By the way, the same argument also shows that the spectrum is bounded. The complex $(\Lambda_n(E/F), \delta_{a-s})$ is exact if and only if $(\Lambda_n(E/F), \delta_{\alpha(s)})$ is exact where

$$\alpha(s) = \frac{a-s}{(1+|s|^2)^{1/2}}.$$

When s is large, $\alpha(s)$ is a perturbation in the sense of Proposition 5 of $s_0 = s/|s|$, and $\delta_{\alpha(s)}$ is a perturbation of $\delta_{s/|s|}$. We know that $s/|s|$ is a non-zero system of scalars, is therefore regular. The complex $(\Lambda_n(E/F), \delta_{a-s})$ is therefore exact.

PROPOSITION 7. *Assume that $(a_1, \ldots, a_n)$ is a regular system of strict commuting endomorphisms of E/F, a_i induced by $a_i^1\colon E \to E$. Let $b_1, \ldots, b_n$ be new commuting endomorphisms of E/F, induced by b_i^1, $E \to E$. Assume that*

$$\|a_i^1 - b_i^1\|_{\mathscr{L}(E,E)} < \varepsilon, \quad \|a_i^1 - b_i^1\|_{\mathscr{L}(F,F)} < \varepsilon, \quad \|a_i^1 a_j^1 - a_j^1 a_i^1 + b_j^1 b_i^1 - b_i^1 b_j^1\|_{\mathscr{L}(E,F)} < \varepsilon,$$

with ε small. Then $(b_1, \ldots, b_n)$ is regular.

For the proof to go through, at least, we must assume not only that $(b_1, \ldots, b_n)$ is near to $(a_1, \ldots, a_n)$ in the Banach algebra of linear transformations of E which leave F invariant. We must also assume that the commutation $[a_i, a_j]$ are near to $[b_i, b_j]$ in the two-sided ideal of linear mappings of E into F.

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PROPERTIES OF THE SPECTRAL RADIUS IN BANACH ALGEBRAS

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CONTENTS

Introduction

In this article we treat a number of topics, as indicated in the table of contents, on the interrelation between the algebraic properties of the spectral radius and those of the algebra. Originally these investigations were motivated by the well-known contrast between commutative and non-commutative Banach algebras. For example, spectra of elements behave much better in commutative algebras than in non-commutative algebras. Therefore we tried to explain to what extent spectral properties (nice in some reasonable sense) of elements in the algebra can effect commutativity, the essence of the Gelfand theory. It emerged that the key to this mystery is contained just in the notion of the spectral radius. It is indeed interesting to find if the properties of the spectral radius can give information about the properties of the whole spectrum and, moreover, about the structure of the whole algebra. Of course, we obtained these results first for Banach algebras over the complex field because some of the crucial steps were based on complex analytic tools like the Cauchy integral formula and the Beurling–Gelfand formula for the spectral radius. The work culminated in the dissertation [25].

Here we provide another approach, also simple, which is more algebraic and avoids the preceding analytic techniques. It consists in a more ingenious applica-

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tion of the classical Jacobson density theorem with the recent improvement for Banach algebras by A. M. Sinclair [18], p. 36. This makes it possible to extend the results to, say, real Banach algebras.

Thus the non-emptiness of the spectrum and the particularly simple form of the Jacobson density theorem for Banach algebras, which we owe, of course, to the characterization of normed division algebras by the Frobenius and Mazur–Gelfand theorems, are the only essential facts from the Banach algebra theory on which we rely. It is curious that the Beurling–Gelfand formula will never be used in this paper. In this way the actual part of the algebraic and analytic features of the subject becomes more obvious. In the last section we have found a remarkable connection with the Kaplansky theorem on locally algebraic operators [10], p. 41. The significance of the concept of completeness in the sense of Banach will be demonstrated especially in the characterization of the Jacobson radical by quasi-nilpotent perturbations, cf. the proof of Theorem 3.3. In reading the text, a number of open questions may arise, but we point out explicitly only some of them.

As we have already mentioned, the results of the present paper are true essentially for both real and complex Banach algebras. Certainly, a complex algebra can be considered as a real algebra, but we prefer to give first the proofs in the complex case because they are often more transparent and easier. So in the first five sections the algebras will be tacitly assumed to be over the complex field while the corresponding modifications for the real case will be suggested in the last section.

Also the presence of a unit in the algebra will not be assumed in general. In any case, whenever an algebra without unit may occur, the situation can easily be adapted to the unitization, the irreducible representations extended to it, etc.

The results of the first four sections formed a part of the thesis [25] and also the last two sections are closely related to it. I would like to record here my gratitude to the Institute of Mathematics of the Polish Academy of Sciences and, in particular, to Professor Wiesław Żelazko for all their help and encouragement. Some results were also obtained during my stay at the Stefan Banach International Mathematical Center in Warszawa in the Autumn of 1977.

1. The main theorem

We shall be considering throughout an arbitrary Banach algebra A, mostly over the complex field. For x in A we denote by $\sigma(x)$ the spectrum and by $|x|_\sigma$ the spectral radius of the element x. If the algebra A does not have a unit, the spectrum is to be understood with respect to the unitization A_1. For real Banach algebras the spectrum is defined with respect to the complexification [4], p. 70. The Jacobson radical of the algebra A will be abbreviated to $\operatorname{rad} A$; recall that it is defined as the intersection of the primitive ideals, i.e. the kernels of the irreducible representations.

For every x in A put

$$s(x) = \sup |x - u^{-1}xu|_\sigma,$$

where u runs over the invertible elements in A (or, in A_1 if A does not have a unit).

