THEOREM 11. *If* $q^i(x+a^i n) = q^i(x)$, $a^i < a$, $\int_0^a |q^i|^2 dx < N$ $(i = 1, 2)$, *then*

$$\delta_{(\alpha,\beta)}\big(\sigma(\overset{(1)}{\mathfrak{A}}), \sigma(\overset{(2)}{\mathfrak{A}})\big) < \sqrt{a}\,\frac{2N+1}{1-e^{-\alpha}} M(\alpha, \beta)\|q^1-q^2\|$$

where

$$M(\alpha, \beta) = \sup_{\alpha\leqslant\lambda\leqslant\beta} \big(|\lambda|^2+1\big)^{-1/2}.$$

The perturbation properties of the type (10) in the case of periodic q follow from some estimates of classical eigenfunctions and from the fact that the eigenelements of $\mathfrak{A}$ span the space $B^2(\boldsymbol{R}^3)$.

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Presented to the semester
Spectral Theory
September 23–December 16, 1977

SPECTRAL THEORY
BANACH CENTER PUBLICATIONS, VOLUME 8
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1982

THE SPECTRAL CUTTING PROBLEM

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The Schrödinger operator $L = -\Delta+q$ with an almost periodic potential q (in symbols: $q \in \mathrm{A.P.}(\boldsymbol{R}^n)$) is of interest in the quantum theory of disordered systems (alloys, liquids).

Let $\varrho(x)$ denote a function for which the set $\operatorname{supp}\varrho \cap \operatorname{supp}(\varrho-1)$ is not too large; we shall call the operator

$$L_{\mathrm{cut}} = -\Delta+\varrho q$$

the *Schrödinger operator* with potential "cut by ϱ". By A and A_{cut} we shall denote the unique self-adjoint extensions in $L^2(\boldsymbol{R}^n)$ of the operators L and L_{cut} defined in $D_L = D_{L_{\mathrm{cut}}} = C_0^2(\boldsymbol{R}^n)$. Now, the spectral cutting problem may be formulated as follows: for which cut ϱq do the spectra $\sigma(A)$, $\sigma(A_{\mathrm{cut}})$ satisfy the inclusion

$$\sigma(A) \subset \sigma(A_{\mathrm{cut}})? \tag{1}$$

This inclusion has the following physical interpretation. Suppose that an electron is moving in the medium defined by the potential q. Then the set of numbers $\sigma(A)$ represents the admissible levels of energy. In other words, $\sigma(A)$ represents the energetical characteristics of an electron moving in the medium considered. The potential ϱq defines the medium arising as the portion cut from the original medium defined by q. The inclusion (1) means that the electron in a piece cut from the medium may have the same energy levels as one moving in the whole medium.

We will use the following norms:

$$\square f \square^2 = \limsup_{T\to\infty} \frac{1}{T^n} \int_{|x|<T} |f|^2 dx, \tag{2}$$

$$\|f\|^2 = \lim_{T\to\infty} \frac{1}{T^n} \int_{|x|<T} |f|^2 dx, \tag{3}$$

$$\|f\|_{L^2}^2 = \int_{\boldsymbol{R}^n} |f|^2 dx, \tag{4}$$

where $f(x)$, $x \in \boldsymbol{R}^n$, is a complex-valued function.

In this paper we shall give the proof of the following theorem:

THE CUTTING THEOREM. *If* $q \in \text{A.P.}(\boldsymbol{R}^n)$,

(5) $$\varrho \in C^2(\boldsymbol{R}^n), \quad ||\varrho|| \neq 0, \quad ||D^\alpha \varrho|| = 0 \text{ for } |\alpha| = 1, 2,$$

$||\varrho(1-\varrho)|| = 0$, *and there exist numbers* $\beta_1 \geqslant \beta_2 > 0$ *such that for each* $u \in \text{A.P.}(\boldsymbol{R}^n)$:

