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APPLICATIONES MATHEMATICAE
XVI, 3 (1979)

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CONSTRUCTION OF A CLASS OF STATIONARY PROCESSES WITH APPLICATIONS IN RELIABILITY

1. INTRODUCTION

Stochastic processes composed in a natural manner of so-called phases or cycles often appear in applications, especially in queueing and reliability theory. The regenerative processes with mutually independent and identically distributed phases are typical examples for this class of processes.

In generalization of the recent paper [7] by Nawrotzki, in Section 3 of the present paper the construction of stationary processes of the above-mentioned kind is given. For this purpose we consider a strictly stationary sequence of phases of the process to be constructed and associate it with the so-called Palm version of a stationary random marked point process. Then the stationary distribution of the process can be obtained by means of the well-known "inversion formula" from the theory of point processes. Section 2 contains the necessary basic definitions and notation from the theory of random marked point processes; it is based on the paper [5], where the reader can find a more detailed treatment of this subject.

In Section 4 the above-mentioned sequence of phases is supposed to form a time-homogeneous Markov chain (MC). Making use of this assumption, we offer a new approach to stationary semi-Markov processes and semi-Markov processes with auxiliary paths and a general state space (see [3], [8], and [9]) and, in particular, to stationary regenerative processes (see [10] and [2]). The class of processes considered in that section includes also the piecewise Markov processes introduced by Kuczura [6] and generalized by Jankiewicz and Rolski [4]. In case of a piecewise Markov process the phases (or auxiliary paths) form a time-homogeneous Markov process and one can obtain deeper results.

Section 5 contains two examples of the application in reliability theory.

2. RANDOM MARKED POINT PROCESSES

Let $[K, \mathfrak{K}]$ be a measurable space, the so-called mark space. We denote by M_K the set of all sequences $\varphi = \{[t_n, k_n]\}_{n=-\infty}^{+\infty}$ of marked points $[t_n, k_n] \in R^1 \times K$, where $R^1 = (-\infty, +\infty)$, $\ldots < t_{-1} < t_0 \leqslant 0 < t_1 < \cdots$, and $t_n \to \pm\infty$ as $n \to \pm\infty$. For every Borel set $B \subseteq R^1$ and $L \in \mathfrak{K}$ we use the notation

$$\varphi(B \times L) = \operatorname{card}\big(\varphi \cap (B \times L)\big).$$

Let $\mathfrak{M}_K$ be the σ-field of subsets of M_K generated by the sets $\{\varphi \in M_K: \varphi(B \times L) = j\}$, $j = 0, 1, \ldots$, where B is a Borel set of R^1, and $L \in \mathfrak{K}$.

Definition 1. A *random marked point process* (rmpp) on R^1 with the mark space K is a random variable (rv) Φ taking values in M_K, i.e. a probability space of the form $\Phi \sim [M_K, \mathfrak{M}_K, P]$.

The elements $\varphi \in M_K$ are realizations of Φ, and P is the probability distribution of Φ.

We will use also the representation $\Phi = \{[t_n, k_n]\}$ of an rmpp, where $t_n = t_n(\Phi)$ is the random position, and $k_n = k_n(\Phi)$ is the random mark of the n-th point of Φ.

Definition 2. An rmpp $\Phi = \{[t_n, k_n]\}$ with the distribution P is called *stationary* if for every $t \in R^1$ the translated rmpp

$$T_t\Phi = \{[t_n - t, k_n]\}$$

has the same distribution P. The distribution P of a stationary rmpp is called also *stationary*.

The expression

$$\lambda_P = \mathrm{E}_P\{\varphi([0,1] \times K)\} = \sum_{i=1}^{\infty} iP\big(\varphi([0,1] \times K) = i\big)$$

defines the *intensity* of a stationary rmpp Φ with the distribution P. The distribution P of a stationary rmpp Φ with $\lambda_P < \infty$ corresponds in a uniquely determined and inversible manner to its Palm distribution P_0 on $\mathfrak{M}_K$. The rmpp Φ_0 with the distribution P_0 is called the *Palm version* of Φ. The following properties are characteristic of P_0 and Φ_0:

$P_0\big(t_0(\Phi_0) = 0\big) = 1$,

$T_{t_1(\Phi_0)}\Phi_0$ has the same distribution P_0 as Φ_0,

$$\Delta = \mathrm{E}_{P_0}\{t_1(\Phi_0)\} = \frac{1}{\lambda_P} < \infty. \tag{2.1}$$

Furthermore,

$$P_0\big(k_0(\Phi_0) \in A\big) = \frac{\lambda_P(A)}{\lambda_P}, \quad A \subseteq K, \tag{2.2}$$

is valid, where $\lambda_P(A) = \mathrm{E}_P\{\varphi([0,1]\times A)\}$, $A \in \mathfrak{K}$. The stationary distribution P can be obtained from P_0 by means of the following inversion formula:

$$(2.3)\qquad P(C) = \lambda_P \int_0^\infty P_0\big(t_1(\Phi_0) > t,\ T_t\Phi_0 \in C\big)\,dt, \qquad C \in \mathfrak{M}_K.$$