The main result can simply be expressed just in terms of this function. In this context, there will appear naturally also the centre modulo the radical, i.e. the set

$$Z(A) = \{x \in A:\ ax - xa \in \operatorname{rad} A \text{ for all } a \in A\}.$$

Clearly, $Z(A)$ is a closed subalgebra containing $\operatorname{rad} A$. For semi-simple algebras this concept coincides with the usual centre.

Before stating the main theorem let us quote a proposition which will be crucial in the proof.

PROPOSITION 1.1. *Let A be a complex Banach algebra and let π be an irreducible representation of A on a complex linear space X. If $\xi_1, \ldots, \xi_n$ are linearly independent in X and if $\eta_1, \ldots, \eta_n$ are arbitrary in X, then there exists an element a in A such that $\pi(a)\xi_i = \eta_i$ for $i = 1, \ldots, n$. Moreover, if $\eta_1, \ldots, \eta_n$ are also linearly independent, then the element a can be chosen so as to be invertible (in A, or in A_1 if A does not have a unit).*

Proof. The first part is the Jacobson density theorem. For the second part, see A. M. Sinclair [18], p. 36.

Now we have got to the core of the subject. The idea of the proof below is the counterpoint of the entire work.

THEOREM 1.2. *Let A be a complex Banach algebra. Let r be a given element in A. Then*

(i) *$r \in Z(A)$ if and only if $s(r) < \infty$;*

(ii) *$r \in \operatorname{rad} A$ if and only if $s(r) < \infty$ and $|r|_\sigma = 0$.*

Proof. The "only if" parts are obvious by considering the spectrum in the algebra $A/\operatorname{rad} A$. The point is in the reverse implications.

Suppose without loss of generality that $s(r) < 1$. Let π be an irreducible representation of A on X. If necessary, π can be extended to be an irreducible representation of A_1 on X. Now it is sufficient to show that $\pi(r) = \lambda\pi(1)$ for some complex λ. If not, there exists a vector ξ in X such that ξ and $\eta = \pi(r)\xi$ are linearly independent. Consequently, by Proposition 1.1 there exists an invertible element a such that $\pi(a)\xi = \xi$ and $\pi(a)\eta = \xi + \eta$. Then we have

$$\pi(r - a^{-1}ra)\xi = \pi(a^{-1})\pi(ar - ra)\xi = \pi(a^{-1})\xi = \xi \neq 0,$$

which means that 1 belongs to the spectrum of $\pi(r - a^{-1}ra)$. This being contained in the spectrum of $r - a^{-1}ra$, we get a contradiction.

Remark 1.3. In the above characterizations, in fact, $s(r) = 0$. But the formally weaker condition $s(r) < \infty$ can practically be verified in various situations.

It is obvious from Sinclair's argument in [18], p. 37 that the characterizations of Theorem 1.2 remain true if in the definition of the function s we let merely $u = e^b$, where b runs over the whole of A.

Also, the same argument as above works if we put even $s(r) = \sup |r + \tau u^{-1}ru|_\sigma$, where $\tau \neq 0$ is a fixed number. It suffices to choose the element a in such a manner that $\pi(a)\xi = \xi$ and $\pi(a)\eta = \xi - \tau\eta$. This observation may be useful in some applications. Originally we proved the result with $\tau = -1$ by an analytic argument, cf. [23] or [24]. See also the proof of Lemma 5 in [20]. For this survey, the case $\tau = \pm 1$ will be sufficient. However, in the real case one must be more careful, cf. Proposition 6.1.

QUESTION 1.4. Let D be a derivation on A. Suppose that $\sup|x^{-1}Dx|_\sigma < \infty$, x invertible. Does it follow that $D(A) \subset \operatorname{rad} A$? If D is inner, this is proved just in the assertion (i) of Theorem 1.2.

We finish this section by listing some corollaries. Let $B(X)$ be the algebra of bounded linear operators on a complex Banach space X.

COROLLARY 1.5. *Let $C \in B(X)$ be such that*

$$\sup\{|C-T|_\sigma\colon T \text{ is similar to } C\} < \infty,$$

or

$$\sup\{|C+T|_\sigma\colon T \text{ is similar to } C\} < \infty.$$

Then C is a scalar multiple of the identity on X.

Proof. Use (i) of Theorem 1.2, Remark 1.3 and the Schur lemma.

COROLLARY 1.6. *Let $C \in B(X)$ be given. Suppose there is a constant γ such that*

$$|CT|_\sigma \leqslant \gamma|C|_\sigma|T|_\sigma \quad \text{for all } T \in B(X).$$

Then C is a scalar multiple of the identity on X.

Proof. We shall verify, for instance, the second condition of Corollary 1.5. Without loss of generality let $|C|_\sigma < 1$. Take a complex number λ with $|\lambda| > |T|_\sigma + \gamma$. Then $\operatorname{dist}(\lambda, \sigma(T)) > \gamma$ and

$$\lambda-(C+T) = (\lambda-T)-C = (1-C(\lambda-T)^{-1})(\lambda-T),$$

where

$$|C(\lambda-T)^{-1}|_\sigma \leqslant \gamma|C|_\sigma|(\lambda-T)^{-1}|_\sigma = \frac{\gamma|C|_\sigma}{\operatorname{dist}(\lambda, \sigma(T))} \leqslant |C|_\sigma < 1;$$

so that $\lambda \notin \sigma(C+T)$. This means, however, that $|C+T|_\sigma \leqslant |T|_\sigma + \gamma = |C|_\sigma + \gamma$ if T is similar to C.

