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ON RANDOM DISCRETE DISTRIBUTIONS

1. Introduction*. Limit theorems for the Poisson-Dirichlet distribution and other related distributions were investigated by Kingman in [4]. Suppose $\zeta(t)$, $t \geqslant 0$, is a subordinator such that $\zeta(0) = 0$ and ∇ is the space of infinite sequences $(p_1, p_2, \ldots)$ satisfying

$$p_1 \geqslant p_2 \geqslant \ldots \geqslant 0, \qquad \sum_{j=1}^{\infty} p_j = 1.$$

Then define random variables ζ_{nj} as follows:

$$\zeta_{nj} = \frac{\zeta(jn^{-1}) - \zeta\big((j-1)n^{-1}\big)}{\zeta(1)} \quad (j = 1, \ldots, n;\ n = 1, 2, \ldots). \tag{1}$$

When $\zeta_{n1}, \ldots, \zeta_{nn}$ are arranged in descending order, followed by zeros, we obtain a random element Φ_n on ∇. As $n \to \infty$, Φ_n converge in distribution to a limit which is a random element of ∇. Recall that the *Poisson-Dirichlet distribution* $\mathscr{PD}(\theta)$ is a distribution on ∇ which is the limiting of Φ_n when the distribution of $\zeta(t)$ is gamma with probability density

$$\theta^t x^{t-1} e^{-\theta x} / \Gamma(t), \qquad x \geqslant 0.$$

One can generalize this problem in a natural way by considering instead of the array given in (1) the array

$$\pi_{nj} = \frac{\xi_{nj}}{\sum_{j=1}^{n} \xi_{nj}} \quad (j = 1, \ldots, n;\ n = 1, 2, \ldots),$$

where ξ_{nj} satisfy the following conditions:

(i) the random variables ξ_{nj} $(j = 1, \ldots, n;\ n = 1, 2, \ldots)$ are positive and independent on a probability space $(\Omega, \mathscr{F}, \Pr)$;

* The paper was partially written while the author stayed at Oxford University.

(ii) $\max_{1\leqslant j\leqslant n}\Pr(\xi_{nj}\geqslant\varepsilon)\to 0$ as $n\to\infty$ $(\varepsilon>0)$;

(iii) $\sum_{j=1}^{n}\xi_{nj}$ converges in distribution to a limit.

To get the analog of Kingman's theorem we must know the joint convergence of

$$\sum_{j=1}^{n}\xi_{nj},\ \xi_{(n,1)},\ \ldots,\ \xi_{(n,n)},\ 0,\ 0,\ \ldots,$$

where $\xi_{(n,1)}\geqslant\ldots\geqslant\xi_{(n,n)}$ are the random variables ξ_{nj} $(j=1,\ldots,n)$ arranged in descending order. This was solved, under the assumption of continuity of $F_{nj}(x)=\Pr(\xi_{nj}\leqslant x)$, by Loève in [6]. In the paper there is given an alternative proof without the continuity assumption (Theorem 1). For the sake of clarity the proof is done for identically distributed, in each row n, random variables ξ_{nj} $(j=1,\ldots,n)$. The idea of the proof can be passed for the general case but computations become tedious. However, we demonstrate the general proof of convergence of $\left(\sum_{j=1}^{n}\xi_{nj},\ \xi_{(n,1)}\right)$. Under an additional natural assumption the sequence π_{nj} $(j=1,\ldots,n)$ arranged in descending order converges in distribution to a limit which is a proper distribution on V (Theorem 2).

If the random variables ξ_{nj} $(j=1,\ldots,n)$ are identically distributed for each n, we obtain a nice explanation of our result, in terms of random distributions, by application of the work by Kallenberg [3]. He proved that convergence in distribution of $\sum_{j\leqslant nt}\pi_{nj}$ $(0\leqslant t\leqslant 1)$ occurs if and only if convergence in distribution of the sequences of π_{nj} arranged in descending order occurs.

All results of this paper are stated in Section 2 and the proofs are given in Section 3.

For all concepts connected with weak convergence of probability measures we refer to [2].

