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# WEAK INEQUALITIES FOR EXIT TIMES AND ANALYTIC FUNCTIONS*

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We give a method for obtaining inequalities of the form

$$\sup_{\lambda>0}\Phi(\lambda)P(f>\lambda)\leqslant c\sup_{\lambda>0}\Phi(\lambda)P(g>\lambda) \tag{1}$$

and then use it to obtain some results about exit times of Brownian motion and analytic functions supplementing those of [3].

Let $\Phi$ be a continuous nondecreasing function from $[0, \infty]$ into $[0, \infty]$ with $\Phi(0) = 0$ and

$$\Phi(2\lambda)\leqslant c\Phi(\lambda),\quad \lambda>0. \tag{2}$$

In particular, $\Phi$ could be any positive power: $\Phi(\lambda) = \lambda^p$ $(0 < p < \infty)$. Throughout the paper $c$ and $C$ denote positive real numbers not necessarily the same from one use to the next.

LEMMA 1. *Suppose that $f$ and $g$ are nonnegative measurable functions on a probability space and $\beta > 1$, $\delta > 0$, $\varepsilon > 0$ are real numbers such that*

$$P(f>\beta\lambda, g\leqslant\delta\lambda)\leqslant\varepsilon P(f>\lambda),\quad \lambda>0. \tag{3}$$

*Let $\gamma$ and $\eta$ be real numbers satisfying*

$$\Phi(\beta\lambda)\leqslant\gamma\Phi(\lambda),\quad \Phi(\delta^{-1}\lambda)\leqslant\eta\Phi(\lambda),\quad \lambda>0. \tag{4}$$

*Also suppose that $\gamma\varepsilon < 1$. Then* (1) *holds with*

$$c=\gamma\eta/(1-\gamma\varepsilon). \tag{5}$$

Under the same conditions and with the same choice of $c$, the "strong" inequality

$$E\Phi(f)\leqslant cE\Phi(g) \tag{6}$$

also holds; see [2] and, for an earlier version, [4]. Inequality (1) is, of course, of particular interest in those cases for which $E\Phi(f)$ is infinite but the right-hand side of (1) is finite. If, instead of being a probability measure, $P$ is an arbitrary positive measure, the conclusion of the lemma still holds provided the left-hand side of (1)

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is finite. Note that the existence of $\gamma$ and $\eta$ satisfying (4) is assured by the growth condition (2).

*Proof of Lemma* 1. We can and do assume in the proof that the left-hand side of (1) is finite. For, if $f$ satisfies (3), then so does $f \wedge n$; if (1) holds for $f \wedge n$, $n \geqslant 1$, then it holds for $f$.

Condition (3) implies that

$$P(f > \beta\lambda) \leqslant \varepsilon P(f > \lambda) + P(g > \delta\lambda).$$

Multiplying both sides by $\Phi(\beta\lambda) = \Phi(\beta\delta^{-1}\delta\lambda)$ and using (4), we obtain

$$\Phi(\delta\lambda)P(f > \beta\lambda) \leqslant \gamma\varepsilon\Phi(\lambda)P(f > \lambda) + \gamma\eta\Phi(\delta\lambda)P(g > \delta\lambda).$$

Taking the least upper bound of each term with respect to $\lambda$ and using the finiteness of the left-hand side of (1), we obtain (1) with $c$ given by (5).

We shall need the following application of Lemma 1.

LEMMA 2. *Let $X = \{X_t, 0 \leqslant t < \infty\}$ be a Brownian motion in $\mathbf{R}^n$ $(n \geqslant 1)$ starting at $x$. Let $\tau$ be a stopping time of $X$ and*

$$X_\tau^* = \sup_t |X_{\tau\wedge t}|.$$

*Then*

$$c\sup_\lambda \Phi(\lambda)P_x([n\tau + |x|^2]^{1/2} > \lambda) \leqslant \sup_\lambda \Phi(\lambda)P_x(X_\tau^* > \lambda) \leqslant C\sup_\lambda \Phi(\lambda)P_x([n\tau + |x|^2]^{1/2} > \lambda) \tag{7}$$

*with the choice of $c$ and $C$ depending only on $\Phi$ and $n$.*

*Proof.* Let $f = X_\tau^*$ and $g = [n\tau + |x|^2]^{1/2}$. Then by (2.8) and (2.9) of [3], condition (3) of Lemma 1 is satisfied with $\varepsilon < \gamma^{-1}$ provided $\delta$ is chosen small enough. So (1) holds. The reverse inequality also holds in view of (2.11) and (2.12) of [3]. This completes the proof.