(6) $$\beta_1 ||u|| \geqslant ||\varrho u|| \geqslant \beta_2 ||u||,$$

then

$$\sigma(A) \subset \sigma(A_{\text{cut}}).$$

Let $\Gamma \subset \boldsymbol{R}^n$ denote a cone satisfying the condition

$$\int\limits_{\{|x|<1\}\cap\Gamma} \varrho\, dx \neq 0.$$

Then condition (6) is satisfied if, for example, there exist numbers $a > b > 0$ such that the following inequalities are fulfilled:

$$a \underset{x\in \boldsymbol{R}^n}{>} \varrho(x) \underset{x\in\Gamma}{>} b.$$

Our proof of the cutting theorem is an application of the idea that in investigating spectral properties of L in $L^2(\boldsymbol{R}^n)$ it can be useful to operate simultaneously with spectral representations of L in other (possibly different) functional Hilbert spaces. In this case we have at our disposal several different points of view on the spectral properties of the operator L. This may enable us to obtain better spectral information in the basic space $L^2(\boldsymbol{R}^n)$. From this point of view the nonseparable functional spaces $\mathfrak{H}(\boldsymbol{R}^n)$ are very useful because: 1. the essential spectra of L in $L^2(\boldsymbol{R}^n)$ and $\mathfrak{H}(\boldsymbol{R}^n)$ are in many cases equal, 2. the spectral resolutions of L in $L^2(\boldsymbol{R}^n)$ and $\mathfrak{H}(\boldsymbol{R}^n)$ are quite different, in particular the square-nonintegrable eigenfunctions of L belong to $\mathfrak{H}(\boldsymbol{R}^n)$.

We shall start with the construction of certain spaces $\mathfrak{H}(\boldsymbol{R}^n)$. Let us put $M = \{f\colon \Box f \Box < +\infty\}$. Then, with the relation: $f \sim g$ iff $\Box f-g \Box = 0$, the linear space

$$\mathfrak{M} = M/\sim$$

is a complete Banach space (see [7]). The elements of $\mathfrak{M}$ will be denoted by capital letters $F, U, \ldots$ If $f \in M$ represents an element $F \in \mathfrak{M}$, we write $F = (f)$.

We look for Hilbert spaces $\mathfrak{H} \subset \mathfrak{M}$. As scalar product we will take

(7) $$(U, V) = \lim_{T\to\infty} \frac{1}{T^n} \int\limits_{|x|<T} u\bar{v}\, dx, \quad U = (u), \ V = (v).$$

Hence it is natural to consider the set

$$P = \{u\colon u \in M, ||u|| \text{ exists}\}.$$

But there arises the trouble that P is not a linear set (see [1]). Two functions $f, g \in P$ will be called *comparable* if $f+g \in P$. Obviously, f, g are comparable iff the scalar

product (f, g) exists. Similarly, we shall say that a function and a subset P are comparable or that two subsets of P are comparable. There holds the following theorem (see [1]).

THEOREM 1. *If* E *is a linear subset of* P, $f_n \in E$, $||f_n - f_m|| \to 0$, *then for* $f \in M$ *such that* $\Box f - f_n \Box \to 0$ *we have*:

$$f \in P, \quad f \text{ is comparable with } E, \quad ||f - f_n|| \to 0.$$

Hence every functional Hilbert space $\mathfrak{H} \in \mathfrak{M}$ with the scalar product (7) is of the form

$$\mathfrak{H} = \bar{E}/\sim$$

where $\bar{E}$ denotes the closure in the norm (3) of some linear subset of P. We have, e.g.,

$$B^2(\boldsymbol{R}^n) = \bar{T}/\sim\, = \overline{\text{A.P.}(\boldsymbol{R}^n)}/\sim$$

where T denotes the set of trigonometric polynomials.

In order to define other linear subsets $E \subset P$ we shall use various sets of complex valued functions u_λ, $\lambda \in \Lambda$, satisfying the following two conditions:

1. $u_\lambda \in P$,
2. $u_\lambda, u_{\lambda'}$ are comparable for each $\lambda, \lambda' \in \Lambda$.

Then we put

$$E = \Big\{u\colon u = \sum_{\mu=1}^{q} a_\mu u_{\lambda_\mu},\ q < +\infty,\ \lambda_\mu \in \Lambda,\ a_\mu \text{ complex numbers}\Big\}.$$

A function u_λ will be called a *weak eigenfunction* and the number λ a *weak eigenvalue* of the operator $L = -\Delta + q$, $\sup\limits_{\boldsymbol{R}^n} |q| < +\infty$, if the following conditions are fulfilled:

(a) $u_\lambda \in C^2(\boldsymbol{R}^n) \cap P$, $||u_\lambda|| \neq 0$,

(b) $\sup\limits_{\boldsymbol{R}^n}(|u_\lambda|, |Du_\lambda|) < +\infty$,

(c) $||(L-\lambda)u_\lambda|| = 0$.