If $\Phi_0 = \{[t_n, k_n]\}$ is the Palm version of a stationary rmpp Φ, then the sequence $\{[\tau_n, k_n]\}$, $\tau_n = t_{n+1}-t_n$, is strictly stationary. Conversely, every stationary sequence $\{[\tau_n, k_n]\}$ of random elements on $R^+ \times K$, $R^+ = [0, +\infty)$, satisfying the properties $\mathrm{E}\{\tau_n\} < \infty$ and $P(\tau_n > 0) = 1$, corresponds to the following Palm version of a stationary rmpp Φ:

$$(2.4)\qquad \Phi_0 = \{[t_n, k_n]\}_{n=-\infty}^{+\infty}, \quad t_0 = 0,\ t_n = \sum_{i=0}^{n-1} \tau_i,\ t_{-n} = \sum_{i=1}^{n} \tau_{-i},\ n \geqslant 1.$$

3. CONSTRUCTION OF STATIONARY PROCESSES

Let $[L, \mathfrak{L}]$ and $[X, \mathfrak{F}]$ be measurable spaces. We denote by X_t the set of all functions on $[0, t)$ taking values in X, and by $\mathfrak{F}_t$ a σ-field of subsets of X_t, for example the σ-field generated by cylindrical sets. Furthermore, let D be the space of all couples $[t, z]$, $t \in R^+$, $z \in X_t$, and let $\mathfrak{D}$ be the minimal σ-field of subsets of D generated by sets $A_{x,C} = \{[t, z]: t > x,\ e_x z \in C\}$, $x \geqslant 0$, $C \in \mathfrak{F}_x$, where $e_x z$ denotes the restriction of z on $[0, x)$. We set $K = L \times D$ and $\mathfrak{K} = \mathfrak{L} \otimes \mathfrak{D}$. Let $\{[\tau_n, k_n]\}_{n=-\infty}^{+\infty}$ be a strictly stationary sequence of random elements on a basic probability space $[\Omega, \mathfrak{A}, P]$ taking values in $R^+ \times K$ and satisfying the properties

$$(3.1)\qquad P(\tau_n > 0) = 1, \quad \mathrm{E}\{\tau_n\} = \Delta < +\infty.$$

Every k_n is of the form $[\eta_n, (\tau_n, \zeta_n)]$, where η_n takes values in L, and (τ_n, ζ_n) in D. Hence ζ_n is a stochastic process on $[0, \tau_n)$ with the state space X. The Palm version $\Phi_0 = \{[t_n, k_n]\}_{n=-\infty}^{+\infty}$ of a stationary rmpp Φ on R^1 with the mark space K is defined by (2.4). We denote by P_0 and P the distributions of Φ_0 and Φ, respectively.

We are interested in the behaviour of the two-dimensional stationary process

$$(3.2)\qquad z(t, \Phi) = [\eta(t, \Phi), \zeta(t, \Phi)] = \big[\eta_n(\Phi), \zeta_n\big(t-t_n(\Phi)\big)\big], \quad t \in \big[t_n(\Phi), t_{n+1}(\Phi)\big),$$

which is composed of phases. The n-th phase begins at the moment $t_n(\Phi)$ and has the random sojourn time $\tau_n(\Phi)$. The first component $\eta(t, \Phi)$

is constant in the n-th phase; the behaviour of the second component $\zeta(t, \Phi)$ on $[t_n, t_n+\tau_n)$ equals the behaviour of ζ_n on $[0, \tau_n)$.

Furthermore, we consider the three-dimensional stationary processes

$$v(t, \Phi) = [z(t, \Phi), a(t, \Phi)], \tag{3.3}$$

$$w(t, \Phi) = [z(t, \Phi), r(t, \Phi)], \tag{3.4}$$

where $a(t, \Phi) = t - t_n(\Phi)$ and $r(t, \Phi) = t_{n+1}(\Phi) - t$ denote the "age" and the "remaining life time" of the present phase of $z(t, \Phi)$ at every moment $t \in [t_n(\Phi), t_{n+1}(\Phi))$. In a manner analogous to (3.2)-(3.4) the processes $z(t, \Phi_0)$, $v(t, \Phi_0)$ and $w(t, \Phi_0)$ corresponding to Φ_0 can be introduced. In general, these latter processes are not stationary, but their distributions are invariant according to the random translation $T_{t_1(\Phi_0)}$.

