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# PROBABILISTIC AND ANALYTIC FORMULAS FOR THE PERIODIC SPLINES INTERPOLATING WITH MULTIPLE NODES

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Let on the one-dimensional torus $T$ a fixed partition $\{s_1, \ldots, s_n\}$ be given. Formulas for periodic splines of degree $2p+1$ interpolating at the nodes $s_j$ of multiplicity $\alpha_j$, $1 \leqslant \alpha_j \leqslant p+1$, are derived. The results are obtained with the help of suitably constructed on $T$ Markovian Gaussian random field. The natural interplay between this random field and splines on $T$ is explored.

## 1. Introduction

The idea contained in this paper is very simple and it can be explained already in the case of interpolation by splines of degree 1, i.e. in the case of $p = 0$. Let $\pi$: $0 = s_0 < \ldots < s_n = 1$ be a given partition of the torus $T = \langle 0, 1)$. Then the spline of degree 1 interpolating given function $u$ on $T$ at the nodes $s_j$ with multiplicities 1 is simply the piecewise linear in each $(s_{j-1}, s_j)$ and continuous on $T$ function $u_0$ such that $u(s_j) = u_0(s_j), j = 1, \ldots, n$. With this interpolation problem in a natural way is connected the Brownian motion $\{X(t), t \in T\}$ or, more precisely, the Brownian bridge, i.e. a continuous Gaussian process on $T$ with mean zero and the covariance given by formula (4.5). Now, the relation between Brownian bridge and the interpolation is given by the formula

$$u_0(t) = E\{X(t) | X(s_1) = u(s_1), \ldots, X(s_n) = u(s_n)\}.$$

However, the Brownian bridge is Markovian and therefore for $t \in \langle s_{j-1}, s_j \rangle$ we have

$$u_0(t) = E\{X(t) | X(s_{j-1}) = u(s_{j-1}), X(s_j) = (s_j)\}.$$

The aim of this paper is to extend this approach in order to obtain formulas for splines of degree $2p+1$ interpolating at nodes $s_j$ with multiplicities $\alpha_j$, $1 \leqslant \alpha_j \leqslant p+1, j = 1, \ldots, n$.

The proper Markovian Gaussian random field one obtaines by path-wise $p$-fold periodic integration of the Brownian bridge.

The considerations were inspired mainly by the works of L. D. Pitt [7], H. B. Curry and I. J. Schoenberg [4], and the author's investigations (see [2]).

## 2. Preliminaries

The unit interval and the one-dimensional torus are denoted by $I$ and $T$, respectively, i.e. $I = \langle 0, 1\rangle$, $T = \langle 0, 1)$. In the real space $L^2(T)$ the scalar product is given as follows

$$(f, g) = \int_T fg. \tag{2.1}$$

For a given integer $p \geq 1$, the *Sobolew space* $H^p(T)$ is defined as the set of all functions $u$ from $C^{p-1}(T)$ with absolutely continuous $D^{p-1}u$ and $D^p u \in L^2(T)$; here and later on $D$ stands for the differentiation operator. The space $H^p(T)$ equipped with the scalar product

$$\langle u, v\rangle = u(0)v(0)+(D^p u, D^p v) \tag{2.2}$$

becomes a Hilbert space.

The Bernoulli periodic polynomials play in this paper a special rôle. They are defined as in [8], i.e.

$$B_k(t) \sim \sum_{\nu=-\infty}^{\infty}{}' \frac{e^{2\pi i\nu t}}{(2\pi i\nu)^k}, \quad k = 0, 1, \dots, \tag{2.3}$$

where the prime means that the summation is taken over all $\nu \neq 0$. In the case of $k = 0$ series (2.3) is understood in the generalized sense and therefore $B_0 = \delta_0 - 1$, where $\delta_0$ is the $\delta$-Dirac concentrated at 0. For later use we list some of the basic properties of periodic Bernoulli polynomials:

1. $B_k \in C^{k-2}(T)$ for $k \geq 1$,
2. $DB_k = B_{k-1}$ for $k \geq 1$,
3. $(B_k, 1) = 0$ for $k \geq 0$,
4. $B_{k+r} = B_k * B_r$ for $k \geq 0, r \geq 0$,
5. $B_k(-t) = (-1)^k B_k(t)$.