2. Theorems. For each n, let $\xi_{n1},\ldots,\xi_{nk_n}$ be positive, independent random variables on a probability space $(\Omega,\mathscr{F},\Pr)$ and let

$$F_{nj}(x)=\Pr(\xi_{nj}\leqslant x).$$

Throughout the paper we assume that the following conditions (A), (B), (C) hold:

(A) $$\lim_{n\to\infty}\inf_{1\leqslant j\leqslant k_n}F_{nj}(x)=1,\qquad x>0,$$

which is equivalent to

(A′) $$\lim_{n\to\infty}\sup_{1\leqslant j\leqslant k_n}|\Phi_{nj}(t)-1|=0,\qquad t\in R,$$

where $\Phi_{nj}(t) = \int\limits_0^\infty e^{-tx} F_{nj}(dx)$;

(B) $$\lim_{n\to\infty} \sum_{j=1}^{k_n} \big(F_{nj}(x) - 1\big) = N(x), \quad x > 0,\ N(x+) - N(x-) = 0;$$

(C) $$\limsup_{n\to\infty} \sum_{j=1}^{k_n} \mathrm{E}(\xi_{nj};\ \xi_{nj} \leqslant \varepsilon) = a, \quad \varepsilon > 0.$$

According to the author's knowledge, Kallenberg [2] was the first to point out explicitly that under (A) the assumptions (B) and (C) are equivalent to

(D) $$\sum_{j=1}^{k_n} \xi_{nj} \to \sigma(a, N),$$

where $\to$ denotes convergence in distribution and

$$-\lg \mathrm{E} e^{-t\sigma(a,N)} = at + \int\limits_0^\infty (1 - e^{-tx}) N(dx).$$

Denote by $\xi_{(n,i)}$ the i-th greatest element of the sequence $\xi_{n1}, \ldots, \xi_{nk_n}$; let us put

$$X_{(n)} = (\xi_{(n,1)}, \ldots, \xi_{(n,k_n)}, 0, 0, \ldots) \quad \text{and} \quad \sigma_n = \sum_{j=1}^{k_n} \xi_{nj}.$$

THEOREM 1 (Loève [6]). *We have*

$$(\sigma_n, X_{(n)}) \to (\sigma, X_{(\infty)}) \quad \text{as } n \to \infty.$$

The distribution of $(\sigma, X_{(\infty)})$ depends only on a and N given by (C) and (B), respectively. Its finite-dimensional distributions are given in (7). The coordinates of $X_{(\infty)}$ form a Poisson process with intensity measure N.

COROLLARY 1. *For each natural L we have*

$$\Big(\sum_{j=1}^{k_n} \xi_{nj}^1, \sum_{j=1}^{k_n} \xi_{nj}^2, \ldots, \sum_{j=1}^{k_n} \xi_{nj}^L, X_{(n)}\Big) \to \Big(\sigma, \sum_{j=1}^{\infty} \xi_{(\infty,j)}^2, \ldots, \sum_{j=1}^{\infty} \xi_{(\infty,j)}^L, X_{(\infty)}\Big) \quad \text{as } n \to \infty.$$

The next theorem deals with random elements on V (random elements on V were first investigated by Kingman in [4]).

THEOREM 2. *If* (C) *holds with $a = 0$, then*

$$X_{(n)}/\sigma_n \to P_N \quad \text{as } n \to \infty,$$

where P_N is a probability measure on V depending on the function N only.

Remark 1. As a special case we get the result obtained by Kingman in [4].

Let

$$\xi_{nj} = \xi(ja_n) - \xi\big((j-1)a_n\big),$$

where $\xi(t)$, $t \geqslant 0$, is a subordinator which has no deterministic drift with distributions determined by the Lévy formula

$$\mathrm{E}e^{-x\xi(t)} = e^{-t\Psi(x)}, \quad \text{where } \Psi(x) = \int_0^\infty (1-e^{-xy})N(dy).$$

If $na_n \to \lambda$, then $\sigma_n = \xi(na_n) \to \xi(\lambda)$. Thus (B) and (C) with $a = 0$ hold and, by Theorem 2, $X_{(n)}/\sigma_n$ converges in distribution to a limit.