The inversion formula (2.3) provides the following relation between the finite-dimensional distributions of $\zeta(t, \Phi)$ and $\zeta(t, \Phi_0)$:

$$\begin{aligned} (3.5)\quad & P\big((\zeta(t_1, \Phi), \dots, \zeta(t_m, \Phi)) \in B\big) \\ & = \frac{1}{\Delta} \int_0^\infty P_0\big((t_1(\Phi_0) > x, \zeta(t_1+x, \Phi_0), \dots, \zeta(t_m+x, \Phi_0)) \in B\big) dx, \\ & \qquad -\infty < t_1 < t_2 < \dots < t_m < \infty. \end{aligned}$$

In the case of regenerative processes (L contains a single element and the τ_n are mutually independent) formula (3.5) is well known (see [10]). It is possible to derive similar formulas for the processes $z(t, \Phi)$, $v(t, \Phi)$ and $w(t, \Phi)$. In particular, the one-dimensional distributions of $v(t, \Phi)$ and $w(t, \Phi)$ are of the form

$$\begin{aligned} (3.6)\quad & P\big(v(0, \Phi) \in A \times C \times [0, x)\big) = P\big(\eta(0, \Phi) \in A, \zeta(0, \Phi) \in C, a(0, \Phi) < x\big) \\ & = P\big(\eta_0(\Phi) \in A, \zeta_0(-t_0(\Phi)) \in C, -t_0(\Phi) < x\big) \\ & = \frac{1}{\Delta} \int_0^\infty P_0\big(t_1(\Phi_0) > t, \eta_0(T_t\Phi_0) \in A, \zeta_0(-t_0(T_t\Phi_0)) \in C, -t_0(T_t\Phi_0) < x\big) dt \\ & = \frac{1}{\Delta} \int_0^x P_0\big(t_1(\Phi_0) > t, \eta_0(\Phi_0) \in A, \zeta_0(t) \in C\big) dt, \qquad A \in \mathfrak{L}, C \in \mathfrak{F}, x \in [0, \infty), \end{aligned}$$

$$\begin{aligned} (3.7)\quad & P\big(w(0, \Phi) \in A \times C \times [0, x)\big) \\ & = P\big(\eta(0, \Phi) \in A, \zeta(0, \Phi) \in C, r(0, \Phi) < x\big) \\ & = \frac{1}{\Delta} \int_0^\infty P_0\big(t < t_1(\Phi_0) < t+x, \eta_0(\Phi_0) \in A, \zeta_0(t) \in C\big) dt, \\ & \qquad A \in \mathfrak{L}, C \in \mathfrak{F}, x \in [0, \infty), \end{aligned}$$

respectively.

4. SEMI-MARKOV PROCESSES WITH AUXILIARY PATHS

4.1. General state space L. This section is devoted to a special case of our model, which is of importance in applications, and for which formulas (3.6) and (3.7) can be written in an essentially simpler form. Suppose that the sequence $\{[\tau_n, k_n]\}_{n=-\infty}^{+\infty}$ forms a time-homogeneous MC with the kernel

(4.1) $$P(\eta_{n+1} \in A, (\tau_{n+1}, \zeta_{n+1}) \in B \mid \eta_n = l, (\tau_n, \zeta_n))$$
$$= P(\eta_{n+1} \in A, (\tau_{n+1}, \zeta_{n+1}) \in B \mid \eta_n = l) = \int_A Q_j(B) p(l, dj),$$
$$A \in \mathfrak{L}, B \in \mathfrak{D}, j \in L,$$

where

(4.2) $$p(l, A) = P(\eta_{n+1} \in A \mid \eta_n = l), \quad l \in L, A \in \mathfrak{L},$$

is the kernel of the embedded MC $\{\eta_n\}_{n=-\infty}^{+\infty}$ and

(4.3) $$Q_l(B) = P((\tau_n, \zeta_n) \in B \mid \eta_n = l), \quad l \in L, B \in \mathfrak{D},$$

is a set of distributions on $\mathfrak{D}$. The index n in (4.3) can be omitted and in the sequel we will use the notation

$$Q_l(B) = P((\tau, \zeta) \in B \mid \eta = l), \quad l \in L, B \in \mathfrak{D}.$$

The structure of the process $z(t, \Phi)$ in the case considered is very simple. The values η_n of the first component $\eta(t, \Phi)$, which is constant in a phase, form a time-homogeneous MC. The distribution of the sojourn time τ_n in the n-th phase and of the behaviour of the second component $\zeta(t, \Phi)$ in this phase depends only on the value η_n. In accordance with Pyke and Schaufele [8], we call $z(t, \Phi)$ a *semi-Markov process with auxiliary paths.*

The structure of the stationary distributions according to the Markov kernel (4.1) is described by the following

THEOREM 1. (a) *Every invariant probability measure*

$$Q(A \times B) = P(\eta_n \in A, (\tau_n, \zeta_n) \in B), \quad A \in \mathfrak{L}, B \in \mathfrak{D},$$

with respect to kernel (4.1) *is of the form*

(4.4) $$Q(A \times B) = \int_A Q_l(B) \pi(dl),$$

where $\pi(\cdot)$ *is an invariant probability measure with respect to kernel* (4.2), *i.e.* $\pi(\cdot)$ *is a probability solution of the equation*

(4.5) $$\pi(\cdot) = \int_L p(l, \cdot) \pi(dl).$$

(b) *If* $\pi(\cdot)$ *is a probability solution of* (4.5), *then* (4.4) *determines an invariant probability measure* Q *with respect to kernel* (4.1).