There is a natural unitary isomorphism $U_p\colon L^2(T) \to H^p(T)$, $p \geq 1$, given as follows:

$$\begin{aligned}(U_p f)(t) &= (f, 1)+\int_0^t (f * B_{p-1})(s)\,ds, \quad f \in L^2(T),\\ (U_p^{-1}u)(t) &= u(0)+(D^p u)(t), \quad u \in H^p(T).\end{aligned} \tag{2.4}$$

## 3. Splines with multiple knots

The results contained in this section are in principle known (cf. [1], [4]) and they are included for the sake of completeness.

First let us recall the main properties of the $B$-splines with simple knots. Let $\pi = \{s_j, j = 0, \pm 1, \dots\}$ be a given partition of the real line such that $0 = s_0 < \dots < s_n = 1$, $s_{j+kn} = s_j + k$, $j = 0, \dots, n-1$; $k = \pm 1, \pm 2, \dots$ To each such $\pi$ there corresponds a sequence of $B$-splines of degree $p \geq 0$



$$N_j(t) = (s_{j+p}-s_{j-1})[s_{j-1}, \dots, s_{j+p}; (s-t)_+^p]. \tag{3.1}$$

Note that we have imposed on $\pi$ the periodicity for later use only and it is not required in definition (3.1).

It is well known (cf. [4]) that the $B$-splines $N_j$ are non-negative, and those not identically equal to zero in a given interval are in that interval linearly independent. They are normalized in such a way that for all $t$,

$$\sum_j N_j(t) = 1.$$

The space of splines of degree $p$ corresponding to $\pi$ and restricted to $I$ is denoted by $S_\pi^p(I)$. It follows from the properties of the $B$-splines that $\dim S_\pi^p(I) = p+n$ and

$$S_\pi^p(I) = V\{N_j, j = -p+1, \dots, n\},$$

where $V$ stands here and later on for linear span.

Since the partition $\pi$ is periodic, the periodic $B$-splines, of period 1 and degree $p$ on $T$, are now defined by the formula

$$T_j(t) = \sum_{k=0,\pm 1,\dots} N_j(t+k). \tag{3.2}$$

These functions have similar properties, they are linearly independent and non-negative. If the space of all periodic splines of degree $p$ on $T$ is denoted by $S_\pi^p(T)$, then $\dim S_\pi^p(T) = n$ and

$$S_\pi^p(T) = V\{T_j, j = 1, \dots, n\}. \tag{3.3}$$

For the periodic $B$-splines one would expect a formula analogous to (3.1). Indeed, it can be seen with the help of (3.2) and the properties of the $N_j$'s that

$$T_j(t) = \frac{s_{j+p}-s_{j-1}}{p+1}\{1+(p+1)!\,[s_{j-1}, \dots, s_{j+p}; B_{p+1}(s-t)-B_{p+1}(-t)]\}. \tag{3.4}$$

It now follows from (3.3) and (3.4) that

$$S_\pi^p(T) = V\{1+B_{p+1}(s_j-t)-B_{p+1}(-t), j = 1, \dots, n\}. \tag{3.5}$$

We are now ready to pass to the case of splines with multiple knots. To each points $s_j \in \pi$ there is assigned a multiplicity $\alpha_j$, i.e. an integer such that $1 \leq \alpha_j \leq p+1$, $p \geq 0$. The sequence of multiplicities is denoted by $\alpha$, i.e. $\alpha = \{\alpha_j, j = 0, \pm 1, \dots\}$. Following H. B. Curry and I. J. Schoenberg we say that a function on $(-\infty, \infty)$ is a *spline of degree $p$ with multiplicities $\alpha$* if in each interval $(s_{j-1}, s_j)$ it is a polynomial of degree at most $p$ and at each point $s_j$ it is of the class $C^{p-\alpha_j}$.

Since now we impose on $\alpha$ the periodicity condition: $\alpha_j = \alpha_{j+kn}$ for $k = \pm 1, \pm 2, \dots$