Remark 2. If we put

$$\xi_{nj} = \sigma(a/n, N/n) \quad (j = 1, \ldots, n;\ n = 1, 2, \ldots),$$

then σ is distributed as $\sigma(a, N)$ but $\sum_{j=1}^{\infty} \xi_{(\infty,j)}$ is distributed as $\sigma(a, N)$. This shows that Theorem 2 is false for $a > 0$.

Put

$$\pi_{nj} = \xi_{nj}/\sigma_n \quad (j = 1, \ldots, k_n;\ n = 1, 2, \ldots), \tag{2}$$

$$\Pi_n(t) = \sum_{j \leqslant k_n t} \pi_{nj}, \quad 0 \leqslant t \leqslant 1,$$

and

$$\Pi(t) = \sum_{j=1}^{\infty} \pi_j 1_{[0,1]}(t-\tau_j), \quad 0 \leqslant t \leqslant 1, \tag{3}$$

where $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots)$ is a random element on V with distribution P_N, and τ_j $(j = 1, 2, \ldots)$ are independent random variables uniformly distributed on $[0, 1]$ and independent of $\boldsymbol{\pi}$. From Theorem 2 and the results of Kallenberg in [3] we have

COROLLARY 2. *Suppose additionally that ξ_{nj} $(j = 1, \ldots, n)$ are identically distributed for each n. Then $\Pi_n \to \Pi$.*

Remark 3. The uniform distribution U on $[0, 1]$ is the limit of distributions M_n $(n = 1, 2, \ldots)$ assigning mass $1/n$ to the point i/n $(i = 1, \ldots, n)$. Repeating this argument with the sequence of random distributions Π_n assigning mass π_{nj} to the point i/n (π_{nj} are defined in (2), and $\mathrm{E}\pi_{nj} = 1/n$ since ξ_{nj} $(j = 1, \ldots, n;\ n = 1, 2, \ldots)$ are identically distributed), we obtain Π defined in (3) as the limiting distribution. Notice that realizations of Π are discrete distributions with probability 1 but the expected limiting distribution $\mathrm{E}\Pi = U$ is absolute continuous. If f is a continuous function on $[0,1]$, then

$$\int_0^1 f(t)\Pi_n(dt) \to \int_0^1 f(t)\Pi(dt).$$

Knowledge of the mean and variance of $\int_0^1 f(t)\Pi(dt)$ is sometimes interesting. After standard calculations we obtain

$$\mathrm{E}\int_0^1 f(t)\Pi(dt) = \int_0^1 f(t)\,dt = \mathrm{E}f(\tau_1),$$

$$\mathrm{Var}\int_0^1 f(t)\Pi(dt) = \mathrm{Var}f(\tau_1)\mathrm{E}\sum_{j=1}^{\infty}\pi_j^2.$$

If π has the Poisson-Dirichlet distribution $\mathscr{PD}(\theta)$, the quantity $\mathrm{E}\sum_{j=1}^{\infty}\pi_j^2$ has a nice genetical interpretation (see [5] and [7]). Using arguments from [5], the higher moments of $\int_{[0}^{1} f(t)\Pi(dt)$ are also available.

3. Proofs. Proof of Theorem 1. For a function $G(t)$, $t \geqslant 0$ and $0 < x < y$, we put

$$G_{|x}(t) = \begin{cases} G(t)/G(x), & 0 \leqslant t < x, \\ G(x), & x \leqslant t, \end{cases}$$

$$G^x(t) = \begin{cases} G(t) & 0 \leqslant t < x, \\ G(x), & x \leqslant t, \end{cases}$$

$$G_{|xy}(t) = \begin{cases} 0, & 0 \leqslant t < x, \\ \dfrac{G(t)-G(x)}{G(y)-G(x)}, & x \leqslant t < y, \\ 1, & y \leqslant t. \end{cases}$$

In the case of a function G_n, the notations above take the forms $G_{n|x}$, G_n^x and $G_{n|xy}$, respectively.