The following lemma is true:

LEMMA 1. *If* $u_\lambda, u_{\lambda'}$, $\lambda \neq \lambda'$, $\operatorname{Im}\lambda = \operatorname{Im}\lambda' = 0$, *are weak eigenfunctions, then* $(u_\lambda, u_{\lambda'}) = 0$, *and thus they are comparable.*

Proof. Let us put $(u, v)_T = \dfrac{1}{T^n} \displaystyle\int\limits_{|x|<T} u\bar{v}\, dx$. For bounded functions u, v we have:

$$\sup_T |\operatorname{Re}(u, v)_T| < +\infty, \quad \sup_T |\operatorname{Im}(u, v)_T| < +\infty.$$

Thus we may define:

$$[u, v]^0 = \max\{\limsup_{T\to\infty} \operatorname{Re}(u, v)_T, \limsup_{T\to\infty} \operatorname{Im}(u, v)_T\},$$

$$[u, v]_0 = \min\{\liminf_{T\to\infty} \operatorname{Re}(u, v)_T, \liminf_{T\to\infty} \operatorname{Im}(u, v)_T\}.$$

For the proof it suffices to show that

$$[u_\lambda, u_{\lambda'}]^0 = [u_\lambda, u_{\lambda'}]_0 = 0.$$

Integrating by parts we obtain for u, v satisfying the conditions (a) and (b):

$$[u, Lv]^0 = [Lv, u]^0.$$

Moreover, for real α we have

$$[\alpha u, v]^0 = [u, \alpha v]^0 = \alpha[u, v]^0.$$

and

$$[f, g]^0 = [f, h]^0$$

if f, g, h are bounded and $\|g-h\| = 0$. Hence, in view of the condition (c), we may write

$$\lambda[u_\lambda, u_{\lambda'}]^0 = [Lu_\lambda, u_{\lambda'}]^0 = [u_\lambda, Lu_{\lambda'}]^0 = \lambda'[u_\lambda, u_{\lambda'}]^0,$$

i.e. $[u_\lambda, u_{\lambda'}]^0 = 0$. In the same way we obtain $[u_\lambda, u_{\lambda'}]_0 = 0$, which ends the proof of the lemma.

Now we shall prove the following simple lemma:

LEMMA 2. *If $\varrho(x)$ satisfies the conditions* (5), *then*

$$\varrho(x)e^{i(\mu,x)} = u_{|\mu|^2}(x), \quad \mu \in \boldsymbol{R}^n$$

is a weak eigenfunction of the operator $L = -\Delta$ and $|\mu|^2$ is the corresponding weak eigenvalue.

The proof follows from the fact that

$$-\Delta u - |\mu|^2 u = -e^{i(\mu,x)}\Delta\varrho + 2i(\operatorname{div}\varrho, \mu)e^{i(\mu,x)}.$$

We shall be working in the three following spaces:

$$L^2(\boldsymbol{R}^n), \quad B^2(\boldsymbol{R}^n), \quad \mathfrak{R} = \overline{T}\varrho/\sim$$

where $T\varrho = \{u\colon u = \varrho v, v \in T\}$ and ϱ satisfies (5). Let us write

$$\text{A.P.}^k(\boldsymbol{R}^n) = \{u\colon u \in \text{A.P.}(\boldsymbol{R}^n),\ D^\alpha u \in \text{A.P.}(\boldsymbol{R}^n),\ |\alpha| \leqslant k\}$$

and

$$\text{A.P.}^k_\varrho(\boldsymbol{R}^n) = \{u\colon u = \varrho v,\ v \in \text{A.P.}^k(\boldsymbol{R}^n)\}.$$

We shall prove the following lemmas:

LEMMA 3. *The operator $L = -\Delta + q$, $q \in \text{A.P.}(\boldsymbol{R}^n)$, $D_L = \text{A.P.}^2(\boldsymbol{R}^n)$, is in the space $B^2(\boldsymbol{R}^n)$ essentially self-adjoint.*

We will denote the unique self-adjoint extension by $\mathfrak{A}$.