COROLLARY 1.7. *If $R \in B(X)$ is such that $|R+Q|_\sigma = 0$ for all Q quasinilpotent, then $R = 0$.*

Proof. By (ii) of Theorem 1.2.

Remark 1.8. For semi-simple algebras the present Theorem 1.2 improves considerably Theorem 3 of I. N. Herstein [7].

2. Spectral radius characterizations of commutativity

We say that the spectral radius is *subadditive* on a Banach algebra A if there exists a (positive) constant α such that

$$|x+y|_\sigma \leqslant \alpha(|x|_\sigma + |y|_\sigma)$$

for all x, y in A. Similarly, the spectral radius is *submultiplicative* on A if

$$|xy|_\sigma \leqslant \beta|x|_\sigma|y|_\sigma$$

for all x, y in A, with some (positive) constant β. Clearly, if the algebra is not radical, the constants must be $\geqslant 1$.

It follows from Gelfand's theory of commutative algebras that the spectrum of elements in such algebras possesses some nice algebraic and continuity properties (namely, subadditivity, submultiplicativity and uniform continuity). In view of the general importance of the notion of spectrum it should be desirable to know, conversely, to what extent these nice properties are characteristic for commutativity. The following theorem gives a very striking answer.

THEOREM 2.1. *Let A be a complex Banach algebra. The following four conditions are equivalent:*

(i) *the spectral radius is subadditive on A;*

(ii) *the spectral radius is submultiplicative on A;*

(iii) *the spectral radius is uniformly continuous on A;*

(iv) *the algebra $A/\operatorname{rad} A$ is commutative.*

Of course, purely spectral characterizations of commutativity are to be expected merely modulo the radical since in radical algebras, the spectrum being identically zero, no further distinction between the points is possible if we take into account only their spectral behaviour. Nevertheless this only underlines, as we shall see throughout, the importance of the concept of the Jacobson radical. The corresponding result for real Banach algebras will be given in the last section.

The simplicity of the proof of Theorem 2.1 fully corresponds to the simplicity of its statement. Indeed, the implication (i) $\Rightarrow$ (iv) is an immediate consequence of Theorem 1.2, assertion (i). So to complete the proof it is enough to show that (ii) implies (i) and that (iii) implies $s(x) < \infty$ for all x in A. This is done in the two lemmas that follow.

LEMMA 2.2. *If the spectral radius is submultiplicative on A, then it is also subadditive on A.*

Proof. The idea is similar to that used in Corollary 1.6, but since the algebra A may have no unit, we must proceed more carefully. Let x, y be given elements in A. Take a complex number λ such that

$$|\lambda| > |y|_\sigma + \beta|x|_\sigma.$$

We shall prove that $\lambda \notin \sigma(x+y)$. This will mean that

$$|x+y|_\sigma \leqslant |y|_\sigma + \beta|x|_\sigma \leqslant \beta(|x|_\sigma + |y|_\sigma).$$

With this end in view, put $u = x/\lambda$ and $v = y/\lambda$. Then $|u|_\sigma < 1/\beta \leqslant 1$ and $|v|_\sigma < 1$, so that we can write

$$1-(u+v) = (1-u)[1-(1-u)^{-1}uv(1-v)^{-1}](1-v).$$

But here

$$|(1-u)^{-1}uv(1-v)^{-1}|_\sigma \leqslant \beta|(1-u)^{-1}u|_\sigma|v(1-v)^{-1}|_\sigma$$
$$\leqslant \beta\,\frac{|u|_\sigma}{1-|u|_\sigma}\,\frac{|v|_\sigma}{1-|v|_\sigma} \leqslant \frac{\beta|u|_\sigma|v|_\sigma}{(1-\beta|u|_\sigma)(1-|v|_\sigma)} < 1.$$

This estimate is correct even if the algebra A does not have a unit because the factors $(1-u)^{-1}u$ and $v(1-v)^{-1}$ are both in A, and so we are using the submultiplicativity only on A. Thus the element $1-(u+v)$, being represented as a product of three invertible elements, is invertible as well. Hence $1 \notin \sigma(u+v)$, i.e. $\lambda \notin \sigma(x+y)$ as claimed.

LEMMA 2.3. *If the spectral radius is uniformly continuous on A, then there exists a constant γ such that $s(x) \leqslant |x|_\sigma+\gamma\,||x||$ for all x in A.*

Proof. Let $\varepsilon > 0$ be such that $||a-b|| \leqslant \varepsilon$ implies $\big||a|_\sigma-|b|_\sigma\big| \leqslant 1$. Given arbitrary elements $x \neq y$ in A, put $a = \varepsilon x/||x-y||$ and $b = \varepsilon y/||x-y||$. Then $||a-b|| \leqslant \varepsilon$ and the result follows with $\gamma = 1/\varepsilon$.

Remark 2.4. Thus we have proved that both properties of the spectral radius, subadditivity and submultiplicativity, always occur either simultaneously with $\alpha = \beta = 1$ (if $A/\mathrm{rad}\,A$ is commutative) or never with any finite constants (if $A/\mathrm{rad}\,A$ is non-commutative). In the complex case, also a direct proof of the implication (i) $\Rightarrow$ (ii), quite analogous to that of (ii) $\Rightarrow$ (i), is available [17].

