ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XVII, 1 (1980)

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A NOTE ON A TWO-PERSON ZERO-SUM GAME STIMULATED BY A MARKOV CHAIN

1. Description of the game. Consider a Markov chain $\{x_n\}_{n \in N}$, $N = \{0, 1, \ldots\}$, where, for every $n \in N$, x_n equals either 1 or 0. The transition probabilities are given as follows:

$$\begin{aligned} &P\{x_{n+1} = 1 \mid x_n = 1\} = p_n, \quad &P\{x_{n+1} = 0 \mid x_n = 1\} = q_n, \\ &P\{x_{n+1} = 1 \mid x_n = 0\} = 0, \quad &P\{x_{n+1} = 0 \mid x_n = 0\} = 1. \end{aligned} \tag{1}$$

We also assume that $P\{x_0 = 1\} = 1$, $p_n > 0$ and $p_n + q_n = 1$, $n \in N$. Let us assume that players A and B observe a realization of the chain and each of them may stop his observation at any moment of time $n \in N$. Each of the players tries to continue the observation longer than his opponent but both players are interested in stopping the observation before the event $\bigcup_{n=0}^{\infty} \{x_n = 0\}$ occurs. No player has any information about his opponent's behaviour.

Now, let us define the pay-off for the game. Let $m, n \in N$ be the moments of time at which players A and B, respectively, stop their observations. If $m < n$, then player A wins a unit value $+1$ in case $x_m = 1$ and $x_n = 0$, and A loses the value when $x_n = 1$. If $m = n$, then the pay-off equals zero for both players. If $m > n$, then A loses the value $+1$ provided $x_n = 1$ and $x_m = 0$, and in case $x_m = 1$ this player wins $+1$. Here, we assume that the game is a zero-sum one and the pay-off for B can easily be found. The expectation of the pay-off will result in the pay-off function for the game.

Taking into account the description of the game we see that a two-person zero-sum game $\Gamma = \langle N, N, K \rangle$ is defined, where $N = \{0, 1, \ldots\}$ and, for $m, n \in N$,

$$K(m, n) = \begin{cases} a_m - 2a_n & \text{if } m < n, \\ 0 & \text{if } m = n, \\ 2a_m - a_n & \text{if } m > n \end{cases} \tag{2}$$

with $a_0 = 1$ and $a_k = p_{k-1}a_{k-1}$ for $k \geqslant 1$. Here we assume that $p_0 > 0.5$, for otherwise both players would have optimal strategies $m = n = 0$.

In Section 3 we give a practical interpretation of the game and some examples of numerical solutions.

2. Optimal strategies in $\Gamma = \langle N, N, K\rangle$. Since the game is symmetric, we shall consider only the space of mixed strategies for player A. The same can be done for the second player. In the space $\mathscr{X}$ of all mixed strategies for player A we define a subspace $\mathscr{X}_l$ in the following way:

$$\bar{x} \in \mathscr{X}_l \quad \text{if } \bar{x} = (x_0, x_1, \ldots, x_l, 0, 0, \ldots), \qquad x_i \geqslant 0,\ i = 0, 1, \ldots, l,\ \sum_{i=0}^{l} x_i = 1, \tag{3}$$

where x_i is the probability that A stops the observation at the moment $i \in N$.

We shall prove in the sequel that under some assumptions concerning the transition probabilities there exists an integer l such that the optimal mixed strategies in Γ belong to $\mathscr{X}_l$. Notice that the value for the game must be zero.

At first, we use the necessary optimality condition for a strategy $\bar{x} \in \mathscr{X}_l$. An optimal mixed strategy must be an *equalizer strategy*, i.e.,

$$K(\bar{x}, s) = \sum_{i=0}^{l} K(i, s)x_i = 0 \quad \text{for } s = 0, 1, \ldots, l. \tag{4}$$

Hence, by equality (2), we obtain the following system of equations:

$$\begin{aligned} K(\bar{x}, 0) &= \sum_{i=1}^{l} (2a_i - 1)x_i = 0, \\ K(\bar{x}, s) &= \sum_{i=0}^{s-1} (a_i - 2a_s)x_i + \sum_{i=s+1}^{l} (2a_i - a_s)x_i = 0 \\ &\qquad\qquad \text{for } s = 1, 2, \ldots, l-1, \\ K(\bar{x}, l) &= \sum_{i=0}^{l-1} (a_i - 2a_l)x_i = 0. \end{aligned} \tag{5}$$

The determinant of system (5) is skew-symmetric. Thus, if we put $l = 2m$, the determinant is zero and the existence of a non-trivial solution of system (5) is proved.