### Exit times of Brownian motion in $\mathbf{R}^n$

Let $R$ be an open, connected subset of $\mathbf{R}^n$ $(n \geqslant 2)$, $X$ a Brownian motion starting at a point $x$ in $R$, and $\tau$ the first time $X$ leaves $R$:

$$\tau(\omega) = \inf\{t > 0: X_t(\omega) \notin R\}.$$

If $0 < p < \infty$ and $u$ is a function harmonic in $R$ such that

$$|x|^p \leqslant u(x), \quad x \in R,$$

then

$$cE_x(n\tau + |x|^2)^{p/2} \leqslant u(x), \quad x \in R, \tag{8}$$

with the choice of $c$ depending only on $p$ and $n$. If $u$ is the least harmonic majorant of $|x|^p$ in $R$, the reverse inequality also holds (with a different constant). These and related results are proved in [3].



Consider the following example. Fix $\alpha \in (0, \pi]$ and let

$$R = \{x \in \mathbf{R}^n: x \neq 0, 0 \leqslant \theta < \alpha\},$$

where $\theta$ is the angle between the vectors $x$ and $(1, 0, \ldots, 0)$. Let

$$h(\theta) = F(-p, p+n-2; (n-1)/2; (1-\cos\theta)/2),$$

where $F(a, b; c; t)$ is the hypergeometric function. (If $n = 2$, $h(\theta) = \cos p\theta$.) Then the inequality (8) holds with $u(x) = |x|^p h(\theta)/h(\alpha)$, the least harmonic majorant of $|x|^p$ in $R$, provided $\alpha$ is less than the first positive zero of $h$: $h(\theta) > 0$, $0 < \theta \leqslant \alpha$. (That $u$ majorizes $|x|^p$ implies a fact that we shall need later: $h$ is nonincreasing, so $h \leqslant h(0) = 1$, up to the first positive zero.) On the other hand, if $h$ has at least one zero in $(0, \alpha]$, then the left-hand side of (8) is infinite. (If $n \geqslant 3$, $h$ has at least one zero in $(0, \pi)$; if $n = 2$, $h$ has at least one zero in $(0, \pi]$ provided $p \geqslant 1/2$.) See [3] for further details.

Robert Kaufman has asked me whether a weak inequality holds in the borderline case. We shall show here that, indeed, a weak inequality does hold. Specifically, if $h(\theta) > 0$, $0 < \theta < \alpha$, but $h(\alpha) = 0$, then

$$c\sup_\lambda \lambda^p P_x([n\tau + |x|^2]^{1/2} > \lambda) \leqslant |x|^p, \quad x \in R, \tag{9}$$

with the choice of $c$ depending only on $p$ and $n$.

We can describe this result in another way. The region $R$ has a positive real number $e(R)$ associated with it such that, for all $x \in R$,

$$E_x\tau^{p/2} < \infty \Leftrightarrow 0 < p < e(R).$$

For example, if $n = 2$, then $e(R) = \pi/(2\alpha)$. Here is an equivalent form of our result: If $p = e(R)$, then (9) holds.

By Lemma 2, the proof of (9) can be reduced to proving a similar inequality for the maximal function $X_\tau^*$:

$$c\sup_\lambda \lambda^p P_x(X_\tau^* > \lambda) \leqslant |x|^p, \quad x \in R. \tag{10}$$