A characterization of such algebraic property as commutativity modulo the radical should not depend on the (possibly non-equivalent) Banach algebra topologies which the algebra A can share. But condition (iii) in Theorem 2.1 apparently does depend on the topology. However, this may be explained as follows. First, it is well-known that the spectrum of an element in A is the same as the spectrum of the corresponding class in $A/\mathrm{rad}A$. Secondly, the algebra $A/\mathrm{rad}\,A$ being semi-simple, its topology is determined uniquely — according to the celebrated theorem of B. E. Johnson [9] — by the algebraic structure of A alone. Therefore continuity or uniform continuity of the spectrum or of the spectral radius on A, which is easily seen to be the same as that on $A/\mathrm{rad}\,A$, is, in fact, an algebraic property. We have just shown that the uniform continuity (of both the spectral radius and the spectrum) corresponds exactly to the commutativity of the algebra modulo its radical.

It would be interesting to find the algebraic properties which correspond to the ordinary continuity of the spectral radius or the spectrum. However, C. Apostol [1] has constructed a Banach algebra (even a C^*-algebra) in which the spectral radius is continuous while the spectrum is discontinuous as a set-valued function in the Hausdorff metric. Thus, in general, not all properties of the spectral radius are so strong as to affect the behaviour of the holes inside the spectrum. Paper [1] contains also other relevant suggestions concerning this problem.

Remark 2.5. Theorem 2.1 was obtained independently by B. Aupetit [2] and the present author [20]. Both these papers were influenced by C. Le Page [14] and R. A. Hirschfeld and W. Żelazko [8], where some important particular cases were discovered.

3. Relations between the radical and the kernel of the spectral radius

In the preceding section we have studied global properties of the spectral radius on the whole algebra. In this section we shall be interested in analogous properties concerning only the kernel of the spectral radius, i.e. the set

$$N = \{x \in A\colon\ |x|_\sigma = 0\}$$

of quasi-nilpotent elements in the algebra. This analogy we observed first in [20], and then developed in [19], [21] and [22]. The present approach is founded again on the general Theorem 1.2.

It is well-known that $N \supset \mathrm{rad}\,A$, but this inclusion may often be proper. A natural characterization of algebras in which equality occurs is provided by the following

THEOREM 3.1. *Let A be a Banach algebra. The following three conditions are equivalent*:

(i) *$x, y \in N$ implies $x+y \in N$*;

(ii) *$x, y \in N$ implies $xy \in N$*;

(iii) $N = \mathrm{rad}\,A$.

Proof. Clearly, (iii) implies both (i) and (ii) since the radical is a subalgebra. The implication (i) $\Rightarrow$ (iii) is an immediate consequence of Theorem 1.2, assertion (ii).

Thus it remains to prove that (ii) implies (i). Hence suppose (ii) and take arbitrary elements x, y in N. If $\lambda \neq 0$, we have to show that $\lambda \notin \sigma(x+y)$. Putting $u = x/\lambda$ and $v = y/\lambda$, it suffices to prove that $1 \notin \sigma(u+v)$. But we have

$$1-(u+v) = (1-u)(1-pq)(1-v),$$

where

$$p = (1-u)^{-1}u \quad \text{ and } \quad q = v(1-v)^{-1}$$

are in N. Thus, by (ii), also $pq \in N$, so that $1-(u+v)$ is a product of three invertible elements. Hence $1 \notin \sigma(u+v)$ as claimed. The proof is complete.

Remark 3.2. It is notable that the purely algebraic conditions (i) or (ii) in Theorem 3.1 have a topological effect, the closedness of the set N in the norm (since the radical is always closed). It would be interesting to characterize, in general, the Banach algebras in which the set N is closed, or to describe its closure. Also, from (i) or (ii) it follows already that N is a two-sided ideal (since the radical is).

The implication (i) $\Rightarrow$ (ii) can be proved directly, without passing through (iii) by an analogous argument as above with the decomposition

$$1-uv = (1-u)(1+p+q)(1-v).$$

Note that the equivalence of (i) and (ii) in Theorem 2.1 is established directly, without passing through commutativity, just by a more detailed quantitative investigation of this type of decompositions, cf. the proof of Lemma 2.2.

Of course, the first two conditions in Theorem 3.1 are weakenings of the corresponding conditions in Theorem 2.1. These two weakenings do not imply the commutativity any longer but they characterize a class of algebras which lies strictly between commutative and general Banach algebras. The spectral radius need not be continuous in algebras of this class [15].

Theorem 3.1 gives global conditions for the set N to coincide with the radical. We can formulate also conditions for a single element in N to belong to the radical. The following one is not the strongest possible, but in any event it seems to be worth mentioning.

THEOREM 3.3. *Let A be a Banach algebra. Let r be a given element in A. Then $r \in \operatorname{rad} A$ if and only if $r+N \subset N$.*

Proof. This result is contained in Theorem 1.2, assertion (ii). However, we can give even a simpler proof not using Sinclair's remark to the Jacobson density theorem. Let ξ and η be as in the proof of Theorem 1.2. By the classical Jacobson density theorem we can choose an element a in A such that $\pi(a)\xi = \xi$ and $\pi(a)\eta = \xi+\eta$. If this a is invertible, we may proceed as in the proof of Theorem 1.2 to show that $1 \in \sigma(r-a^{-1}ra)$, a contradiction.

If this a is not invertible, then — because $\sigma(a)$ is bounded — we can take a sufficiently large number ω such that the element $u = a+\omega$ will be invertible. Then $ur-ru = ar-ra$, so that $\pi(r-u^{-1}ru)\xi = \pi(u^{-1})\xi = \xi/(1+\omega)$; hence $1/(1+\omega)$ belongs to $\sigma(r-u^{-1}ru)$, a contradiction.

Thus we conclude that $\pi(r) = \lambda\pi(1)$. But as $r \in N$, this λ must be zero.