The space of splines of degree $p$ corresponding to the pair $(\pi, \alpha)$ and restricted to $I$ is denoted by $S^p_{\pi,\alpha}(I)$. The $B$-splines corresponding to $(\pi, \alpha)$ are obtained as follows. Consider the partition $\pi' = \{s'_k\}$ of period $N = \alpha_1 + \ldots + \alpha_n$ such that $0 = s'_0 \leqslant \ldots \leqslant s'_N = 1$ and for $1 \leqslant j \leqslant n$,

$$s'_k = s_j \quad \text{if} \quad \alpha_1 + \ldots + \alpha_{j-1} < k \leqslant \alpha_1 + \ldots + \alpha_j,$$

where $\alpha_1 + \ldots + \alpha_{j-1} = 0$, by definition for $j = 1$. Now, the notion of divided difference can be extended naturally to the partition $\pi$ (cf. [5]), whence formula (3.1) can be used to define the $B$-splines corresponding to $\pi'$. This procedure, in combination with (3.1), allows to find that

$$S^p_{\pi,\alpha}(I) = V\{1, \ldots, t^p, (t-s_j)^p_+, \ldots, (t-s_j)^{p-\alpha_j+1}_+, \ j = 1, \ldots, n-1\},$$

whence we infer

$$\dim S^p_{\pi,\alpha}(I) = p+1+\alpha_1 + \ldots + \alpha_{n-1}.$$

The subspace of $S^p_{\pi,\alpha}(I)$ of periodic splines corresponding to $(\pi, \alpha)$ is

$$S^p_{\pi,\alpha}(T) = \{u \in S^p_{\pi,\alpha}(I)\colon D^k u(s_0) = D^k u(s_n), \ k = 0, \ldots, p-\alpha_0\}.$$

Moreover, it follows from the definition of the corresponding to $\pi'$ the $T'_j$ $B$-splines and from (3.4) that

$$S^p_{\pi,\alpha}(T) = V\{1+B_{p+1}(s_j-t)-B_{p+1}(-t), \ B_{p+1-k}(s_j-t), \quad 1 \leqslant k \leqslant \alpha_j - 1, \ j = 1, \ldots, n\}. \tag{3.6}$$

Consequently,

$$\dim S^p_{\pi,\alpha}(T) = \alpha_1 + \ldots + \alpha_n. \tag{3.7}$$

Let us construct now a sequence of partitions $\pi_j$, $j = 1, 2, \ldots$, of $T$ with the following properties:

(i) $\pi_j = \{s_{j,k}, \ k = 0, \pm 1, \pm 2, \ldots\}$;

(ii) $s_{j,0} = 0$, $s_{j,n} = 1$, $s_{j,k-1} < s_{j,k}$;

(iii) $s_{j,k+hn} = s_{j,k}+h$, $h = \pm 1, \pm 2, \ldots$; $k = 1, \ldots, n$;

(iv) $\pi_j \subset \pi_{j+1}$ and $\pi_{j+1} \backslash \pi_j$ is one-point set;

(v) $\pi_n = \pi$.

The space of splines of degree $p$ on $T$ corresponding to $\pi_j$ with multiplicities equal to 1 at all partition points is denoted by $S^p_j(T)$. Clearly, $\dim s^p_j(T) = j$. Now, if $\bigcup_j \pi_j$ is dense in $T$, then

$$L^2(T) = V\{S^p_j(T), \ j = 1, 2, \ldots\}. \tag{3.8}$$

It is useful to notice that for $p \geqslant 0$,

$$U_{p+1}\colon S^p_{\pi,\alpha}(T) \xrightarrow{\text{onto}} S^{2p+1}_{\pi,\alpha}(T).$$

This is a consequence of (2.4) and (3.6). Let now $P$ denote the orthogonal projection of $L^2(T)$ onto $S^p_{\pi,\alpha}(T)$ and $Q = U_{p+1} P U^{-1}_{p+1}$ the orthogonal projection of $H^{p+1}(T)$ onto $S^{2p+1}_{\pi,\alpha}(T)$.

THEOREM 3.1. *Let $(\pi, \alpha)$ and $p \geqslant 0$ be given as above and let $u \in H^{p+1}(T)$. Then $v = Qu$ if and only if it satisfies the following properties:*



(i) $v \in S^{2p+1}_{\pi,\alpha}(T)$,

(ii) $D^k v(s_j) = D^k u(s_j)$, $0 \leqslant k \leqslant \alpha_j - 1$, $j = 1, \ldots, n$.