To prove (10), we here let

$$u(x) = \begin{cases} |x|^p h(\theta), & x \in R, \\ 0, & x \in \partial R, \end{cases}$$

$$u_\tau^* = \sup_t u(X_{\tau\wedge t}),$$

$$Y_x = [|x|^p u_\tau^*/u(x)]^{1/p}, \quad x \in R.$$

Then $u$ is continuous on $R \cup \partial R$, harmonic in $R$, and $u(x) \leqslant |x|^p$. Also, assuming as usual that the Brownian motion $X$ starts at $x$, we have

$$u_\tau^* \leqslant Y_x^p, \quad |x| \leqslant Y_x, \quad x \in R. \tag{11}$$

Let $x \in R$. Since

$$E_x u(X_{\tau\wedge t}) \leqslant E_x|X_{\tau\wedge t}|^p \leqslant E_x|X_t|^p < \infty,$$

the stochastic process $\{|x|^p u(X_{\tau\wedge t})/u(x),\ t \geqslant 0\}$, which starts at $|x|^p$, is a nonnegative martingale (see [7]). By definition, $Y_x^p$ is its maximal function. Therefore,

$$\lambda^p P_x(Y_x > \lambda) = \lambda^p P_x(Y_x^p > \lambda^p) \leqslant |x|^p. \tag{12}$$

The main step in the proof of (10) is to show that, for $0 < \delta < 1 < \beta$,

$$P_x(X_\tau^* > \beta\lambda,\ Y_x \leqslant \delta\lambda) \leqslant \varepsilon P_x(X_\tau^* > \lambda), \qquad \lambda > 0, \tag{13}$$

where $\varepsilon$ depends only on $R$, $\beta$, $\delta$ and $\varepsilon \to 0$ as $\delta \to 0$.

By Lemma 1, the two inequalities (12) and (13) imply (10).

To prove (13), we can assume that $|x| < \lambda$ since otherwise, by (11), the left-hand side of (13) is zero. Let

$$\mu = \inf\{t > 0\colon |X_{\tau\wedge t}| > \lambda\}.$$

Then

$$\{\mu < \infty\} = \{X_\tau^* > \lambda\} \subset \{|X_\mu| = \lambda,\ X_\mu \in R\}.$$

Therefore, by (11),

$$\begin{aligned} P_x(X_\tau^* > \beta\lambda,\ Y_x \leqslant \delta\lambda) &\leqslant P_x(\mu < \infty,\ \sup_{\mu<t\leqslant\tau} |X_t| > \beta\lambda,\ \sup_{\mu<t\leqslant\tau} u(X_t) \leqslant \delta^p\lambda^p) \\ &\leqslant \sup_{\substack{|y|=\lambda \\ y\in R}} P_y(X_\tau^* > \beta\lambda,\ u_\tau^* \leqslant \delta^p\lambda^p) P_x(\mu < \infty) \\ &= \varepsilon P_x(X_\tau^* > \lambda). \end{aligned}$$

We now show that $\varepsilon$ as defined has the desired properties. By homogeneity,

$$\varepsilon = \sup_{\substack{|y|=1 \\ y\in R}} P_y(X_\tau^* > \beta,\ u_\tau^* \leqslant \delta^p),$$

that is, $\varepsilon$ does not depend on $\lambda$. Consider the open, connected set

$$G = \{y \in R\colon |y| < \beta,\ u(y) < 2\delta^p\}$$

and

$$A = \{y \in R\colon |y| = \beta,\ u(y) < 2\delta^p\}.$$

Then $A \subset \partial G$ and, for $y \in G$,

$$P_y(X_\tau^* > \beta,\ u_\tau^* < 2\delta^p) \leqslant P_y(X \text{ hits } A \text{ before } \partial G \setminus A) = w(y),$$

where $w(y)$ is the harmonic measure of $A$ with respect to $G$ at the point $y$. Let $w(y) = 0$ for all $y$ outside of $G$. Then, since the boundary of $G$ is sufficiently smooth, $w$ is continuous on the compact set $|y| = 1$. Furthermore, $w(y)$ is monotone in $\delta$. In fact, $w(y) \downarrow 0$ as $\delta \downarrow 0$, $|y| = 1$, since

$$w(y) \leqslant P_y(u_{\tau\wedge\nu}^* < 2\delta^p) \downarrow 0 \quad \text{as} \quad \delta \downarrow 0,\ y \in R,$$

where $\nu = \inf\{t > 0\colon |X_t| = \beta\}$. Therefore, by Dini's theorem, the convergence is uniform on the set $|y| = 1$ so

$$\varepsilon \leqslant \sup_{|y|=1} w(y) \downarrow 0 \quad \text{as} \quad \delta \downarrow 0.$$

This completes the proof of (13), hence (9).