Remark 3.4. The above argument shows very well in what way completeness in the sense of Banach is important in this circle of results. Namely, the boundedness of spectra is just a consequence of the completeness (the Neumann series converges).

If the spectra were, say, the whole complex plane, then nothing could be said in terms of them. Indeed, T. J. Laffey has constructed a normed (incomplete) algebra in which the characterization of the radical by the property "$r+N \subset N$" fails. I am indebted to M. R. F. Smyth for this information.

A characterization of elements in the radical by invertible perturbations has been known (cf. [16] and [13]): An element r in A belongs to the radical if and only if $r+u$ is invertible whenever u is invertible. Clearly, this result is an immediate consequence of Theorem 3.3.

COROLLARY 3.5. *Let A be a Banach algebra. Let r be a given element in A. The following three conditions are equivalent:*

(i) $\sigma(a+r) = \sigma(a)$ *for all a in A;*

(ii) $|a+r|_\sigma = |a|_\sigma$ *for all a in A;*

(iii) $r \in \operatorname{rad} A$.

Remark 3.6. Theorem 1.2 relates the characterizations of the radical to the characterizations of commutativity. It may be said that the Jacobson radical is just the set of those elements perturbations by which leave the spectrum completely invariant while the centre modulo the radical, a larger set, consists of exactly those elements perturbations by which can give rise to merely bounded changes of the spectrum. In other words, any element outside the centre can cause arbitrarily large changes of the spectral radius.

A classical characterization due to Jacobson says that $r \in \operatorname{rad} A$ if and only if $|ar|_\sigma = 0$ for all a in A. This is improved as follows.

THEOREM 3.7. *Let A be a Banach algebra. Let r be a given element in A. Then $r \in \operatorname{rad} A$ if and only if $|(1+q)r|_\sigma = 0$ for all q in N.*

Proof. The condition of Theorem 3.3 can easily be verified by using the decomposition

$$1+a+r = (1+a)\{1+[1-(1+a)^{-1}a]r\}$$

with a in N.

COROLLARY 3.8. *Let A be a semi-simple Banach algebra. Then the closed subalgebra of A generated by the set N is also semi-simple.*

COROLLARY 3.9. *If $N \neq \operatorname{rad} A$, then sums as well as products of two quasi-nilpotent elements may have arbitrarily large spectral radii. In particular, this refers to quasi-nilpotent operators on Banach spaces.*

4. Spectral radius characterization of two-sided ideals

Let I be a closed two-sided ideal in a Banach algebra A. Denote by $\sigma(x+I)$ the spectrum of a class $x+I$ in the Banach algebra A/I and by $|x+I|_\sigma$ the corresponding spectral radius. If the original algebra A does not have a unit, these spectra have to be considered with respect to A_1/I, even if A/I could have its own unit; this convention will simplify the language but does not effect the essence of the results which will still be expressible in terms of the spectral radii. It is thus obvious that $\sigma(x+I) \subset \sigma(x)$ for all x in A.

Hence if $r \in I$, then

$$\sigma(a+I) \subset \sigma(a+r) \quad \text{for all } a \text{ in } A. \tag{1}$$

Thus the natural question arises whether, conversely, an element r in A satisfying conditon (1) must belong to the ideal I. The particular case where $I = \operatorname{rad} A$ was studied in detail in the preceding section. For a general ideal, however, the answer may be negative: the zero ideal in a non-zero radical algebra can serve as the simplest example. A less trivial example is the ideal of compact operators relative to the algebra of bounded operators on the Banach space $L(0, 1)$; cf. [4], p. 135, and Theorem 4.2 below. Nevertheless, it turns out that there is an important class of ideals which admit this natural spectral characterization. Namely, we shall show that, in general, the set of all elements $r \in A$ satisfying condition (1) coincides with $\ker(\operatorname{hul}(I))$, the intersection of the primitive ideals containing I. Moreover, we shall see that an element $r \in A$ satisfies (1) if and only if it satisfies merely

$$|a+I|_\sigma \leqslant |a+r|_\sigma \quad \text{for all } a \text{ in } A, \tag{2}$$

the corresponding inequality for the spectral radii.

Thus the ideal I admits spectral characterization (that is, it coincides with the set of all elements $r \in A$ satisfying condition (1) or, which is equivalent, condition (2)) if and only if the algebra A has spectral synthesis at I in the sense that $I = \bigcap P$, where P runs over the primitive ideals containing I. In other words, this is the same as saying that the algebra A/I is semi-simple. This happens, for example, if A is a C^*-algebra with arbitrary I.

In the case where I is the ideal of compact operators on a Banach space, such characterizations have been known in terms of the so-called essential or Weyl spectrum, cf. K. Gustafson [6]. The present results improve the earlier ones in two respects. First, the characterization is expressed purely in terms of the spectral radius (it does not matter what happens inside the spectra!) and, secondly, they are valid in an arbitrary Banach algebra (the nature of operators is not essential).

The point is in the case of primitive ideals with which we begin.

THEOREM 4.1. *Let P be a primitive ideal of a Banach algebra A. Then an element $r \in A$ belongs to P if and only if the inequality*

$$|a+P|_\sigma \leqslant |a+r|_\sigma$$

holds for all $a \in A$.