*Proof.* Suppose that $v = Qu$ and let $f = U^{-1}_{p+1} u$ and $g = U^{-1}_{p+1} v$. Then $g \in S^p_{\pi,\alpha}(T)$ and $h \overset{\text{df}}{=} f - g \in L^2 \ominus S^p_{\pi,\alpha}(T)$. Moreover,

$$w(t) \overset{\text{df}}{=} (u-v)(t) = U_{p+1} h(t) = \int_T h(s)[1+B_{p+1}(t-s)-B_{p+1}(-s)]\,ds,$$

and

$$D^k w(t) = \int_T h(s) B_{p+1-k}(t-s)\,ds, \qquad k > 0.$$

Since $h \in S^p_{\pi,\alpha}(T)$, we find, using (3.6), that $D^k w(s_j) = 0$ for $0 \leqslant k \leqslant \alpha_j - 1$, $j = 1, \ldots, n$. On the other hand, $u = v + w$ and therefore (ii) follows. Condition (i) is satisfied by the very definition of $v$.

Conversely, let us assume that for given $u \in H^{p+1}(T)$ the function $v$ satisfies (i) and (ii). If we show that $w \perp u - v$ for $w \in S^{2p+1}_{\pi,\alpha}(T)$, then by the uniqueness of orthogonal decompositions it will follow that $v = Qu$. The orthogonality is proved as follows:

$$\begin{aligned}
\langle u, w\rangle &= u(0)w(0) + (D^{p+1}u, D^{p+1}w) \\
&= v(0)w(0) + \sum_{j=1}^{n} \int_{s_{j-1}}^{s_j} D^{p+1}u D^{p+1}w \\
&= v(0)w(0) + \sum_{j=1}^{n}\sum_{k=0}^{p} (-1)^k [D^{p-k}u(s_j) D^{p+1+k}_{-} w(s_j) - \\
&\qquad\qquad - D^{p-k}u(s_{j-1}) D^{p+1+k}_{+} w(s_{j-1})] \\
&= v(0)w(0) + \sum_{j=1}^{n} (-1)^p \big(u(s_j) - u(s_{j-1})\big) D^{2p+1} w\left(\frac{s_{j-1}+s_j}{2}\right) + \\
&\quad + \sum_{j=1}^{n} \Big[\sum_{k=0}^{p-\alpha_j} (-1)^k D^{p-k}u(s_j) D^{p+1+k} w(s_j) - \\
&\qquad\qquad - \sum_{k=0}^{p-\alpha_{j-1}} (-1)^k D^{p-k} u(s_{j-1}) D^{p+k+1} w(s_{j-1})\Big] + \\
&\quad + \sum_{j=1} \Big[\sum_{k=p-\alpha_j+1}^{p-1} (-1)^k D^{p-k} u(s_j) D^{p+1+k}_{-} w(s_j) - \\
&\qquad\qquad - \sum_{k=p-\alpha_{j-1}+1}^{p-1} (-1)^k D^{p-k} u(s_{j-1}) D^{p+1+k}_{+}(s_{j-1})\Big].
\end{aligned}$$

In the first sum and in the third one $u$ can be replaced by $v$ and the middle sum cancels out. Thus, $\langle u, w\rangle = \langle v, w\rangle$ and the proof is complete.

COROLLARY 3.1. *Let* $V\colon S^{2p+1}_{\pi,\alpha}(T) \to \mathbf{R}^N$, $N = \alpha_1 + \ldots + \alpha_n$, *be the mapping defined by the formula*

$$Vv = \big(D^k v(s_j),\ 0 \leqslant k \leqslant \alpha_j - 1,\ j = 1, \ldots, n\big).$$

*Then $V$ is an isomorphism.*

To prove this it is sufficient to show that $V$ maps onto $\mathbf{R}^N$. Let $\{a_{k,j},\ 0 \leqslant k \leqslant p,\ j = 1, \ldots, n\}$ be a given table of real numbers. In each interval $(s_{j-1}, s_j)$ we solve the following two-point Hermite interpolation problem: $D^k w_j(s_j) = a_{k,j}$, $D^k w_j(s_{j-1}) = a_{k,j-1}$, $k = 0, \ldots, p$. There is a unique polynomial $w_j$ of degree not exceeding $2p+1$ with these properties. Let $w$ be the piecewise Hermite solution, i.e. $w(t) = w_j(t)$ for $t \in \langle s_{j-1}, s_j)$, $j = 1, \ldots, n$. It is clear that $w \in H^{p+1}(T)$. Put $v = Qw$. According to Theorem 3.1 we have $D^k v(s_j) = a_{k,j}$, $0 \leqslant k \leqslant \alpha_j - 1$, $j = 1, \ldots, n$, and this completes the proof.

