x+g(x,u)}(s)\big)dw_n(s)\colon \sigma^*(w) > h(u)^{-2}\Big]\frac{du}{|u|^2}.\end{aligned} \tag{5.14}$$

Then, by the same estimate as in the proof of (5.6), we can easily prove the following

THEOREM 5.3. *$\beta_2(x) = (\beta_2^i(x))_{i=1}^{n-1}$ is bounded and Lipschitz continuous on ∂D.*

6. Construction of path functions

Now, we are going to construct our diffusion process corresponding to the analytic data (A, L) given in § 3. We prepare, on a suitable quadruplet $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$, the following:

(i) $\mathscr{G}_t \subset \mathscr{F}_0$: an increasing family of sub-σ-fields of $\mathscr{F}_0$ and an n-dimensional $\mathscr{G}_t$-Wiener process $B = (B_k(t))_{k=1}^n$,

(ii) an $(n-1)$-dimensional $\mathscr{F}_t$-Wiener process $B^* = (B_k^*(t))_{k=1}^{n-1}$,

(iii) an $\mathscr{F}_t$-stationary Poisson point process p_1 on $\mathscr{W}_0$ with the characteristic measure Q and $\mathscr{F}_t$-stationary Poisson point process p_2 on $\mathscr{W}_1^*\cup(\boldsymbol{R}\setminus\{0\})$ with the characteristic measure $P_1\times\dfrac{du}{|u|^2}$ such that they are mutually independent, i.e. mutually independent $\mathscr{F}_t$-Poisson point processes of Brownian excursions of the first and second kinds.

N.B. We have automatically that $\{B(t)\}$, $\{B^*(t)\}$, p_1 and p_2 are mutually independent. This is well known and an easy consequence of Itô's formula.

Let $x \in D$ be given as the starting point of the diffusion process to be constructed. Let $\eta^x = (\eta^x(t))$ be the solution of the stochastic differential equation

$$\begin{aligned}d\eta^i(t) &= \sum_{k=1}^n \sigma_k^i(\eta(t))dB_k(t)+b^i(\eta(t))\,dt, \qquad i = 1, 2, \ldots, n,\\ \eta(0) &= x.\end{aligned} \tag{6.1}$$

Clearly, there exists the unique solution which is $\mathcal{G}_t$-adapted and let

$$\sigma^*_{\partial D} = \inf\{t \geqslant 0;\ \eta^x(t) \in \partial D\} \tag{6.2}$$

and

$$\xi_0 = \eta^x(\sigma^*_{\partial D}) \in \partial D. \tag{6.3}$$

Note that ξ_0 is $\mathcal{F}_0$-measurable.

Let

$$\tilde{\beta}^i(x) = \beta^i(x) + \beta^i_1(x) + \beta^i_2(x), \quad i = 1, 2, \ldots, n-1,\ x \in \partial D, \tag{6.4}$$

where $\beta(x)$ is given in § 3, $\beta_1(x)$ in (4.24) and $\beta_2(x)$ in (5.14). By Theorems 4.3 and 5.3, $\tilde{\beta}(x)$ is bounded and Lipschitz continuous on ∂D. We consider the following stochastic differential equation of jump-type for the process $\xi(t)$ on ∂D:

$$\begin{aligned}\xi^i(t) = {} & \xi^i_0 + \sum_{k=1}^{n-1}\int_0^t \tau^i_k(\xi(s))\,dB^*_k(s) + \int_0^t \tilde{\beta}^i(\xi(s))\,ds + \\ & + \int_0^{t+}\int_{\mathcal{W}_0} I_{[\sigma(w)\leqslant 1]}\varphi_i(\xi(s-), w)[N_{p_1}(ds, dw) - ds\,dQ] + \\ & + \int_0^{t+}\int_{\mathcal{W}_0} I_{[\sigma(w)\leqslant 1]}c\varphi_i(\xi(s-), w)N_{p_1}(ds, dw) + \\ & + \int_0^{t+}\int_{\mathcal{W}^*_1\times(R\setminus\{0\})} I_{[\sigma^*(w)\leqslant h(u)^{-2},\ |u|\leqslant 1]}\psi_i(\xi(s-), w, u)\left[N_{p_2}(ds, dw\,du) - ds\,P_1(dw)\frac{du}{|u|^2}\right] + \\ & + \int_0^{t+}\int_{\mathcal{W}^*_1\times(R\setminus\{0\})} I_{[\sigma^*(w)\leqslant h(u)^{-2},\ |u|\leqslant 1]}c\psi_i(\xi(s-), w, u)N_{p_2}(ds, dw\,du), \\ & \qquad i = 1, 2, \ldots, n-1, \\ \xi_n(t) \equiv {} & 0.\end{aligned} \tag{6.5}$$

By Theorems 4.2 and 5.2, the coefficients of (6.5) satisfy the conditions like (2.4)–(2.8) of § 2 and hence, there exists the unique $\mathcal{F}_t$-adapted solution $\xi(t)$ on ∂D. $\{\xi(t)\}$ is called the *process on the boundary* of the diffusion to be constructed.