Proof. If $r \in P$, then the inequality holds as we have already mentioned. Conversely, let $r \in A$ be an element that satisfies the above inequality for all $a \in A$. Putting $a = -r$ we get $|r+P|_\sigma = 0$. Moreover, if b runs over A, we have

$$\sup |r - e^{-b} r e^{b} + P|_\sigma = \sup |-r + e^{-b} r e^{b} + P|_\sigma \leqslant |r|_\sigma$$

by assumption. Thus by Theorem 1.2 (ii) (cf. also Remark 1.3) we conclude that the class $r+P$ belongs to $\mathrm{rad}(A/P) = 0$, i.e. $r \in P$.

The general case easily follows from the primitive case if we recall two general facts. First, if $P \supset I$, then $\sigma(x+P) \subset \sigma(x+I)$. Secondly, we have the formula

$$\sigma(x+I) = \bigcup_P \sigma(x+P),$$

where P runs over the primitive ideals containing I.

THEOREM 4.2. *Let I be a closed two-sided ideal in a Banach algebra A. Then the set of all elements $r \in A$ satisfying the condition*

$$|a+I|_\sigma \leqslant |a+r|_\sigma \quad \text{for all } a \text{ in } A$$

coincides with $\ker(\mathrm{hul}(I))$, the intersection of the primitive ideals containing I.

Remark 4.3. The results of this section appeared first in [24].

5. Characterizations of central idempotents

Let $E = \{e \in A\colon e^2 = e\}$ denote the set of idempotents in a Banach algebra A. Theorem 1.2 provides simple characterizations of idempotents in $Z(A)$.

THEOREM 5.1. *Let e be an idempotent in a complex Banach algebra A. The following four conditions are equivalent:*

(i) $e \in Z(A)$;

(ii) $\sup\{|ef|_\sigma\colon f \in E\} < \infty$;

(iii) $\sup\{|e-f|_\sigma\colon f \in E\} < \infty$,

(iv) $\sup\{|e+f|_\sigma\colon f \in E\} < \infty$;

Proof. Clearly (i) implies all the other conditions. Also (iii) implies (i) by Theorem 1.2 and (iv) implies (i) by Remark 1.3. It is thus sufficient to show that (ii) implies (iii).

Denote by $\varkappa$ the finite supremum in (ii). Given an idempotent f it will be convenient to show that $|e-f|_\sigma \leqslant (1+\varkappa)^{1/2}$. To this end let $|\lambda| > (1+\varkappa)^{1/2}$. Then we may write

$$\lambda - (e-f) = \frac{1}{\lambda}(\lambda - e)[1 + (\lambda-e)^{-1} ef (\lambda+f)^{-1}](\lambda+f).$$

To prove that λ is not in the spectrum of $e-f$ it is enough to show that the middle factor is invertible. We have

$$\begin{aligned}(\lambda-e)^{-1} ef(\lambda+f)^{-1} &= \left(\frac{1}{\lambda} + \frac{e}{\lambda^2} + \frac{e}{\lambda^3} + \ldots\right) ef \left(\frac{1}{\lambda} - \frac{f}{\lambda^2} + \frac{f}{\lambda^3} - \ldots\right) \\ &= \left(\frac{e}{\lambda} + \frac{e}{\lambda^2} + \frac{e}{\lambda^3} + \ldots\right)\left(\frac{f}{\lambda} - \frac{f}{\lambda^2} + \frac{f}{\lambda^3} - \ldots\right) \\ &= \left(\frac{1}{\lambda} + \frac{1}{\lambda^2} + \frac{1}{\lambda^3} + \ldots\right)\left(\frac{1}{\lambda} - \frac{1}{\lambda^2} + \frac{1}{\lambda^3} - \ldots\right) ef = \frac{1}{\lambda^2-1}\, ef.\end{aligned}$$

Since $|ef|_\sigma \leqslant \varkappa$, we conclude that

$$|(\lambda-e)^{-1} ef (\lambda+f)^{-1}|_\sigma \leqslant |\varkappa/(\lambda^2-1)| < 1.$$

This completes the proof.

Now let $C(A) = \{x \in A\colon ax = xa \text{ for all } a \text{ in } A\}$ denote the usual centre of the algebra A. For semi-simple algebras we have $C(A) = Z(A)$ and in this case another equivalent condition for an idempotent e to be central is $eE \subset E$. This characterization can be proved and generalized in a purely algebraic setting; we refer to [27].

For (real or complex) Banach algebras idempotents in $C(A)$ are characterized just as isolated points of the set E in norm topology. In the case of C^*-algebras this result has been obtained first by Y. Kato [11]. We give here a general proof.

THEOREM 5.2. *Let A be a Banach algebra. An idempotent $e \in E$ belongs to the centre $C(A)$ if and only if e is an isolated point in the set E. In other words, the centre of a Banach algebra meets the set of idempotents just in its isolated points.*

Proof. It is well-known and easy to show that the distance between any two distinct commuting idempotents is at least 1. Thus a central idempotent is certainly isolated in the set E. Conversely, suppose that e is isolated in E. For every x in A the element $f(x) = e + ex - exe$ is an idempotent and since the mapping $x \mapsto f(x)$

is continuous and $f(0) = e$, we conclude that $f(x)$ is constant. Hence $ex = exe$ for all x in A. Similarly, using the idempotents $e+xe-exe$ we get that also $xe = exe$. Thus e is central in A.

For a non-central idempotent we can estimate from below its distance from the centre. The corresponding proof relies only on the triangle inequality and so it is valid in an arbitrary normed ring.

THEOREM 5.3. *Let A be a Banach algebra. If an idempotent e is not in the centre $C(A)$, then its distance from $C(A)$ is at least* $1/2$.