### Analytic functions and exit times

Here let $R$ be an open, connected subset of the complex plane, $B$ a complex Brownian motion starting at a point $b$ in $R$, and $\tau$ the first time $B$ leaves $R$: $\tau = \inf\{t > 0\colon B_t \notin R\}$. If $F$ is a function analytic in the open unit disc $D$, let $N(F)$ denote its nontangential maximal function: $N(F)(\theta) = \sup|F(z)|$, where the supremum is taken with respect to all $z$ in the interior of the smallest convex set containing the disc $|z| < 1/2$ and the point $e^{i\theta}$. (The number $1/2$ has no special significance; any other number in $(0, 1)$ would do as well.) Let $m$ be Lebesgue measure on $[0, 2\pi)$ and $\Phi$ any function as in (2).

THEOREM 1. *If $F$ is analytic in $D$ with $F(D) \subset R$ and $F(0) = b$, then*

$$\sup_\lambda \Phi(\lambda) m\big(N(F) > \lambda\big) \leqslant c \sup_\lambda \Phi(\lambda) P_b([2\tau + |b|^2]^{1/2} > \lambda) \tag{14}$$

*with the choice of $c$ depending only on $\Phi$.*

THEOREM 2. *If $F$ is analytic in $D$ with $F(0) = b$ and, for almost all $\theta$, the nontangential limit of $F$ at $e^{i\theta}$ exists and belongs to the complement of $R$, then*

$$\sup_\lambda \Phi(\lambda) P_b([2\tau + |b|^2]^{1/2} > \lambda) \leqslant c \sup_\lambda \Phi(\lambda) m\big(N(F) > \lambda\big) \tag{15}$$

*with the choice of $c$ depending only on $\Phi$.*

The corresponding strong inequalities also hold. For connections with $H^p$, see [3].

COROLLARY 1. *If $F$ is analytic and univalent in $D$ with $F(D) = R$ and $F(0) = b$, then both* (14) *and* (15) *hold.*

*Proof.* By the univalence, the condition of Theorem 2 is satisfied.

These two theorems have another immediate consequence: If $F$ and $G$ are analytic in $D$ with $F(0) = G(0)$ and, for almost all $\theta$, the nontangential limit of $G$ at $e^{i\theta}$ exists and belongs to the complement of $F(D)$, then

$$\sup_\lambda \Phi(\lambda) m\big(N(F) > \lambda\big) \leqslant c \sup_\lambda \Phi(\lambda) m\big(N(G) > \lambda\big).$$

However, much more is true:

THEOREM 3. *If $F$ and $G$ are analytic in $D$ with $F(0) = G(0)$ and, for almost all $\theta$, the nontangential limit of $G$ at $e^{i\theta}$ exists and belongs to the complement of $F(D)$, then*

$$m\big(N(F) > \lambda\big) \leqslant c\, m\big(N(G) > \lambda\big), \qquad \lambda > 0. \tag{16}$$

*The choice of $c$ is independent of $F$, $G$, and $\lambda$.*

*Proof of Theorem* 1. We can assume that $F$ is nonconstant. Then there exist two complex Brownian motions $Z$ and $W$ defined on the same probability space, $Z$ starting at $0$ and $W$ starting at $b = F(0)$, such that

$$F(Z_s) = W_{\beta(s)}, \qquad 0 \leqslant s < \mu, \tag{17}$$

where $\mu = \inf\{s > 0\colon |Z_s| = 1\}$ and

$$\beta(s) = \int_0^s |F'(Z_t)|^2 dt, \qquad 0 \leqslant s \leqslant \mu.$$

This follows from Lévy's principle of the conformal invariance of Brownian motion; see McKean [9]. With probability one, $\beta$ is a strictly increasing continuous function on $[0, \mu]$. Let $\nu = \beta(\mu)$ and $\sigma = \inf\{t > 0: W_t \notin R\}$. Then $B_\tau^*$ and $W_\sigma^*$ have the same distribution. Also, with probability one $\nu \leqslant \sigma$: Let $t = \beta(s)$; then, by (17),

$$t < \nu \Rightarrow s < \mu \Rightarrow F(Z_s) \in R \Rightarrow W_t \in R.$$

So, under the conditions of Theorem 1,

$$P(F^* > \lambda) \leqslant P_b(B_\tau^* > \lambda), \quad \lambda > 0, \tag{18}$$

where $F^* = \sup_{s<\mu} |F(Z_s)|$, since, by (17),

$$P(F^* > \lambda) = P(W_\nu^* > \lambda) \leqslant P(W_\sigma^* > \lambda) = P_b(B_\tau^* > \lambda).$$