Next, we set, for $t > 0$,

$$\begin{aligned}A(t) = {} & \sigma^*_{\partial D} + \int_0^{t+}\int_{\mathcal{W}_0} \sigma[\Phi(\xi(s-), w)]N_{p_1}(ds, dw) + \\ & + \int_0^{t+}\int_{\mathcal{W}^*_1\times(R\setminus\{0\})} \sigma^*[\Psi(\xi(s-), w, u)]N_{p_2}(ds, dw\,du) + \int_0^t \varrho(\xi(s))\,ds \\ = {} & \sigma^*_{\partial D} + \int_0^{t+}\int_{\mathcal{W}_0} \mu^2(\xi(s-))\sigma(w)N_{p_1}(ds, dw) + \\ & + \int_0^{t+}\int_{\mathcal{W}^*_1\times(R\setminus\{0\})} g^n(\xi(s-), w, u)^2\sigma^*(w)N_{p_2}(ds, dw\,du) + \int_0^t \varrho(\xi(s))\,ds.\end{aligned} \tag{6.6}$$

$A(t)$ is called the *inverse local time* on ∂D of the diffusion to be constructed.

THEOREM 6.1. *With probability one, $t \to A(t)$ is right continuous, strictly increasing and $\lim_{t\uparrow\infty} A(t) = \infty$.*

Proof. By assumption (3.9), it is easy to see that $t \to A(t)$ is strictly increasing a.s. By Itô's formula and (3.10),

$$\begin{aligned}& E(e^{-A(t)}) - E(e^{-\sigma^*\partial D}) \\ & = -\int_0^t E(e^{-A(s)}\varrho(\xi(s))\,ds - \int_0^t\int_{\mathcal{W}_0} E(e^{-A(s)}(1 - e^{-\mu^2(\xi(s))\sigma(w)}))Q(dw) - \\ & \quad - \int_0^t\int_{\mathcal{W}^*_1\times(R\setminus\{0\})} E(e^{-A(s)}\{1 - e^{-g^n(\xi(s), u)^2\sigma^*(w)}\}P_1(dw)\frac{du}{|u|^2} \\ & \leqslant -K\int_0^t E\left(e^{-A(s)}\left\{\varrho(\xi(s)) + \mu^2(\xi(s)) + \int_{R\setminus\{0\}}[g^n(\xi(s), u)^2 \wedge 1]\frac{du}{|u|^2}\right\}\right) \\ & \leqslant -K'\int_0^t E(e^{-A(s)})\,ds.\end{aligned}$$

Hence, $1 \geqslant K'\int_0^t E(e^{-A(s)})$ implying $\lim_{t\uparrow\infty} E(e^{-A(t)}) = 0$, i.e. $\lim_{t\uparrow\infty} A(t) = \infty$ a.s.

For every $t \geqslant 0$, there exists unique $s := \varphi(t)$ such that $A(s-) \leqslant t \leqslant A(s)$.

DEFINITION 6.1. $(\varphi(t))$ is called the *local time on the boundary* of the diffusion to be constructed.

If $s := \varphi(t) = 0$, i.e. $0 \leqslant t \leqslant \sigma^*_{\partial D}$, we set

$$X^x(t) = \eta^x(t). \tag{6.7}$$

If $s := \varphi(t) > 0$ and $A(s-) < A(s)$, then this implies that just one of the following cases occurs:

(i) $s \in D_{p_1}$ and $\mu(\xi(s-)) > 0$,

(ii) $s \in D_{p_2}$ and, if we set $p_2(s) = (w^s, u^s)$, $g^n(\xi(s-), u^s) > 0$.

We set in the case of (i) (by setting $p_1(s) = w^s$)

$$X^x(t) = \Phi(\xi(s-), w^s)\,(t - A(s-)) \tag{6.8}$$

and in the case of (ii)

$$X^x(t) = \Psi(\xi(s-), w^s, u^s)\,(t - A(s-)). \tag{6.9}$$

If $s > 0$ and $A(s-) = A(s)$ $(= t)$, then just one of the following cases occurs:

(i) $s \in D_{p_2}$ and, if we set $p_2(s) = (w^s, u^s)$, then $g(\xi(s-), u^s) \neq 0$ and $g(\xi(s-), u^s) \in \partial D$,

(ii) otherwise, i.e. $s \notin D_{p_1} \cup D_{p_2}$, or $s \in D_{p_1}$ and $\mu(\xi(s-)) = 0$, or $s \in D_{p_2}$ and $g(\xi(s-), u^s) = 0$.

In the case (i), we have $\xi(s) = \xi(s-)+g(\xi(s-), u^s)$ and we set

$$X^x(t) = \xi(s). \tag{6.10}$$

In the case (ii), $\xi(s-) = \xi(s)$ and we set

$$X^x(t) = \xi(s). \tag{6.11}$$

Thus, we have defined, for almost all elements in Ω, a map $t \in [0, \infty) \to X^x(t)$ which is clearly right continuous with left-hand limits and $X^x(t) \neq X^x(t-)$ occurs only when $X^x(t-) \in \partial D$.

7. Concluding remarks

The stochastic process $X^x = (X^x(t))$ constructed in § 6 is a diffusion process on D described by the given analytic data (A, L) in the sense that, for every $f \in C_b^2(D)$,

$$f(X(t))-f(X(0)) = \text{a martingale}+\int_0^t (Af)(X(s))\,ds+\int_0^t (Lf)(X(s))\,d\varphi(s), \tag{7.1}$$

$$\int_0^t I_{\partial D}(X(s))\,ds = \int_0^t \varrho(X(s))\,d\varphi(s), \tag{7.2}$$

where $\varphi(t)$ is the local time (cf. Definition 6.1). Furthermore, we can prove, under the assumption of § 3, the uniqueness of the above martingale problem.

The proof of these facts requires formulas like excursion formulas or the last exit formula as in [9], [10] (which are obtained immediately by the way of our construction) and some formulas on stochastic sum over excursions (such a formula was discussed in [10]). Details will be given in a future publication.

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