Proof. Take a non-central idempotent e and consider the inner derivation implemented by it, that is $Dx = xe-ex$ for all x in A. A standard application of the triangle inequality yields that $\|D\| \leqslant 2\mathrm{dist}(e, C(A))$. On the other hand, a direct calculation shows that $D^3 = D$. Hence $\|D\| \geqslant 1$ as e is not central. The two estimates give the result.

We refer to [26] for additional information about the set E. In particular, it is proved there that the set E is locally arcwise connected and the structure of its connected components is investigated in more detail. Each component turns out to be either a singleton or unbounded, and the distance between any two distinct components is shown to be $\geqslant 1$. A link with the principal component of the group of invertible elements is also found and a number of corollaries are listed. We point out here only one of them: if the set E is at most countable, then $E \subset C(A)$.

We conclude this section by suggesting another direction of possible further development.

QUESTION 5.4. It may be of some interest to study spectra of products of non-commuting idempotents. Clearly, a finite product of idempotents is always non-invertible. Thus in the extreme case (when the idempotents are "very non-commuting") these products can fill in the whole set of non-invertible elements. It was proved by J. A. Erdos [5] that this is the case for the full matrix algebras over a field. Thus the natural question arises to what extent this property characterizes the full matrix algebras among all Banach algebras.

6. Real Banach algebras

In this section we show that most of the preceding results have a natural interpretation also in real Banach algebras. Instead of the spectral radius we introduce another function which turns out to be more convenient for this general case. Moreover, the results formulated in terms of that function will often be even stronger than the previous ones.

Notation. Let A be a real Banach algebra. For every x in A put $|x|_\varrho = 0$ whenever there are no real points in the spectrum of x, and otherwise put $|x|_\varrho = \sup|\lambda|$, where λ runs over the real points in the spectrum of x. Clearly, $|x|_\varrho \leqslant |x|_\sigma$.

If π is an irreducible representation of A on a real linear space X, we denote by Q_π the commutant of $\pi(A)$ in the algebra of linear transformations on X. It is

well-known that Q_π is a real division algebra, cf. [4], p. 127. Consequently, Q_π is isomorphic to $\boldsymbol{R}$, $\boldsymbol{C}$ or to $\boldsymbol{K}$, the algebra of quaternions. For complex Banach algebras Q_π is always isomorphic to $\boldsymbol{C}$. When π is given we shall often write simply Q for Q_π.

For the investigation of real Banach algebras the following modification of Proposition 1.1 is appropriate, cf. [3].

PROPOSITION 6.1. *Let A be a real Banach algebra and let π be an irreducible representation of A on a real linear space X. If $\xi_1, \ldots, \xi_n$ are Q_π-independent in X and if $\eta_1, \ldots, \eta_n$ are arbitrary in X, then there exists an element a in A such that $\pi(a)\xi_i = \eta_i$ for $i = 1, \ldots, n$. Moreover, if Y is the real linear hull of $\xi_1, \ldots, \xi_n$ and if there exists an invertible linear transformation T from Y into Y such that $T^2\xi_i = \eta_i$ for $i = 1, \ldots, n$, then the element a can be chosen so as to be invertible (in A or A_1).*

Proof. Using the functional calculus it is easy to see that $T^2 = e^R$, where R is a real linear transformation from Y into Y. By the classical Jacobson density theorem there exists a b in A such that $\pi(b)\xi_i = R\xi_i$ for $i = 1, \ldots, n$. Now the proof can be finished as in [18], p. 37.

In contrast to the complex case, the real situation sometimes gives rise to certain new difficulties which can be reduced to the following purely algebraic result of I. Kaplansky.

PROPOSITION 6.2. *Let T be a linear transformation on a real linear space X. Suppose there exists an integer m such that for every ξ in X the vectors $\xi, T\xi, \ldots$ $\ldots, T^m\xi$ are linearly dependent. Then T is algebraic over $\boldsymbol{R}$.*

Proof. We refer the reader to [10], p. 41.

The real analogy of the main Theorem 1.2 is contained in the following two results. For r in A we put, this time,

$$q(r) = \sup|r-u^{-1}ru|_\varrho,$$

where u runs over the invertible elements in A (or, in A_1). Thus $q(r) \leqslant s(r)$.

THEOREM 6.3. *Let A be a real Banach algebra. Let r be in A such that $q(r) < \infty$. Then, for every irreducible representation π of A, the operator $\pi(r)$ is algebraic over $\boldsymbol{R}$.*

Proof. Let π be an irreducible representation of A on X. Without loss of generality assume that $q(r) < 1$.

Suppose first that there exists a vector ξ in X such that ξ and $\eta = \pi(r)\xi$ are Q-independent. Let Y be the real linear hull of ξ and η. Define the linear operator T on Y by $T\xi = \xi$ and $T\eta = \dfrac{\xi}{2}+\eta$. Then by Proposition 6.1 there exists an invertible element a such that $\pi(a)\xi = \xi$ and $\pi(a)\eta = \xi+\eta$. We finish as in the proof of Theorem 1.2 to get a contradiction.

Thus for every ξ in X the vectors ξ and $\eta = \pi(r)\xi$ are Q-dependent. That means, for every ξ in X there exists an element $w(\xi)$ in Q such that $\pi(r)\xi = w(\xi)\xi$.

As Q is isomorphic to $\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{K}$, let $w^*(\xi)$ denote the natural conjugate of $w(\xi)$. Then we have

$$\big(\pi(r)-w^*(\xi)\big)\big(\pi(r)-w(\xi)\big)\xi = 0;$$

hence ξ is annihilated by a polynomial in $\pi(r)$ with real coefficients and degree 2. By Proposition 6.2 we conclude that $\pi(r)$ is algebraic over $\boldsymbol{R}$, as was to be proved.