Proof. (a) Let Q be an invariant probability measure with respect to kernel (4.1), i.e.

$$Q(A \times B) = \int_D \int_L \int_A Q_l(B) p(y, dl) Q(dy \times du) \tag{4.6}$$

$$= \int_L \int_A Q_l(B) p(y, dl) Q(dy \times D).$$

Then $\pi(\cdot) = Q\big((\cdot) \times D\big)$ is, in view of our assumption, a stationary distribution of the time-homogeneous MC $\{\eta_n\}$, i.e. a probability solution of (4.5). Hence from (4.6) we obtain

$$Q(A \times B) = \int_A Q_l(B) \int_L p(y, dl) \pi(dy) = \int_A Q_l(B) \pi(dl).$$

(b) Let $\pi(\cdot)$ be a probability solution of (4.5). Then it follows from representation (4.4) that

$$Q(A \times B) = \int_A Q_l(B) \pi(dl) = \int_A Q_l(B) \int_L p(y, dl) \pi(dy).$$

For $B = D$, (4.4) implies $Q(A \times B) = \pi(A)$. Thus

$$Q(A \times B) = \int_A Q_l(B) \int_L p(y, dl) Q(dy \times D)$$

$$= \int_L \int_A Q_l(B) p(y, dl) Q(dy \times du),$$

i.e. Q is a stationary distribution according to (4.1).

If we introduce the notation

$$F_l(x) = P(\tau < x \mid \eta = l) = Q_l(\tau < x), \quad \Delta_l = \int_0^\infty x \, dF_l(x), \; l \in L, \; x \in R^+, \tag{4.7}$$

then conditions (3.1) change into

$$F_l(+0) = 0, \; l \in L, \qquad \Delta = \int_L \Delta_l \pi(dl) < \infty.$$

THEOREM 2. *If the sequence $\{[\tau_n, k_n]\}_{n=-\infty}^{+\infty}$ is a time-homogeneous MC with kernel (4.1)-(4.3) and there exists a uniquely determined stationary distribution $\pi(\cdot)$ according to kernel (4.2), then the one-dimensional distri-*

butions of the stationary processes $v(t, \Phi)$ *and* $w(t, \Phi)$ *are of the form*

(4.8) $$P\big(v(0, \Phi) \in A \times C \times [0, x)\big)$$
$$= \frac{1}{\Delta} \int_0^x \int_A Q_l\big(\tau > t, \zeta(t) \in C\big)\pi(dl)\,dt, \quad A \in \mathfrak{L}, C \in \mathfrak{F}, x \in [0, \infty),$$

(4.9) $$P\big(w(0, \Phi) \in A \times C \times [0, x)\big)$$
$$= \frac{1}{\Delta} \int_0^\infty \int_A Q_l\big(t < \tau < t+x, \zeta(t) \in C\big)\pi(dl)\,dt, \quad A \in \mathfrak{L}, \ C \in \mathfrak{F}, \ x \in [0, \infty),$$

respectively.

Proof of (4.8). According to (3.6) we have

$$P\big(v(0, \Phi) \in A \times C \times [0, x)\big) = \frac{1}{\Delta} \int_0^x P_0\big(t_1(\Phi_0) > t, \eta_0(\Phi_0) \in A, \zeta_0(t) \in C\big)\,dt.$$

The event

$$\{t_1(\Phi_0) > t, \eta_0(\Phi_0) \in A, \zeta_0(t) \in C\} = \{\tau_0(\Phi_0) > t, \eta_0(\Phi_0) \in A, \zeta_0(t) \in C\}$$

is a statement only about the phase (τ_0, k_0) of the stationary Markov chain $\{[\tau_n, k_n]\}_{n=-\infty}^{+\infty}$ with distribution (4.4), which is uniquely determined in view of our assumption and Theorem 1. Hence

$$P\big(t_1(\Phi_0) > t, \eta_0(\Phi_0) \in A, \zeta_0(t) \in C\big) = \int_A Q_l\big(\tau > t, \zeta(t) \in C\big)\pi(dl).$$

In an analogous manner one can prove (4.9).

It can easily be seen that the component $\eta(t, \Phi)$ of $z(t, \Phi)$ is a semi-Markov process (SMP) taking values in the general state space L and having the semi-Markov kernel

(4.10) $$P(\eta_{n+1} \in A, \tau_{n+1} < x \mid \eta_n = y, \tau_n) = \int_A F_z(x)p(y, dz),$$
$$y \in L, \ x \in [0, \infty), \ A \in \mathfrak{L},$$

where $p(\cdot, \cdot)$ and $F_z(\cdot)$ are defined by (4.2) and (4.7), respectively (see [3]). From (4.8) we derive the one-dimensional distribution of the stationary SMP with kernel (4.10):

$$P\big(\eta(0, \Phi) \in A, a(0, \Phi) < x\big) = \frac{1}{\Delta} \int_0^x P_0\big(t_1(\Phi_0) > t, \eta_0(\Phi_0) \in A\big)\,dt$$
$$= \frac{1}{\Delta} \int_0^x \int_A \big(1 - F_l(t)\big)\pi(dl)\,dt, \quad A \in \mathfrak{L}, \ x \in [0, \infty).$$

Remark. It is possible to investigate two generalizations of our model.