We denote the fundamental $B$-splines corresponding to $(\pi, \alpha)$ by $v_{k,j}$, $0 \leqslant k \leqslant \alpha_j - 1$, $j = 1, \ldots, n$. According to Corollary 3.1, they are characterized by the following conditions:

$$D^i v_{k,j}(s_h) = \delta_{ik}\delta_{hj}, \quad 0 \leqslant i \leqslant \alpha_h - 1,\ h = 1, \ldots, n, \quad v_{k,j} \in S^{2p+1}_{\pi,\alpha}(T). \tag{3.9}$$

COROLLARY 3.2. *The set of splines $\{v_{k,j},\ 0 \leqslant k \leqslant \alpha_j - 1, j = 1, \ldots, n\}$ is a basis in $S^{2p+1}_{\pi,\alpha}(T)$ and for every $v \in S^{2p+1}_{\pi,\alpha}(T)$ we have*

$$v = \sum_{j=1}^{n} \sum_{k=0}^{\alpha_j - 1} \big(D^k v(s_j)\big) v_{k,j}. \tag{3.10}$$

Notice that (3.10) is implied by (3.9) and Corollary 3.1.

COROLLARY 3.3. *Let $u \in H^{p+1}(T)$ be given. Then*

$$\langle Qu, Qu\rangle = \inf\{\langle v, v\rangle\colon D^k v(s_j) = D^k u(s_j),\ 0 \leqslant k \leqslant \alpha_j - 1,\ j = 1, \ldots, n;\ v \in H^{p+1}(T)\},$$

*and $Qu$ is the unique solution of this variational problem.*

## 4. The reproducing kernel for the Hilbert space $H^{p+1}(T)$

For every integer $p \geqslant 0$ the following kernel is defined

$$\begin{aligned} R_p(t, s) &= \int_T [1 + B_{p+1}(t-u) - B_{p+1}(-u)]\,[1 + B_{p+1}(s-u) - B_{p+1}(-u)]\,du \\ &= 1 + (-1)^{p+1}[B_{2p+2}(s-t) + B_{2p+2}(0) - B_{2p+2}(t) - B_{2p+2}(s)]. \end{aligned} \tag{4.1}$$

The kernel $R_p$ has the following properties

$$\begin{aligned} &R_p(\cdot, s) \in H^{p+1}(T), \quad s \in T; \\ &\langle R_p(\cdot, s), R_p(\cdot, t)\rangle = R_p(s, t), \quad s, t \in T; \\ &H^{p+1}(T) = V\{R_p(\cdot, s),\ s \in T\}. \end{aligned} \tag{4.2}$$



The first property follows from the properties of Bernoulli polynomials. To check the remaining two properties we notice that

$$R_p(t, s) = (U_{p+1} f_t)(s), \quad f_t(u) = 1 + B_{p+1}(t-u) - B_{p+1}(-u). \tag{4.3}$$

Since $U_{p+1}\colon L^2(T) \to H^{p+1}(T)$ is unitary, the middle property in (4.2) is a consequence of (4.3) and (4.1).

The third property, according to (4.3), is equivalent to the equality

$$L^2(T) = V\{f_t,\ t \in T\},$$

but this is implied by (3.5) and (3.8).

The properties (4.2) mean exactly that $R_p$ reproduces $H^{p+1}(T)$ or else that $H^{p+1}(T)$ is the reproducing kernel Hilbert space (RKHS) corresponding to $R_p$.

The evaluation functional at a fixed point $t$ is continuous on $H^{p+1}(T)$ and therefore the middle equality in (4.2) extends to

$$u(t) = \langle u, R(\cdot, t)\rangle, \quad u \in H^{p+1}(T),\ t \in T. \tag{4.4}$$

It is a good place to mention that for $p = 0$ we obtain

$$R_0(t, s) = 1 + \min(t, s) - ts, \quad 0 \leqslant s, t < 1. \tag{4.5}$$

This is the covariance of periodic Brownian motion (Brownian bridge) starting at independent random point distributed on the real line according to $N(0, 1)$.