For distribution functions $G_1, \ldots, G_n$ we set

$$\prod_{j=1}^{n}{}^{*} G_j = G_1 * \ldots * G_n,$$

where $*$ is the sign of the convolution operation.

Now we need some lemmas. After the proof of Lemma 1 we sketch the general proof of convergence of $(\sigma_n, \xi_{(n,1)})$ for $n = 1, 2, \ldots$

LEMMA 1. *For all continuity points x of N the relation*

$$\sum_{j=1}^{k_n} F_{nj|x} \to \sigma(a, N^x)$$

holds.

Proof. Let x be a continuity point of N. We suppose that $F_{nj}(x) > 0$, which is possible, because of (A), for sufficiently large n. Write

$$\varphi_{nj|x}(t) = \int_0^\infty e^{-ts} F_{nj|x}(ds) = \int_0^x e^{-ts} F_{nj}(ds)/F_{nj}(x),$$

$$\varphi(t) = \lim_{n\to\infty} \varphi_{nj}(t).$$

We show that

$$\lim_{n\to\infty} \prod_{j=1}^{k_n} \Big(1 - \int_x^\infty e^{-ts} F_{nj}(ds)/\varphi_{nj}(t)\Big) = \exp\Big[-\int_x^\infty e^{-ts} N(ds)\Big]$$

which, combined with (see [6])

$$\prod_{j=1}^{k_n} F_{nj}(x) \to e^{N(x)} \quad \text{as } n \to \infty,$$

yields

$$(4) \quad \lim_{n\to\infty} \prod_{j=1}^{k_n} \varphi_{nj|x}(t)$$
$$= \lim_{n\to\infty} \Big(\prod_{j=1}^{k_n} \varphi_{nj}(t) / \prod_{j=1}^{k_n} F_{nj}(t)\Big) \prod_{j=1}^{k_n} \Big(1 - \int_x^\infty e^{-ts} F_{nj}(ds)/\varphi_{nj}(t)\Big)$$
$$= \varphi(t) \exp\Big[N(x) - \int_x^\infty e^{-ts} N(ds)\Big].$$

To prove (4) we show that if for an array a_{nj} ($j = 1, \ldots, k_n$; $n = 1, 2, \ldots$) of positive numbers the equalities

$$(5) \quad \lim_{n\to\infty} \max_{1\leqslant j\leqslant k_n} a_{nj} = 0,$$

$$(6) \quad \lim_{n\to\infty} \sum_{j=1}^{k_n} a_{nj} = a$$

hold, then

$$\lim_{n\to\infty} \prod_{j=1}^{k_n} (1 + a_{nj}) = e^a.$$

The simple proof of this statement goes by applying the inequality

$$|\lg(1-x)+x| \leqslant x^2, \qquad |x| < \frac{1}{2},$$

to the relation

$$a_{nj} = \int_x^\infty e^{-ts} F_{nj}(ds)/\varphi_{nj}(t).$$

Now, formula (5) follows from (A) (and A′), since

$$\max_{1\leqslant j\leqslant k_n} |a_{nj}| \leqslant \max_{1\leqslant j\leqslant k_n} \big(1-F_{nj}(x)\big)/\min_{1\leqslant j\leqslant k_n} |\varphi_{nj}(t)|,$$

and formula (6) is implied by (A′) and (B) (bearing in mind Helly's theorem), since

$$\Big|\sum_{j=1}^{k_n} \int_x^\infty e^{-ts} F_{nj}(ds)/\varphi_{nj}(t) - \int_x^\infty e^{-ts} N(ds)\Big|$$
$$= \Big|\sum_{j=1}^{k_n} \int_x^\infty e^{-ts} F_{nj}(ds) \sum_{l=0}^{\infty} \big(\varphi_{nj}(t)-1\big)^l - \int_x^\infty e^{-ts} N(ds)\Big|$$
$$\leqslant \Big|\sum_{j=1}^{k_n} \int_x^\infty e^{-ts} F_{nj}(ds) - \int_x^\infty e^{-ts} N(ds)\Big| +$$
$$+\sum_{j=1}^{k_n} \big(1-F_{nj}(x)\big) \max_{1\leqslant j\leqslant k_n} \big(|\varphi_{nj}(t)-1|/|\varphi_{nj}(t)|\big).$$

This completes the proof of the lemma.