(a) The MC $\{[\tau_n, k_n]\}_{n=-\infty}^{+\infty}$ has instead of (4.1)-(4.3) the kernel

$$(4.1\mathrm{a})\qquad P\big(\eta_{n+1}\in A,\,(\tau_{n+1},\zeta_{n+1})\in B\mid k_n=k\big)=\int_A Q_l(B)\,r(k,dl),$$
$$A\in\mathfrak{L},\ B\in\mathfrak{D},\ k\in K,$$

where $Q_l(B)$ is given by (4.3) and

$$(4.2\mathrm{a})\qquad r(k,A)=P(\eta_{n+1}\in A\mid k_n=k),\qquad A\in\mathfrak{L},\ k\in K,$$

is a stochastic kernel from K into L.

In generalization of (4.2), expression (4.2a) means that the probability law of η_{n+1} depends not only on the value of η_n but also on (τ_n, ζ_n).

The structure of the stationary distribution is again described by Theorem 1, where $p(l, A)$ is given by the formula

$$p(l,A)=P(\eta_{n+1}\in A\mid\eta_n=l)=\int_D r\big((l,\omega),A\big)Q_l(d\omega),\qquad l\in L,\ A\in\mathfrak{L}.$$

The statement of Theorem 2 and all its conclusions given in Section 4.2 are also preserved. It is easy to see that the class of piecewise Markov processes considered by Belyayev [1] and Kuczura [6] is a special case of the model given by (4.1a), (4.2a) and (4.3).

(b) We can consider also a model for which the distribution of the couple $[\tau_n, \zeta_n)$ depends not only on the "mark" η_n taken at the beginning of the n-th phase but also on the "mark" η_{n+1} taken at the beginning of the $(n+1)$-st phase. Such a model can be translated into the model considered throughout this section (see [8] for the case of an SMP). For this purpose we set

$$\bar{L}=L\times L,\qquad \bar{K}=\bar{L}\times D,$$
$$\bar{\eta}(t,\Phi)=[\eta_n(\Phi),\eta_{n+1}(\Phi)],\qquad t\in\big[t_n(\Phi),t_{n+1}(\Phi)\big)$$

and obtain a model for which the distribution of (τ_n, ζ_n) depends only on $\bar{\eta}_n$. By (4.8), the one-dimensional distribution of the stationary process

$$\bar{v}(t,\Phi)=[\bar{\eta}(t,\Phi),\zeta(t,\Phi),a(t,\Phi)]$$

can be given by

$$(4.11)\quad P\big(\bar{v}(0,\Phi)\in A\times C\times[0,x)\big)=P\big(\bar{\eta}(0,\Phi)\in A,\zeta(0,\Phi)\in C,a(0,\Phi)<x\big)$$
$$=\frac{1}{\Delta}\int_0^x P_0\big(t_1(\Phi_0)>t,\ \eta_0(\Phi)\in A_1,\ \eta_1(\Phi_0)\in A_2,\ \zeta(t)\in C\big)\,dt$$
$$=\frac{1}{\Delta}\int_0^x\int_{A_1}\int_{A_2} Q_{yz}\big(\tau>t,\ \zeta(t)\in C\big)\,p(y,dz)\,\pi(dy)\,dt,$$
$$A=A_1\times A_2,\ A_1,A_2\in\mathfrak{L},\ C\in\mathfrak{F},\ x\in[0,\infty),$$

where $p(\cdot, \cdot)$ and $\pi(\cdot)$ are the kernel and the stationary distribution of the embedded MC $\{\eta_n\}$, respectively, and

$$\Delta_{y,z} = \int_0^\infty x Q_{yz}(\tau \in dx), \qquad \Delta = \int_L \Delta_{y,z} p(y, dz)\pi(dy),$$

$$Q_{yz}(B) = P\big((\tau_n, \zeta_n) \in B \mid \eta_n = y, \eta_{n+1} = z\big), \qquad y, z \in L,\ B \in \mathfrak{D}.$$

The corresponding formula for the SMP $[\bar{\eta}(t, \Phi), a(t, \Phi)]$ takes, in view of (4.11), the form

$$P\big([\eta(0, \Phi), a(0, \Phi)] \in A \times [0, x)\big)$$
$$= \frac{1}{\Delta} \int_0^x \int_{A_1} \int_{A_2} \big(1 - F_{yz}(t)\big) p(y, dz)\pi(dy)\, dt, \qquad A \in \bar{\mathfrak{L}}, x \in [0, \infty),$$

where $F_{yz}(t) = Q_{yz}(\tau < t), t \geqslant 0$.