## 5. The Gausian random field and its relation to spline interpolation

Suppose that we are given over a probability space $(\Omega, F, P)$ a separable real valued Gaussian random field (GRF) $\{X(t),\ t \in T\}$ such that $(p \geqslant 0)$

$$E\big(X(t)X(s)\big) = R_p(t, s), \quad E\big(X(t)\big) = 0, \quad t, s \in T. \tag{5.1}$$

One can check that

$$\begin{aligned} E|\Delta_\delta^{p+1} X(t)|^2 &= \int_T |\Delta_\delta^{p+1} B_{p+1}(u)|^2 du \leqslant \delta^{2(p-1)} \int_T |\Delta_\delta^2 B_2(u)|^2 du \\ &= O(\delta^{2p+1}), \quad \delta \to 0_+. \end{aligned}$$

It can be applied now a criterion, [3], for regularity of trajectories for GRF to get

$$P\{D^p X(\cdot) \in \mathrm{Lip}(\varepsilon; T)\} = 1 \quad \text{for} \quad 0 < \varepsilon < \tfrac{1}{2}. \tag{5.2}$$

In $L^2(\Omega, F, P)$ the Gaussian subspace generated by $\{X(t),\ t \in T\}$ is denoted by $\mathscr{H}$, i.e. $\mathscr{H} = V\{X(t),\ t \in T\}$. According to (4.2),

$$\langle R(\cdot, s), R(\cdot, t)\rangle = E\big(X(s)X(t)\big),$$

whence we infer that to each $u \in H^{p+1}(T)$ there is exactly one $Ju \in \mathscr{H}$ such that

$$\langle u, R(\cdot, t)\rangle = E\big(JuX(t)\big)$$

and

$$\langle u, v\rangle = E(JuJv).$$

Thus $J\colon H^{p+1}(T)\to\mathscr{H}$ is a unitary isomorphism. Moreover, according to (4.4), we obtain

$$u(t) = E\big(JuX(t)\big), \quad t\in T,\ u\in H^{p+1}(T). \tag{5.3}$$

THEOREM 5.1. *For given $(\pi,\alpha)$ and $t\in T$ we have*

$$E\{X(t)\,|\,D^kX(s_j),\ 0\leqslant k\leqslant \alpha_j-1, j=1,\dots,n\} = \sum_{j=1}^{n}\sum_{k=0}^{\alpha_j-1} D^kX(s_j)v_{k,j}(t), \tag{5.4}$$

*where the $v_{k,j}$'s are the fundamental B-splines defined as in Section* 3.

*Proof.* Let us define a subspace of $\mathscr{H}$ related to $(\pi,\alpha)$ as follows:

$$\mathscr{H}_{\pi,\alpha} = V\{D^kX(s_j),\ 0\leqslant k\leqslant \alpha_j-1,\ j=1,\dots,n\}.$$

According to (5.2) this subspace is well defined.

It will be shown that

$$\mathscr{H}_{\pi,\alpha} = JS_{\pi,\alpha}^{2p+1}(T). \tag{5.5}$$

Notice that $u\perp S_{\pi,\alpha}^{2p+1}(T)$ is equivalent to $Qu=0$ which, in view of Section 3, is equivalent to $D^ku(s_j)=0$, $0\leqslant k\leqslant\alpha_j-1$, $j=1,\dots,n$. However, according to (5.3), this is equivalent to $Ju\perp\mathscr{H}_{\pi,\alpha}$ and this gives (5.5).

Let now, for fixed $t\in T$, $QX(t)$ be an element of $\mathscr{H}_{\pi,\alpha}$ defined as follows:

$$QX(t) = \sum_{j=1}^{n}\sum_{k=0}^{\alpha_j-1} D^kX(s_j)v_{k,j}(t). \tag{5.6}$$

Our aim is to show that $QX(t)$ is the orthogonal projection of $X(t)$ on $\mathscr{H}_{\pi,\alpha}$, and this, by the uniqueness of orthogonal decompositions, will imply (5.4).

In $L^2(T)$ we choose on orthonormal basis $\{f_j,\ j=1,2,\dots\}$ such that

$$S_{\pi,\alpha}^{p}(T) = V\{f_j,\ j=1,\dots,N\}, \tag{5.7}$$

where $N=\alpha_1+\dots+\alpha_n$. Moreover, in $\mathscr{H}$ we choose an orthonormal basis $\{Y_j,\ j\geqslant 1\}$, and then define a GRF $\{Y(t),\ t\in T\}$ as follows:

$$Y(t) = \sum_{j=1}^{\infty} Y_j(U_{p+1}f_j)(t),$$

where the series converges in $\mathscr{H}$. This process has the same characteristics as $\{X(t),\ t\in T\}$, i.e. $E\big(Y(t)Y(s)\big)=R_p(t,s)$, $E\big(Y(t)\big)=0$. It is not hard to see that

$$\mathscr{H} = V\{Y(t),\ t\in T\},$$

and an elementary argument shows the existence of unique orthonormal basis $\{X_j,\ j=1,2,\dots\}$, in $\mathscr{H}$ such that (cf. [2])

$$X(t) = \sum_{j=1}^{\infty} X_j(U_{p+1}f_j)(t). \tag{5.8}$$

The general theorems of K. Ito and M. Nisio (cf. [6]) on convergence with probability one of random series in Banach spaces can be used to prove that (5.8) is $p$-times continuously differentiable, i.e.