By the results of [5],

$$cP(F^* > \lambda) \leqslant m\big(N(F) > \lambda\big) \leqslant CP(F^* > \lambda), \quad \lambda > 0. \tag{19}$$

Applying Lemma 2 to $B$ and $\tau$ and using (18) and (19), we obtain (14). This completes the proof of Theorem 1.

*Proof of Theorem* 2. Keeping the notation of the above proof and recalling that nontangential limits imply Brownian limits (Doob [8]), we see that, with probability one, the limits

$$\lim_{s\uparrow\mu} F(Z_s) = \lim_{t\uparrow\nu} W_t$$

exist and belong to the complement of $R$. So here $\sigma \leqslant \nu$. Therefore,

$$P(W_\sigma^* > \lambda) \leqslant P(W_\nu^* > \lambda), \quad \lambda > 0,$$

which is equivalent to the inequality

$$P_b(B_\tau^* > \lambda) \leqslant P(F^* > \lambda), \quad \lambda > 0. \tag{20}$$

Using (19) and (20) and again applying Lemma 2 to $B$ and $\tau$, we obtain (15). This completes the proof of Theorem 2.

*Proof of Theorem* 3. Let $R = F(D)$; if $F$ is nonconstant, as we can assume, $R$ is an open, connected set. We now apply (18) and (20) to $F$ and $G$. If $G^*$ denotes the Brownian maximal function of $G$:

$$G^* = \sup_{s<\mu} |G(Z_s)|,$$

then, by (20),

$$P_b(B_\tau^* > \lambda) \leqslant P(G^* > \lambda), \quad \lambda > 0. \tag{21}$$

So, by (18), under the conditions of Theorem 3,

$$P(F^* > \lambda) \leqslant P(G^* > \lambda), \quad \lambda > 0. \tag{22}$$

The theorem now follows from (19).

Symmetrization can be used with the above methods to obtain further results. For example, here is the symmetrized version of Theorem 3.



THEOREM 4. *If $F$ and $G$ are analytic in $D$, $F(D) \subset R$, $F(0) = G(0) = 0$, and, for almost all $\theta$, the nontangential limit of $G$ at $e^{i\theta}$ exists and belongs to the complement of $R_s$, the region obtained from the region $R$ by circular symmetrization, then* (16) *holds.*

Recall that if $R$ contains the origin, as it does here, then $R_s$ is also a region containing the origin with the following properties. If $r > 0$ and $\{|z| = r\} \subset R$, then $\{|z| = r\} \subset R_s$. If $r > 0$ and $\{|z| = r\} \not\subset R$, then

$$R_s \cap \{|z| = r\} = \{re^{i\theta}: |\theta| < \alpha\},$$

where $\alpha$ is chosen so that $R_s \cap \{|z| = r\}$ and $R \cap \{|z| = r\}$ have the same circular Lebesgue measure.

*Proof of Theorem* 4. Let $\tau_s$ be the first time the Brownian motion $B$ leaves $R_s$ starting at the origin. Then

$$P_0(B_{\tau_s}^* > \lambda) \leqslant P_0(\beta_{\tau_s}^* > \lambda), \quad \lambda > 0. \tag{23}$$

This is a translation into the language of Brownian motion of an inequality for harmonic measure due to Baernstein [1]. Here (21) takes the form

$$P_0(B_{\tau_s}^* > \lambda) \leqslant P(G^* > \lambda), \quad \lambda > 0. \tag{24}$$

Inequality (22) follows from (18), (23), (24) and gives (16) as before. This completes the proof of Theorem 4.