4.2. Countably infinite or finite state space L. We treat the SMP (Markov renewal process) with auxiliary paths in a way different from that taken by Pyke and Schaufele [8], and Schäl [9] who also investigated such processes. If L is countably infinite or finite, then kernel (4.1)-(4.3) of the MC $\{[\tau_n, k_n]\}_{n=-\infty}^{+\infty}$ can be written in the form

$$P\big(\eta_{n+1} = j, (\tau_{n+1}, \zeta_{n+1}) \in B \mid \eta_n = i, (\tau_n, \zeta_n)\big) = p_{ij} Q_j(B), \tag{4.12}$$
$$i, j \in L,\ B \in \mathfrak{D},$$

$$p_{ij} = P(\eta_{n+1} = j \mid \eta_n = i), \qquad i, j \in L, \tag{4.13}$$

$$Q_j(B) = P\big((\tau_n, \zeta_n) \in B \mid \eta_n = j\big), \qquad B \in \mathfrak{D},\ j \in L. \tag{4.14}$$

If π_i, $i \in L$, is the uniquely determined stationary distribution according to $(p_{ij})_{i,j \in L}$, then from (4.8) we obtain the formula

$$P\big(v(0, \Phi) \in \{k\} \times C \times [0, x)\big) = \frac{1}{\Delta} \pi_k \int_0^x Q_k\big(\tau > t,\ \zeta(t) \in C\big) dt, \tag{4.15}$$
$$k \in L,\ C \in \mathfrak{F},\ x \in [0, \infty).$$

In order to describe the non-stationary behaviour on R^+ of the process $v(t, \Phi)$ defined in the preceding section we choose an initial distribution

$$\bar{Q}_i(B) = P\big((\tau_0, \zeta_0) \in B,\ \eta_0 = i\big), \qquad B \in \mathfrak{D},\ i \in L, \tag{4.16}$$

i.e. a distribution of the initial phase (τ_0, k_0) of the process. We assume that

$$\lim_{t \to \infty} \sum_{i \in L} \bar{Q}_i(\tau_0 < t) = 1 \tag{4.17}$$

is valid. The time-dependent behaviour of $z(t, \Phi)$ is determined by (τ_0, k_0) and by the sequence of phases $\{[\tau_n, k_n]\}_{n \geqslant 1}$ governed by kernel (4.12)-

-(4.14). By $P_{\bar{Q}}$ is meant the distribution of the process $z(t, \Phi)$ on R^+ induced by the initial distribution (4.16) and kernel (4.12)-(4.14).

Now we give a sufficient condition for the strong regularity of $z(t, \Phi)$, $t \geq 0$, i.e. for the validity of

$$P_{\bar{Q}}\big(N(t) = \max\{n: t_n(\Phi) < t\} < \infty\big) = 1, \quad t \in [0, \infty). \tag{4.18}$$

THEOREM 3. *If all probabilities π_i, $i \in L$, are positive, then* (4.18) *is valid.*

Proof. Since

$$P_0\left(\frac{t_n(\Phi_0)}{n} \to \Delta\right) = 1 \quad \text{as } n \to \infty,$$

we have $P_0\big(N(t) < \infty\big) = 1$, $t \in R^+$. Furthermore, the representation

$$P_0\big(N(t) < \infty\big) = \sum_{i \in L} \pi_i P_i\big(N(t) < \infty\big)$$

holds, where $P_i(\cdot) = P_0(\cdot \mid \eta_0 = i)$, $i \in L$. Thus, from the assumption $\pi_i > 0$, $i \in L$, it follows $P_i\big(N(t) < \infty\big) = 1$, $i \in L$, $t \in R^+$. Hence, by (4.17) we have

$$P_{\bar{Q}}\big(N(t) < \infty\big) = \sum_{i \in L} \bar{Q}_i(\tau_0 > t) + \sum_{i,j \in L} p_{ij} \int_0^t P_j\big(N(t-x) < \infty\big) \bar{Q}_i(\tau_0 \in dx) = 1.$$

Under the assumption of Theorem 3, in an analogous manner one can also prove that

$$P_{\bar{Q}}\big(\lim_{t \to \infty} N_i(t) = \infty\big) = 1, \quad i \in L,$$

where $N_i(t)$ denotes the number of points t_n with $\eta_n = i$ in the interval $[0, t)$. These points form a renewal process (process of i-renewals). The distribution function of the distance between two successive i-renewals is denoted by $G_{ii}(x)$. From (2.2) and the identity $\pi_i = P_0(\eta_0 = i)$ it follows that the intensity $\lambda_P(\{i\})$ of the stationary process of i-renewals equals

$$\lambda_P(\{i\}) = \pi_i \lambda_P = \frac{\pi_i}{\Delta}.$$

Therefore, in view of (2.1) the mean μ_{ii} of the distribution function $G_{ii}(x)$ is equal to Δ / π_i.