$$D^kX(t) = \sum_{j=1}^{\infty} X_jD^kU_{p+1}f_j(t), \quad t\in T,\ 0\leqslant k\leqslant p, \tag{5.9}$$

and the series converges uniformly with probability one.

According to (5.7), $f_j\perp S_{\pi,\alpha}^{p}(T)$ for $j>N$, whence $U_{p+1}f_j\perp S_{\pi,\alpha}^{2p+1}(T)$ for $j>N$ or, equivalently, $D^kU_{p+1}f_j(s_i)=0$ for $0\leqslant k\leqslant\alpha_i-1$, $i=1,\dots,n$ and $j>N$. This and (5.9) imply that, with probability one,

$$D^k\sum_{j=1}^{N} X_jU_{p+1}f_j(s_i) = D^kX(s_i),$$

and therefore, by (5.6) and Corollary 3.1,

$$QX(t) = \sum_{j=1}^{N} X_jU_{p+1}f_j(t).$$

Since $X_j=\langle QX(\cdot),\ U_{p+1}f_j(\cdot)\rangle$ for $1\leqslant j\leqslant N$ and $QX(t)\in\mathscr{H}_{\pi,\alpha}$, it follows from (5.6) that $X_j\in\mathscr{H}_{\pi,\alpha}$ for $1\leqslant j\leqslant N$. Thus,

$$\mathscr{H}_{\pi,\alpha} = V\{X_j,\ j=1,\dots,N\}.$$

On the other hand,

$$X(t)-QX(t) = \sum_{j=N+1}^{\infty} X_jU_{p+1}f_j(t)\in\mathscr{H}\ominus\mathscr{H}_{\pi,\alpha}$$

and this completes the proof.

COROLLARY 5.1. *The spline of degree $2p+1$ corresponding to $(\pi,\alpha)$ and interpolating given $u\in H^{p+1}(T)$ at the multiple nodes is given by the formula*

$$Qu(t) = E\{X(t)\,|\,D^kX(s_j)=D^ku(s_j),\ 0\leqslant k\leqslant\alpha_j-1, j=1,\dots,n\}. \tag{5.10}$$

THEOREM 5.2. *Let $p$ and $(\pi,\alpha)$ be qiven as above and let $u\in H^{p+1}(T)$. Then*

$$Qu(t) = \sum_{j=1}^{n}\sum_{k=0}^{\alpha_j-1} y_{k,j}D_s^kR_p(t,s_j), \tag{5.11}$$

*where $\{y_{k,j}\}$ is the solution of the system of equations*

$$\sum_{j=1}^{n}\sum_{k=0}^{\alpha_j-1} y_{k,j}D_t^iD_s^kR_p(s_h,s_j) = D^iu(s_h), \quad 0\leqslant i\leqslant\alpha_h-1,\ h=1,\dots,n. \tag{5.12}$$

To obtain (5.11) and (5.12), it is sufficient to work out the left-hand side of (5.4) simply by finding in $\mathscr{H}_{\pi,\alpha}$ the best approximating element to $X(t)$, and then to compare the solution with the right-hand side of (5.4).

*Remark.* Since $R_p(t,s)$ is explicitely given in (4.1) and the Bernoulli polynomials are easy for computing, formulas (5.11) and (5.12) may prove to be useful in numerical approximation.

COROLLARY 5.2. *The interpolation formulas without multiplicities become now very simple. Since $\alpha_j = 1$ for all $j$, we have*

$$Qu(t) = \sum_{j=1}^{n} y_j R_p(t, s_j),$$

*where the $y_j$ are to be determined from the equations*

$$\sum_{j=1}^{n} y_j R_p(s_h, s_j) = u(s_h), \quad h = 1, \ldots, n.$$

*In particular, for the fundamental splines in this case we have*

$$v_j(t) = \sum_{h=1}^{n} A_{jh} R_p(t, s_h),$$

*where $(A_{j,h})$ is the inverse matrix to $\big(R_p(s_j, s_h)\big)$.*

## 6. The Markov property of the GRF and piecewise Hermite interpolation

The main task of this section is to show that the GRF $\{X(t);\ t \in T\}$ is $p$-Markovian.