From Lemma 1 we obtain immediately the convergence of

$$\Big(\sum_{j=1}^{k_n} \xi_{nj}, \xi_{(n,1)}\Big)$$

if we notice that

$$\Pr\Big(\sum_{j=1}^{k_n} \xi_{nj} \leqslant x, \xi_{(n,1)} \leqslant y\Big)$$
$$= \Pr\Big(\sum_{j=1}^{k_n} \xi_{nj} \leqslant x \mid \xi_{(n,1)} \leqslant y\Big) \Pr(\xi_{(n,1)} \leqslant y)$$
$$= \Pr\Big(\sum_{j=1}^{k_n} \eta_{nj} \leqslant x\Big) \Pr(\xi_{(n,1)} \leqslant y) \qquad (n = 1, 2, \ldots),$$

where η_{nj} ($j = 1, \ldots, k_n$; $n = 1, 2, \ldots$) are independent with $\Pr(\eta_{nj} \leqslant x) = F_{nj|y}(x)$.

To prove the convergence of

$$\Big(\sum_{j=1}^{k_n} \xi_{nj},\ \xi_{(n,1)},\ \ldots,\ \xi_{(n,m)}\Big)$$

we need more complicated arguments. For the sake of clarity we assume hereafter that $k_n = n$, $F_{nj} = F_n$ $(j = 1, \ldots, n)$.

LEMMA 2. *If* $0 < x < y$ *are continuity points of* N, *then*

$$F_{n|xy}(t) \to N_{|xy}(t) \quad \text{as } n \to \infty.$$

Proof. We have for $0 < x \leqslant t < y$

$$\lim_{n\to\infty} F_{n|xy}(t) = \frac{n\big(F_n(t)-1\big) - n\big(F_n(x)-1\big)}{n\big(F_n(y)-1\big) - n\big(F_n(x)-1\big)} = \frac{N(t)-N(x)}{N(y)-N(x)}.$$

LEMMA 3. *If* $v_1 + \ldots + v_l = k$, *then*

$$\lim_{n\to\infty} \frac{n!}{(n-k)!\, n^{v_1} \ldots n^{v_l}} = 1.$$

Some notations are needed. Fix a natural number m and let

$$\infty = y_0 > y_1 > \ldots > y_m > y_{m+1} = 0.$$

Denote by Y the set of all non-decreasing functions y from $\{1, \ldots, m\}$ into $\{y_1, \ldots, y_m\}$ such that $y(i) \leqslant y_i$ $(i = 1, \ldots, m)$. Consider a function $y \in Y$. Suppose that it assumes $l+1$ values $\bar{y}(1), \ldots, \bar{y}(l+1)$ such that

$$\bar{y}(1) > \ldots > \bar{y}(l+1) > \bar{y}(l+2) = 0.$$

The value y_i is assumed v_i times $(i = 1, \ldots, l+1)$. For convenience we put $v_0 = 0$. Notice that $\bar{y}(l+1) = y_m$. Denote by k the greatest i such that $y(i) > y_m$; namely, $k = v_1 + \ldots + v_l$.

Now we are going to show that, for any m, $y_1 > \ldots > y_m > 0$, where each y_j $(j = 1, \ldots, m)$ is a continuity point of N, and that, for any x except of points from a countable set, the sequence

$$\Pr\Big(\sum_{j=1}^{k_n} \xi_{nj} \leqslant x,\ \xi_{(n,1)} \leqslant y_1, \ldots, \xi_{(n,m)} \leqslant y_m\Big) \quad (n = 1, 2, \ldots)$$

converges to a limit. This is sufficient for proving Theorem 1 due to the tightness of the sequence $(\sigma_n, X_{(n)})$, since the sequences σ_n and $X_{(n)}$ are tight.