Let $H_{\bar{Q},i}(t)$ be the renewal function of the (embedded) process of i-renewals induced by the initial distribution $\bar{Q}$ and kernel (4.12)-(4.14). Then the equation

$$R_t(i, x, C) = P\big(\eta(t, \Phi) = i, \alpha(t, \Phi) < x, \zeta(t, \Phi) \in C\big)$$

$$= \int_{t-x}^{t} Q_i\big(\tau > t-u, \zeta(t-u) \in C\big) dH_{\bar{Q},i}(u), \quad i \in L, x > t, C \in \mathfrak{F}, t \in R^+,$$

is valid.

THEOREM 4 (ERGODIC THEOREM). *If all the probabilities π_i, $i \in L$, are positive and the distribution function $G_{ii}(x)$ is non-lattice, then*

$$\lim_{t\to\infty} R_t(i, x, C) = \frac{\pi_i}{\Delta}\int_0^x Q_i(\tau > u,\ \zeta(u) \in C)du, \qquad x \geqslant 0,\ C \in \mathfrak{F},\ i \in L.$$

The statement follows directly from the key renewal theorem.

Theorem 4 states that the one-dimensional distribution of the stationary process $v(t, \Phi)$ determined by (4.15) is a limit distribution for any initial distribution.

5. EXAMPLES FROM RELIABILITY THEORY

The behaviour of a wide class of redundant systems with repair can be described by SMP with auxiliary paths. Here, X is the finite set of all system states and L is a subset of X. In general, L can be taken as the set of all up-states of the system. We consider two redundant systems with repair, the first of them is well known and can be described by a piecewise Markov process (see [1] and [7]). However, we take no advantage of this fact and compute the stationary state probabilities by our formulas without extensive calculations. The second system to consider cannot be described by a piecewise Markov process with a countably infinite state space.

5.1. The system consists of two redundant elements of the same kind standing in parallel and of a single repair facility. If one element fails, then the repair quickly begins and restores completely the properties of the element. After completing the repair, the element is simultaneously switched into the operation. System break-down occurs if both elements fail simultaneously. The element lastly failed must wait until completing the repair of the other element. Switching times are neglectable. The random life-times X_1 and X_2 of the elements and the random time to repair Y of any failed element are mutually independent and have the distribution functions

$$P(X_i < t) = P(X < t) = 1 - e^{-\lambda t}, \qquad t \geqslant 0,\ i = 1, 2,$$
$$P(Y < t) = G(t), \qquad t \geqslant 0,$$

with the mean

$$a = \int_0^\infty x dG(x)$$

and the Laplace-Stieltjes transform

$$\hat{g}(s) = \int_0^\infty e^{-st} dG(t).$$

The set X of the system states is of the form $X = \{0, 1, 2\}$, where any state denotes the number of failed elements.

We determine the transition probabilities of the embedded MC $\{\eta_n\}$ with the state space $L = \{0, 1\}$ as

$$p_{00} = 0, \quad p_{01} = 1, \quad p_{10} = P(X > Y) = \hat{g}(\lambda), \quad p_{11} = 1 - \hat{g}(\lambda)$$

and for the uniquely determined initial distribution of $\{\eta_n\}$ we obtain

$$\pi_0 = \frac{p_{10}}{1+p_{10}}, \qquad \pi_1 = \frac{1}{1+p_{10}}.$$

We state

$$\Delta_0 = \mathrm{E}\{\min(X_1, X_2)\} = \frac{1}{2\lambda}, \qquad \Delta_1 = \mathrm{E}\{Y\} = a,$$

$$\Delta = \pi_0 \Delta_0 + \pi_1 \Delta_1 = \frac{\hat{g}(\lambda) + 2\lambda a}{2\lambda\big(1+\hat{g}(\lambda)\big)}.$$

By (4.15) we obtain the stationary state probabilities q_k for $k \in X$:

$$q_0 = \frac{\pi_0}{\Delta} \int_0^\infty Q_0\big(\tau > t,\ \zeta(t) = 0\big)\,dt = \frac{\pi_0 \Delta_0}{\Delta} = \frac{\hat{g}(\lambda)}{\hat{g}(\lambda) + 2\lambda a},$$

$$q_1 = \frac{\pi_1}{\Delta} \int_0^\infty Q_1\big(\tau > t,\ \zeta(t) = 1\big)\,dt = \frac{\pi_1}{\Delta} \int_0^\infty P(Y > t,\ X > t)\,dt$$

$$= \frac{\pi_1}{\Delta} \int_0^\infty \big(1 - G(t)\big) e^{-\lambda t}\,dt = \frac{\pi_1}{\Delta}\,\frac{1 - \hat{g}(\lambda)}{\lambda} = \frac{2\big(1 - \hat{g}(\lambda)\big)}{\hat{g}(\lambda) + 2\lambda a},$$

$$q_2 = \frac{\pi_1}{\Delta} \int_0^\infty Q_1\big(\tau > t,\ \zeta(t) = 2\big)\,dt = \frac{\pi_1}{\Delta} \int_0^\infty \big(1 - G(t)\big)\big(1 - e^{-\lambda t}\big)\,dt$$

$$= \frac{2\big(\lambda a - 1 + \hat{g}(\lambda)\big)}{\hat{g}(\lambda) + 2\lambda a}.$$