Now, we shall find the solution of system (5). Let us set

$$\xi_i = \sum_{j=0}^{i} x_j \quad \text{for } i = 0, 1, \ldots, 2m.$$

We easily notice that system (5) is equivalent to the following set of equations:

$$(5') \qquad K(\bar{x}, s) - K(\bar{x}, s+1) = a_s(p_s \xi_{s+1} - \xi_{s-1} + p_s - 1) = 0 \quad \text{for } s = 1, 2, \ldots, 2m-1,$$

$$(5'') \qquad K(\bar{x}, 0) = K(\bar{x}, 1) = 0.$$

Now, taking into account equations (5′) and putting

$$\eta_s = 1 + \xi_s,$$

we obtain

$$(6) \qquad p_s \eta_{s+1} - \eta_{s-1} = 0 \quad \text{for } s = 1, 2, \ldots, 2m-1.$$

Equations (6) have the solution

$$(7) \qquad \eta_k = \begin{cases} \eta_0 \prod_{s=1}^{r} \dfrac{1}{p_{2s-1}} & \text{if } k = 2r,\ r = 1, 2, \ldots, m, \\ \eta_1 \prod_{s=1}^{r} \dfrac{1}{p_{2s}} & \text{if } k = 2r+1,\ r = 1, 2, \ldots, m-1, \end{cases}$$

where constants η_0 and η_1 are to be determined. In that order we put

$$a = \sum_{i=0}^{2m} a_i x_i.$$

Thus (5″) yields

$$K(\bar{x}, 0) = 2a - \eta_0 = 0, \qquad K(\bar{x}, 1) = 2a + 1 - \eta_0 - p_0 \eta_1 = 0.$$

Hence we easily obtain $\eta_1 = p_0^{-1}$.

Now, using the normalizing condition $\xi_{2m} = 1$ (or $\eta_{2m} = 2$) as well as formula (7), we obtain

$$\eta_0 = 2 \prod_{s=1}^{m} p_{2s-1} \quad \text{and} \quad a = \prod_{s=1}^{m} p_{2s-1}.$$

Hence we have the solution of (6):

$$(7') \qquad \eta_k = \begin{cases} 2 & \text{if } k = 2m, \\ 2 \prod_{s=r+1}^{m} p_{2s-1} & \text{if } k = 2r,\ r = 0, 1, \ldots, m-1, \\ \prod_{s=0}^{r} \dfrac{1}{p_{2s}} & \text{if } k = 2r+1,\ r = 0, 1, \ldots, m-1. \end{cases}$$

Now, we can easily find the components of $\bar{x}$ as defined in (3):

$$x_0 = \eta_0 - 1,$$

$$x_{2r} = \eta_{2r} - \eta_{2r-1} \quad \text{for } r = 1, 2, \ldots, m,$$

$$x_{2r+1} = \eta_{2r+1} - \eta_{2r} \quad \text{for } r = 0, 1, \ldots, m-1,$$

or, applying (7′),

$$(8)\qquad \begin{aligned} x_0 &= 2\prod_{s=1}^{m} p_{2s-1} - 1, \\ x_{2r} &= 2\prod_{s=r+1}^{m} p_{2s-1} - \prod_{s=0}^{r-1} \frac{1}{p_{2s}} \quad \text{for } r = 1, 2, \ldots, m-1, \\ x_{2r+1} &= \prod_{s=0}^{r} \frac{1}{p_{2s}} - 2\prod_{s=r+1}^{m} p_{2s-1} \quad \text{for } r = 0, 1, \ldots, m-1, \\ x_{2m} &= 2 - \prod_{s=0}^{m-1} \frac{1}{p_{2s}}. \end{aligned}$$

For any arbitrary sequence $\{p_n\}$ it may happen that $\bar{x}$ with components defined in (8) does not belong to $\mathscr{X}_{2m}$. Thus, we should require that

(∗) there exists an $m \geqslant 1$ such that

$$p_0 \prod_{s=1}^{m} p_{2s-1} \leqslant 0.5 \leqslant \prod_{s=1}^{m} p_{2s-1},$$

$$\prod_{s=0}^{r} p_{2s} \prod_{s=r+1}^{m} p_{2s-1} \leqslant 0.5 \leqslant \prod_{s=0}^{r-1} p_{2s} \prod_{s=r+1}^{m} p_{2s-1} \quad \text{for } r = 1, 2, \ldots, m-1,$$

$$0.5 \leqslant \prod_{s=0}^{m-1} p_{2s},$$

and at least one of the inequalities is sharp.