It is assumed that $T$ is split into two closed intervals with common boundary, i.e. $T = I_+ \cup I_-$, $\partial I_\pm = I_+ \cap I_- = \Gamma$.

THEOREM 6.1. *The* GRF $\{X(t),\ t \in T\}$ *is $p$-Markovian, i.e.*

$$(6.1)\quad E\{X(t)|D^kX(s),\ k = 0, \ldots, p;\ s \in I_-\} = E\{X(t)|D^kX(s),\ k = 0, \ldots, p;\ s \in \Gamma\} \quad \text{for } t \in I_+.$$

*Proof.* We need to distinguish between the following two cases: (a) $0 \in \Gamma$ and (b) $0 \notin \Gamma$.

(a) The RKHS is being decomposed orthogonally as follows:

$$(6.2)\qquad H^{p+1}(T) = H_+ \oplus H_\Gamma \oplus H_-,$$

where $\pi = \Gamma$, $\alpha = (p+1, p+1)$ and

$$H_\Gamma = S_{\pi,\alpha}^{2p+1}(T),$$
$$H_\pm = \{u \in H^{p+1}(T):\ \operatorname{supp} u \subset I_\pm\}.$$

Now, let $\mathscr{H}_\Gamma = JH_\Gamma$ and $\mathscr{H}_\pm = JH_\pm$ ; then clearly (6.2) implies

$$(6.3)\qquad \mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_\Gamma \oplus \mathscr{H}_-.$$

It remains to identify $\mathscr{H}_\Gamma$ and $\mathscr{H}_\pm$ by means of the GRF $\{X(t),\ t \in T\}$, but this can be done with the help of formula (5.3). Indeed,

$$u \perp H_\Gamma \Leftrightarrow D^ku(s) = 0,\ s \in \Gamma,\ k = 0, \ldots, p$$
$$\Leftrightarrow Ju \perp V\{D^kX(s),\ s \in \Gamma,\ k = 0, \ldots, p\},$$

whence we infer

$$(6.4)\qquad \mathscr{H}_\Gamma = V\{D^kX(s),\ s \in \Gamma,\ k = 0, \ldots, p\}.$$



Moreover,

$$u \in H_\pm \Leftrightarrow Ju \perp V\{D^kX(s),\ s \in I_\mp\},$$

whence

$$(6.5)\qquad \mathscr{H}_\mp \oplus \mathscr{H}_\Gamma = V\{D^kX(s),\ s \in I_\mp\}.$$

(b) In this case we take $\pi = \{0\} \cup \Gamma$, $s_0 = 0$, $\{s_1, s_2\} = \Gamma$, $\alpha = (1, p+1, p+1)$. The spaces $H_\pm$ are defined exactly in the same way as in the case (a), and

$$H_\Gamma = \{v \in S_{\pi,\alpha}^{2p+1}(T):\ v(0) = 0\}.$$

It is clear that $H_+ \perp H_-$, and if $Q$ denotes as before the orthogonal projection on $S_{\pi,\alpha}^{2p+1}(T)$, then

$$H_\Gamma = \{Qu - Qu(0):\ u \in H^{p+1}(T)\},$$
$$H_+ \oplus H_- = \{u - Qu + u(0):\ u \in H^{p+1}(T)\}.$$

Direct computation implies that $H_\Gamma \perp H_+ \oplus H_-$ and for each $u \in H^{p+1}(T)$ we have $u = \big(Qu - Qu(0)\big) + \big(u - Qu + Qu(0)\big)$. Thus, (6.2) holds in the case (b) as well. Since now, we argue in exactly the same way as in the case (a), and check formulas (6.4) and (6.5).

To complete the proof it is sufficient to notice that $X(t) \in \mathscr{H}_+ \oplus \mathscr{H}_\Gamma$ for $t \in I_+$ and then to use formulas (6.5) and (6.4).

COROLLARY 6.1. *The piecewise Hermite interpolation problem corresponds to $(\pi, \alpha)$ with $\alpha_1 = \ldots = \alpha_n = p+1$, and the $p$-Markov property is equivalent to the uniqueness of the solution of the two-point Hermite interpolation problem.*

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