THEOREM 6.4. *Let A be a real Banach algebra. Let r be in A such that $q(r) < \infty$. Let π be an irreducible representation of A. If $|\pi(r)|_\sigma = 0$, then $\pi(r) = 0$.*

Proof. By Theorem 6.3 we have

$$\sum_{k=0}^{m} \alpha_k \pi(r)^k = 0,$$

where α_k are real coefficients. Let h be the least index such that $\alpha_h \neq 0$. Since $|\pi(r)|_\sigma = 0$, the operator

$$\alpha_h + \sum_{k=h+1}^{m} \alpha_k \pi(r)^{k-h}$$

has spectrum $\{\alpha_h\}$, hence it is invertible. It follows that

$$\pi(r)^h = 0. \tag{3}$$

Also, as in the proof of Theorem 6.3, for every ξ in X there is an element $w(\xi)$ in Q such that

$$\pi(r)\xi = w(\xi)\xi. \tag{4}$$

From (3) and (4) we get

$$w(\xi)^h\xi = \pi(r)^h\xi = 0.$$

Let $\xi \neq 0$ be fixed. Put $S = w(\xi)^h$, an element of Q. Then $S\pi(a) = \pi(a)S$ for each a in A; hence, in particular, at the point ξ we have

$$S\pi(a)\xi = \pi(a)S\xi = 0. \tag{5}$$

But $\pi(A)\xi = X$ by irreducibility of π. Hence (5) gives $S = 0$.

As Q is isomorphic to one of the three division algebras, it does not contain proper nilpotents. Thus we get $w(\xi) = 0$. This being true for any ξ, we conclude that $\pi(r) = 0$ as claimed.

Now we can strengthen the characterizations of the Jacobson radical given in Section 3. Let

$$N_\varrho = \{x \in A\colon |x|_\varrho = 0\},$$

so that $N \subset N_\varrho$. For example, we have

THEOREM 6.5. *Let A be a real Banach algebra. Let r be a given quasi-nilpotent element in A. The following two conditions are equivalent:*

(i) $r+N \subset N_\varrho$;

(ii) $r \in \operatorname{rad} A$.

Proof. An immediate consequence of Theorem 6.4.

It is now obvious that most of the other arguments given in Sections 2–4 can also be modified in this spirit in order to extend the complex results to the real case.

Only the characterizations of commutativity require a few words. Namely, the algebra $\boldsymbol{K}$ of quaternions satisfies $s(x) < \infty$ for every x (because the spectral radius coincides with the norm in $\boldsymbol{K}$) though it is not commutative. The appropriate analogy of Theorem 2.1 was obtained in [3], cf. Theorem 6.7 below. Here we wish to point out that the following proposition is, in fact, sufficient for the proof in place of Lemma 5 in [3].

PROPOSITION 6.6. *Let A be a semi-simple real Banach algebra. Suppose that every element in A is algebraic over $\boldsymbol{R}$. Then A is finite-dimensional.*

Proof. The proof can easily be deduced from the nice ideas in T. J. Laffey [12].

THEOREM 6.7. *Let A be a real Banach algebra. The following conditions are equivalent:*

(i) *for every irreducible representation π, the algebra $\pi(A)$ is isomorphic with Q;*

(ii) *for every r in A we have $s(r) < \infty$;*

(iii) *for every r in A we have $q(r) < \infty$;*

(iv) *the spectral radius is uniformly continuous on A;*

(v) *the spectral radius is subadditive on A;*

(vi) *the spectral radius is submultiplicative on A.*

Proof. Knowing that the spectral radius coincides with the norm on $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{K}$, and that for every x in A there exists an irreducible representation π such that $|x|_\sigma = |\pi(x)|_\sigma$, we conclude that (i) implies all the other conditions. It is also easy to see that each one of (iv) and (v) implies (ii) which in turn implies (iii) trivially. From (vi) we get $q(x+y) \leqslant \beta(|x|_\sigma+|y|_\sigma)$ as in Lemma 2.2 (cf. Lemma 3 in [3]), so that (vi) implies (iii) also.

Hence it remains to prove that (iii) implies (i). Let π be an irreducible representation of A. By Theorem 6.3 we know that $\pi(r)$ is real algebraic. This is true for every r in A, and the image $\pi(A)$ is a real Banach algebra with the norm taken over from $A/\pi^{-1}(0)$. By Proposition 6.6 we see that $\pi(A)$ is finite-dimensional over $\boldsymbol{R}$, and hence over Q as well. By the Jacobson density theorem, $\pi(A)$ is Q-transitive, and thus $\pi(A)$ must be the full matrix algebra $M_n(Q)$ for some integer n. If $n \geqslant 2$, then $M_n(Q)$ contains a subalgebra isomorphic with $M_2(\boldsymbol{R})$, but on this subalgebra we have $\sup|r-u^{-1}ru|_\varrho = \infty$ for

$$r = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}, \qquad k = 1, 2, 3, \ldots,$$

which contradicts condition (iii). Thus $n = 1$.

Remark 6.8. We have already mentioned that, in the complex case, the equivalence of conditions (v) and (vi) can be proved directly without passing through (i). This is not evident in the real case. Moreover, these properties of the spectral radius do not extend to the complexification (even in the simplest case $A = K$), so there is no hope of applying the complex theory (the Liouville theorem etc.). It is therefore all the more interesting to see that these natural properties of the spectral radius are again equivalent also in the real case. Proving this was made possible by the algebraic approach presented in this paper.

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