We have

$$\Pr\Big(\sum_{j=1}^{k_n} \xi_{nj} \leqslant x,\ \xi_{(n,1)} \leqslant y_1, \ldots, \xi_{(n,m)} \leqslant y_m\Big)$$

$$= \sum_{y\in Y} \Pr\Big(\sum_{j=1}^{n} \xi_{nj} \leqslant x,\ y(2) < \xi_{(n,1)} \leqslant y(1), \ldots, y(m) < \xi_{(n,m-1)} \leqslant y(m-1),$$

$$0 \leqslant \xi_{(n,m)} \leqslant y(m)\Big).$$

Each component of the last sum is equal to (remind that l and v_i $(i = 1, \ldots, l+1)$ depend on $y \in Y$)

$$\Pr\Big(\sum_{j=1}^{n} \xi_{nj} \leqslant x,\, y(2) < \xi_{(n,1)} \leqslant y(1), \ldots, y(m) < \xi_{(n,m-1)} \leqslant y(m-1),$$

$$0 \leqslant \xi_{(n,m)} \leqslant y(m)\Big)$$

$$= \frac{n!}{v_1! \ldots v_l!(n-k)!} \Pr\Big(\sum_{j=1}^{n} \xi_{nj} \leqslant x,\, \bigcap_{i=1}^{l} \bigcap_{j=v_{i-1}+1}^{v_i} \{\bar{y}(i+1) < \xi_{nj} \leqslant \bar{y}(i)\},$$

$$\bigcap_{j=l+1}^{n} \{\xi_{nj} \leqslant y_m\}\Big)$$

$$= \frac{n!}{v_1! \ldots v_l!(n-k)!} \Pr\Big(\sum_{j=1}^{n} \xi_{nj} \leqslant x \mid \bigcap_{i=1}^{l} \bigcap_{j=v_{i-1}+1}^{v_i} \{\bar{y}(i+1) < \xi_{nj} \leqslant \bar{y}(i)\},$$

$$\bigcap_{j=k+1}^{n} \{\xi_{nj} \leqslant y_m\}\Big) \Pr\Big(\bigcap_{i=1}^{l} \bigcap_{j=v_{i-1}+1}^{v_i} \{\bar{y}(i+1) < \xi_{nj} \leqslant \bar{y}(i)\} \bigcap_{j=k+1}^{n} \{\xi_{nj} \leqslant y_m\}\Big)$$

$$= \frac{n!}{v_1! \ldots v_l!(n-k)!} \Big(\prod_{j=1}^{l}{}^{*} F_{n|\bar{y}(j+1)\bar{y}(j)}\Big) * F_{n|y_m}^{*(n-k)}(x) \times$$

$$\times \prod_{j=1}^{l} \Big(F_n\big(\bar{y}(j+1)\big) - F_n\big(\bar{y}(j)\big)\Big)^{v_j} F_n^{n-k}(y_m).$$

Applying Lemmas 2 and 3 we get

$$\text{(7)} \quad \lim_{n\to\infty} \sum_{y\in Y} \Pr\Big(\sum_{j=1}^{n} \xi_{nj} \leqslant x,\, \bar{y}(2) < \xi_{(n,1)} \leqslant \bar{y}(1), \ldots,$$

$$\bar{y}(m) < \xi_{(n,m-1)} \leqslant \bar{y}(m-1),\, \xi_{(n,m)} \leqslant \bar{y}(m)\Big)$$

$$= \sum_{y\in Y} \frac{1}{v_1! \ldots v_l!} \Big(\prod_{j=1}^{l}{}^{*} N_{|\bar{y}(j+1)\bar{y}(j)}\Big) * \sigma(a, N^{y_m})(x) \times$$

$$\times \prod_{j=1}^{l} \Big(N\big(\bar{y}(j+1)\big) - N\big(\bar{y}(j)\big)\Big)^{v_j} \exp[N(y_m)].$$

Substituting $x = \infty$ in (7) we obtain

$$\lim_{n\to\infty} \Pr(\xi_{(n,1)} \leqslant x_1, \ldots, \xi_{(n,m)} \leqslant x_m)$$

$$= \sum_{y\in Y} \frac{1}{v_1! \ldots v_l!} \prod_{j=1}^{l} \Big(N\big(\bar{y}(j+1)\big) - \bar{y}(j)\Big)^{v_j} \exp[N(y_m)],$$

which shows that points from $X_{(\infty)}$ form a Poisson process with the intensity measure equal to N.