The stationary system availability A is the sum of the stationary probabilities of up-states:

$$A = q_1 + q_2 = 1 - q_2 = \frac{2 - \hat{g}(\lambda)}{\hat{g}(\lambda) + 2\lambda a}.$$

5.2. The second system to consider differs from the system treated in 5.1 only in two points:

(a) the elements have different reliability characteristics,

(b) if both the elements are available, then one of them is active and the other is in cold standby.

The random life-time X_i and the random repair time Y_i of the element (i) have the distribution functions

$$P(X_i < t) = F_i(t) \quad \text{and} \quad P(Y_i < t) = G_i(t), \quad t \geqslant 0,$$

with the mean

$$b_i = \int_0^\infty x dF_i(t), \quad i = 1, 2.$$

The system states of the set $X = \{1, 2, 3, 4\}$ are determined as follows:

for $i = 1, 2$, the element (i) failed and begins to repair; the element $(3-i)$ switches from the standby into the operation;

for $i = 3, 4$ we have system break-down states: while the element $(5-i)$ is still repaired, the element $(i-2)$ failed and waits for repair.

The transition probabilities of the embedded MC $\{\eta_n\}$ with the state space $L = \{1, 2\}$ take the form $p_{11} = p_{22} = 0$, $p_{12} = p_{21} = 1$.

There exists a uniquely determined initial distribution of the MC $\{\eta_n\}$ such that $\pi_1 = \pi_2 = \frac{1}{2}$. We state

$$\Delta_1 = \mathrm{E}\{\max(X_2, Y_1)\} = \int_0^\infty \big(1 - F_2(t)G_1(t)\big)dt,$$

$$\Delta_2 = \mathrm{E}\{\max(X_1, Y_2)\} = \int_0^\infty \big(1 - F_1(t)G_2(t)\big)dt,$$

$$\Delta = \pi_1\Delta_1 + \pi_2\Delta_2 = \frac{1}{2}(\Delta_1 + \Delta_2).$$

By (4.15) we can obtain the stationary state probabilities q_k for $k \in X$:

$$q_1 = \frac{\pi_1}{\Delta}\int_0^\infty Q_1\big(\tau > t, \zeta(t) = 1\big)dt = \frac{\pi_1}{\Delta}\int_0^\infty P\big(\max(X_2, Y_1) > t, X_2 > t\big)dt$$

$$= \frac{\pi_1}{\Delta}\int_0^\infty P(X_2 > t)dt = \frac{b_2}{\Delta_1 + \Delta_2},$$

$$q_4 = \frac{\pi_1}{\Delta}\int_0^\infty Q_1\big(\tau > t, \zeta(t) = 4\big)dt = \frac{\pi_1}{\Delta}\int_0^\infty P\big(\max(X_2, Y_1) > t, X_2 < t\big)dt$$

$$= \frac{\pi_1}{\Delta}\int_0^\infty P(Y_1 > t,\, X_2 < t)\,dt = \frac{\pi_1}{\Delta}\int_0^\infty P(Y_1 > t)P(X_2 < t)\,dt$$

$$= \frac{1}{\Delta_1+\Delta_2}\int_0^\infty \big(1-G_1(t)\big)F_2(t)\,dt.$$

In an analogous manner we can obtain

$$q_2 = \frac{b_1}{\Delta_1+\Delta_2} \quad \text{and} \quad q_3 = \frac{1}{\Delta_1+\Delta_2}\int_0^\infty \big(1-G_2(t)\big)F_1(t)\,dt.$$

For the value of the stationary system availability we have

$$A = q_1+q_2 = 1-q_3-q_4 = (b_1+b_2)\Big[\int_0^\infty \big(2-F_1(t)G_2(t)-F_2(t)G_1(t)\big)\,dt\Big]^{-1}.$$

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Received on 30. 9. 1976;
revised version on 4. 3. 1977

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KONSTRUKCJA PEWNEJ KLASY STACJONARNYCH PROCESÓW STOCHASTYCZNYCH STOSOWANYCH W TEORII NIEZAWODNOŚCI

STRESZCZENIE

W pracy przedstawiono ogólne podejście do procesów stochastycznych, składających się z tzw. faz. Rozważane procesy są uogólnieniami procesów odnowy, półmarkowskich i przedziałami markowskich. Korzystając z wyników teorii procesów punktowych otrzymuje się rozkład stacjonarny rozważanych procesów. W przypadku struktury półmarkowskiej włożonej wzory stają się stosunkowo proste. Podano także dwa przykłady zastosowań w teorii niezawodności.