The inequalities in (∗) correspond to the following set of inequalities:

$$x_i \geqslant 0 \quad \text{for } i = 0, 1, \ldots, 2m.$$

For example, if $i = 0, 1$, then

$$2\prod_{s=1}^{m} p_{2s-1} - 1 \geqslant 0, \qquad \frac{1}{p_0} - 2\prod_{s=1}^{m} p_{2s-1} \geqslant 0.$$

Let us denote by C the set of all positive integers for which condition (∗) is satisfied for a given sequence $\{p_n\}$.

Remarks. 1. For some sequences of probabilities $\{p_n\}$ other specific structure for the mixed strategy space should be assumed.

2. For a constant sequence $p_n = p$ condition $(*)$ reduces to the following:

there exists an $m \geqslant 1$ such that $p^{m+1} \leqslant 0.5 \leqslant p^m$.

Now, we shall study the sufficient optimality condition for a strategy in $\mathscr{X}_{2m}$. It states that for every $s \in N$

$$K(\bar{x}, s) \geqslant 0, \quad \bar{x} \in \mathscr{X}_{2m}.$$

We have already known that $K(\bar{x}, s) = 0$ for $s = 0, 1, \ldots, 2m$. Further, for $s \geqslant 2m+1$, we obtain

$$K(\bar{x}, s) = a - 2a_s = \prod_{s=1}^{m} p_{2s-1}\Big(1 - 2\prod_{s=0}^{m} p_{2s}p_{2m+1} \ldots p_{s-1}\Big) \tag{9}$$
$$\geqslant \prod_{s=1}^{m} p_{2s-1}\Big(1 - 2\prod_{s=0}^{m} p_{2s}\Big).$$

Now, we shall consider two cases.

(i) At first, let us assume that there is no integer m in C for which

$$\prod_{s=0}^{m-1} p_{2s} = 0.5.$$

Next, we define an integer $m' \in C$ such that

$$\prod_{s=0}^{m'} p_{2s} \leqslant 0.5 < \prod_{s=0}^{m'-1} p_{2s}. \tag{10}$$

Using (9) we see that if m' in (10) exists, then, for $\bar{x} \in \mathscr{X}_{2m'}$ described by (8),

$$K(\bar{x}, s) = 0 \quad \text{if } s = 0, 1, \ldots, 2m',$$
$$K(\bar{x}, s) > 0 \quad \text{if } s \geqslant 2m'+1.$$

Thus $\bar{x} \in \mathscr{X}_{2m'}$ is optimal.

(ii) It may happen that there exists an integer m'' in C such that

$$\prod_{s=0}^{m''-1} p_{2s} = 0.5. \tag{11}$$

In this case, by (9), we have

$$K(\bar{x}, s) = 0 \quad \text{if } s = 0, 1, \ldots, 2m'',$$
$$K(\bar{x}, s) \geqslant 0 \quad \text{if } s \geqslant 2m''+1,$$

and $\bar{x} \in \mathscr{X}_{2m''}$ is optimal. Here we also have

$$\prod_{s=0}^{m''-1} p_{2s} = 0.5 < \prod_{s=0}^{m''-2} p_{2s},$$

and if $m''-1$ also belongs to C, then the strategy $\bar{x} \in \mathscr{X}_{2(m''-1)}$ is also optimal. Thus we can take

$$\bar{x} = a\bar{x}_1 + (1-a)\bar{x}_2, \tag{12}$$

where $\bar{x}_1 \in \mathscr{X}_{2m''}$ and $\bar{x}_2 \in \mathscr{X}_{2(m''-1)}$; and we easily notice that, for every $a \in [0, 1]$, $\bar{x}$ defined by (12) is optimal.

Remark 3. Taking into account Remark 2 we see that, for a constant sequence $p_n = p$, m' is defined by

$$p^{m'+1} < 0.5 < p^{m'},$$

and if there exists an integer m'' such that $p^{m''} = 0.5$, then

$$p^{m''+1} < 0.5 = p^{m''}, \quad p^{m''} = 0.5 < p^{m''-1},$$

and we have the situation described in case (ii) for $\bar{x}_1 \in \mathscr{X}_{2m''}$ and $\bar{x}_2 \in \mathscr{X}_{2(m''-1)}$.

Thus we have proved the following

THEOREM. *If the transition probabilities in* (1) *of the Markov chain* $\{x_n\}_{n\in N}$ *satisfy condition* $(*)$ *for some integers* $m \in N$, *then both players have a unique optimal strategy in* $\mathscr{X}_{2m'}$ *provided case* (i) *occurs, and if case* (ii) *takes place, then their optimal strategies in the game stimulated by* $\{x_n\}_{n\in N}$ *are linear convex combinations of strategies in* $\mathscr{X}_{2m''}$ *and* $\mathscr{X}_{2(m''-1)}$.