Proof of Corollary 1. Following the part of the proof from the Appendix of [4], one can show that

$$f\colon \bar{V}_M \ni (x_1, x_2, \ldots) \to \Big(\sum_{j=1}^{\infty} (x_j)^2, \ldots, \sum_{j=1}^{\infty} (x_j)^l\Big) \in R^{l-1}$$

is a continuous function in the set $\bar{V}_m$ of sequences

$$x_1 \geqslant x_2 \geqslant \ldots \geqslant 0, \qquad \sum_{j=1}^{\infty} x_j \leqslant M,$$

which together with the convergence of $\sum_{j=1}^{n} \xi_{nj}$, by Theorem 5.2 from [1], gives

$$\Big(\sum_{j=1}^{k_n} \xi_{nj}, f(X_{(n)})\Big) \to \big(\sigma, f(X_{(\infty)})\big).$$

Proof of Theorem 2. From Theorem 1 we obtain immediately

$$X_{(n)}/\sigma_n \to \pi = (\pi_1, \pi_2, \ldots),$$

where $\to$ denotes convergence in distribution on $\bar{V}_1$. Thus to prove Theorem 2 it suffices to show that $P_N(V) = 1$ or, equivalently, that

$$\sum_{j=1}^{\infty} \pi_j = 1$$

with probability 1. Fatou's lemma asserts that

$$\sigma \geqslant \sum_{j=1}^{\infty} \xi_{(\infty,j)} \tag{8}$$

with probability 1. Since points from $X_{(\infty)}$ form a Poisson process with intensity measure equal to N, we obtain

$$\mathrm{E}\exp\Big[-t\sum_{j=1}^{\infty} \xi_{(\infty,j)}\Big] = \exp\Big[-\int_0^{\infty} (1-e^{-tx})N(dx)\Big].$$

So we infer that σ is identically distributed as $\sum_{j=1}^{\infty} \xi_{(\infty,j)}$, which by (8) yields

$$1 = \sum_{j=1}^{\infty} (\xi_{(\infty,j)}/\sigma) = \sum_{j=1}^{\infty} \pi_j$$

with probability 1.

References

[1] P. Billingsley, *Convergence of probability measures*, J. Wiley, New York 1968.
[2] O. Kallenberg, *Characterization and convergence of random measures and point processes*, Z. Wahrscheinlichkeitstheorie Verw. Geb. 27 (1973), p. 9-21.
[3] — *On symmetrically distributed random measures*, Trans. Amer. Math. Soc. 202 (1975), p. 105-121.
[4] J. F. C. Kingman, *Random discrete distributions*, J. R. Statist. Soc. B 37 (1975), p. 1-22.
[5] — *The population structure associated with the Ewens sampling formula*, Theor. Pop. Biol. 6 (1977), p. 274-283.
[6] M. Loève, *Ranking limit problem*, Proc. Third Berkeley Symp. 2 (1956), p. 177-194.
[7] G. A. Watterson, *The stationary distribution of the infinitely many neutral alleles diffusion model*, J. Appl. Probability 13 (1976), p. 639-651.

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O LOSOWYCH ROZKŁADACH DYSKRETNYCH

STRESZCZENIE

Kingman [4] sformułował twierdzenie o zbieżności pewnej klasy losowych rozkładów dyskretnych do rozkładów Poissona-Dirichleta. W pracy zauważono, że twierdzenie to jest słuszne dla szerszej klasy losowych rozkładów dyskretnych. Do dowodu użyto twierdzenia Loève'a z [6] o łącznej zbieżności sum i statystyk pozycyjnych wierszy pewnej macierzy trójkątnej, niezależnych w wierszach zmiennych losowych. Podany jest nowy dowód twierdzenia Loève'a, pozwalający opuścić założenie ciągłości dystrybuant zmiennych losowych.