These theorems have many straightforward applications. For example, suppose that $F$ is analytic and univalent in $D$ with $F(0) = 0$ and $F'(0) = 1$. Then

$$m\big(N(F) > \lambda\big) \leqslant c\lambda^{-1/2}, \quad \lambda > 0, \tag{25}$$

where the choice of $c$ is independent of $F$ and $\lambda$. To see this, let $R = F(D)$ and $G(z) = 4z(1-z)^{-2}$. Then the conditions of Theorem 4 are satisfied and (16) becomes (25).

Here is another simple application. Let $R$ be a simply connected region with a nondegenerate boundary $\partial R$. Let $\delta$ denote the distance from $b \in R$ to $\partial R$. Then the first exit time $\tau$ of $R$ satisfies

$$P_b(\tau^{1/2} > \lambda) \leqslant c(\delta/\lambda)^{1/2}, \quad \lambda > 0, \tag{26}$$

where the choice of $c$ is independent of $\lambda$, $R$, and $b$. To prove this, we can assume that $b = 0$. Let $G$ be an analytic and univalent function with $G(0) = 0$ mapping $D$ onto $R$. Then $\delta \leqslant |G'(0)| \leqslant 4\delta$. This is classical; for example, see [6]. Let $F = G/G'(0)$. Applying Corollary 1 to $G$ and using (25), we obtain

$$\begin{aligned} \sup \lambda^{1/2} P_0([2\tau]^{1/2} > \lambda) &\leqslant c \sup \lambda^{1/2} m\big(N(G) > \lambda\big) \\ &= c|G'(0)|^{1/2} \sup \lambda^{1/2} m\big(N(F) > \lambda\big) \\ &\leqslant c\delta^{1/2}, \end{aligned}$$

which implies (26).

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# PROBABILISTIC AND ANALYTIC FORMULAS FOR THE PERIODIC SPLINES INTERPOLATING WITH MULTIPLE NODES

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Let on the one-dimensional torus $T$ a fixed partition $\{s_1, \ldots, s_n\}$ be given. Formulas for periodic splines of degree $2p+1$ interpolating at the nodes $s_j$ of multiplicity $\alpha_j$, $1 \leqslant \alpha_j \leqslant p+1$, are derived. The results are obtained with the help of suitably constructed on $T$ Markovian Gaussian random field. The natural interplay between this random field and splines on $T$ is explored.

## 1. Introduction

The idea contained in this paper is very simple and it can be explained already in the case of interpolation by splines of degree 1, i.e. in the case of $p = 0$. Let $\pi$: $0 = s_0 < \ldots < s_n = 1$ be a given partition of the torus $T = \langle 0, 1)$. Then the spline of degree 1 interpolating given function $u$ on $T$ at the nodes $s_j$ with multiplicities 1 is simply the piecewise linear in each $(s_{j-1}, s_j)$ and continuous on $T$ function $u_0$ such that $u(s_j) = u_0(s_j)$, $j = 1, \ldots, n$. With this interpolation problem in a natural way is connected the Brownian motion $\{X(t), t \in T\}$ or, more precisely, the Brownian bridge, i.e. a continuous Gaussian process on $T$ with mean zero and the covariance given by formula (4.5). Now, the relation between Brownian bridge and the interpolation is given by the formula

$$u_0(t) = E\{X(t)|X(s_1) = u(s_1), \ldots, X(s_n) = u(s_n)\}.$$

However, the Brownian bridge is Markovian and therefore for $t \in \langle s_{j-1}, s_j\rangle$ we have

$$u_0(t) = E\{X(t)|X(s_{j-1}) = u(s_{j-1}), X(s_j) = (s_j)\}.$$

The aim of this paper is to extend this approach in order to obtain formulas for splines of degree $2p+1$ interpolating at nodes $s_j$ with multiplicities $\alpha_j$, $1 \leqslant \alpha_j \leqslant p+1, j = 1, \ldots, n$.

The proper Markovian Gaussian random field one obtaines by path-wise $p$-fold periodic integration of the Brownian bridge.

The considerations were inspired mainly by the works of L. D. Pitt [7], H. B. Curry and I. J. Schoenberg [4], and the author's investigations (see [2]).