3. Some numerical examples. One can find more or less realistic models of the game considered in Section 1. Here we consider two real situations which can be modelled according to the rules of the game.

Assume that two partners use the same energy supply, and a random cut in the energy supply may occur according to the behaviour of the Markov chain $\{x_n\}_{n\in N}$. Each of the opponents may stop using the energy at any moment of time provided the cut did not occur. The one who had used the energy for a longer period of time and had stopped before the cut occurred wins the competition.

Now, let two research centers work on a project of certain device. They are supported by the same bank and the support can be cut at any moment of time $n \in N$ with probability p_n given in (1). The one which continues the research for a longer period of time before the cut appears wins the game.

Let us consider some numerical examples of the game.

Example 1. Let us take $p_n = p$, $n \in N$, for $p = 0.9$. By Remark 2 we observe that for $m = 6$ we have $p^{m'+1} < 0.5 < p^{m'}$ and the components of the optimal strategy can be evaluated according to formula (8). The results are given in Table 1.

TABLE 1

i	x_i	i	x_i
0	0.0628	6	0.0862
1	0.0482	7	0.0661
2	0.0699	8	0.0958
3	0.0536	9	0.0735
4	0.0776	10	0.1065
5	0.0595	11	0.0816
		12	0.1187

Example 2. Here we assume that $p_n = p^{n+1}$, $n \in N$, for $p = 0.9$. Then condition $(*)$ requires that there exists an $m \geqslant 1$ such that

$$p^{m(m+1)+1} \leqslant 0.5 \leqslant p^{m(m+1)},$$

$$p^{m^2+2r+1} \leqslant 0.5 \leqslant p^{m^2} \quad \text{for } r = 1, 2, \ldots, m-1, m.$$

We easily notice that the condition is satisfied for $m' = 2$. Hence, by formula (6), we have

$$\eta_0 = 1.0628, \quad \eta_1 = 1.1111, \quad \eta_2 = 1.3222, \quad \eta_3 = 1.5241, \quad \eta_4 = 2,$$

and the components of the optimal strategy are as follows:

$$x_0 = 0.0628, \quad x_1 = 0.0482, \quad x_2 = 0.2011,$$
$$x_3 = 0.2119, \quad x_4 = 0.4760.$$

Example 3. Now, let $p_0 = 0.8$, $p_1 = 0.6$, $p_2 = 0.625$ and $p_n = 0.9$ for $n \geqslant 3$. We easily notice that $p_0 p_2 = 0.5$ and case (ii) considered in Section 2 occurs. For $m'' = 2$ we have

$$p_0 p_1 p_3 < 0.5 < p_1 p_3, \quad p_0 p_2 p_3 < 0.5 < p_0 p_2,$$

$$p_0 p_2 p_4 < 0.5 = p_0 p_2,$$

and for $m''-1$ we obtain

$$p_0 p_1 < 0.5 < p_1, \quad p_0 p_2 = 0.5 < p_0.$$

Thus, we find that for $m = m'' = 2$

$$\eta_0 = 1.08, \quad \eta_1 = 1.25, \quad \eta_2 = 1.8, \quad \eta_3 = 2, \quad \eta_4 = 2,$$

and for $m''-1 = 1$

$$\eta_0 = 1.2, \quad \eta_1 = 1.25, \quad \eta_2 = 2.$$

Hence the optimal strategy in that case is a convex linear combination of the strategies given in Table 2.

TABLE 2

$m = 1$		$m = 2$	
i	x_i	i	x_i
0	0.20	0	0.08
1	0.05	1	0.17
2	0.75	2	0.55
		3	0.20

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Received on 25. 7. 1977;
revised version on 23. 3. 1978

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UWAGI O GRZE DWUOSOBOWEJ O SUMIE ZERO, ZWIĄZANEJ Z PEWNYM ŁAŃCUCHEM MARKOWA

STRESZCZENIE

W pracy rozważa się grę dwuosobową o sumie zerowej, związaną z dwustanowym łańcuchem Markowa, gdzie jeden ze stanów jest pochłaniający. Gracze obserwują oddzielnie realizację tego łańcucha i mogą w każdej chwili podjąć decyzję o przerwaniu obserwacji przy założeniu, że wcześniej nie nastąpiło pochłonięcie. Wygrywa ten, który dłużej prowadził obserwację, lecz zatrzymał się przed pochłonięciem. Znaleziono optymalne strategie graczy w pewnej podprzestrzeni strategii mieszanych w tej grze.