LEMMA 4. *The operator $L = -\Delta + q$, $q \in \text{A.P.}(\boldsymbol{R}^n)$, $D_L = \text{A.P.}^2_\varrho(\boldsymbol{R}^n)$, is in the space $\mathfrak{R}$ essentially self-adjoint if ϱ satisfies the conditions* (5).

We will denote the corresponding self-adjoint extension by $\underset{1}{\mathfrak{A}}$.

LEMMA 5. *The operator $L_{\text{cut}} = -\Delta + \varrho q$, $q \in \text{A.P.}(\boldsymbol{R}^n)$, $D_{L_{\text{cut}}} = \text{A.P.}^2_\varrho(\boldsymbol{R}^n)$, is essentially self-adjoint in $\mathfrak{R}$ if ϱ satisfies the conditions* (5) *and* $\|\varrho(1-\varrho)\| = 0$.

We will denote the corresponding self-adjoint extension by $\underset{1}{\mathfrak{A}}_{\text{cut}}$.

Proof. The symmetry of the operators defined in the lemmas follows simply by integrating by parts. Let us consider the operator $-\Delta$ in $D_{-\Delta} = T$. Then, after simple computations, we obtain that the set $(-\Delta-\lambda)T$ is dense in $B^2(\boldsymbol{R}^n)$ if $\operatorname{Im}\lambda \neq 0$. Hence $-\Delta$ is essentially self-adjoint in $B^2(\boldsymbol{R}^n)$. But $L-(-\Delta)$ is in $D_{-\Delta} \subset D_L$ bounded and symmetric, hence L is essentially self-adjoint in $B^2(\boldsymbol{R}^n)$. This completes the proof of Lemma 3.

For $v \in \text{A.P.}^2(\boldsymbol{R}^n)$ we have

$$-\Delta(\varrho v) + q(\varrho v) = \varrho(-\Delta v + qv) + v\Delta\varrho + 2(\operatorname{grad}\varrho, \operatorname{grad} v).$$

Hence, in virtue of the conditions (5), we have

$$(L(\varrho v)) = (\varrho Lv) \in \mathfrak{R},$$

i.e. L, $D_L = \text{A.P.}^2_\varrho(\boldsymbol{R}^n)$ is well defined in $\mathfrak{R}$.

Moreover, we may write

$$(L\pm i)\text{A.P.}^2_\varrho(\boldsymbol{R}^n) = \varrho(L\pm i)\text{A.P.}^2(\boldsymbol{R}^n).$$

But the last set is dense in $\mathfrak{R}$ because, according to Lemma 3, the set $(L\pm i)\text{A.P.}^2(\boldsymbol{R}^n)$ is dense in $B^2(\boldsymbol{R}^n)$. This ends the proof of Lemma 4.

The proof of Lemma 5 is now an obvious consequence of the assumption $\|\varrho(1-\varrho)\| = 0$. Indeed, for $u \in \text{A.P.}^n_\varrho(\boldsymbol{R}^n)$, $u = \varrho v$, $v \in \text{A.P.}^2(\boldsymbol{R}^n)$ we have:

$$\begin{aligned}(L_{\text{cut}}u) &= (-\Delta(\varrho v) + \varrho q \varrho v) = (-\Delta(\varrho v) + q\varrho v - qu\varrho(1-\varrho)) \\ &= (-\Delta(\varrho v) - q\varrho v) = \underset{1}{\mathfrak{A}}(u).\end{aligned}$$

Hence, in virtue of Lemma 4, the operator L_{cut} is essentially self-adjoint, which ends the proof of Lemma 5. Moreover, we obtain the equality

$$\underset{1}{\mathfrak{A}} = \underset{1}{\mathfrak{A}}_{\text{cut}}. \tag{8}$$

The proof of the cutting theorem is based on the following chain of equalities and one inclusion.

$$\sigma(A) = \sigma(\mathfrak{A}) = \sigma(\underset{1}{\mathfrak{A}}_{\text{cut}}) \subset \sigma(A_{\text{cut}}). \tag{9}$$

The first equality follows from the general Šubin's theorem (see [8], [9]) stating that elliptic operators with almost periodic coefficients have the same spectra in $L^2(\boldsymbol{R}^n)$ and $B^2(\boldsymbol{R}^n)$.

According to (8) for the proof of the second equality of the chain it suffices to show that $\sigma(\mathfrak{A}) = \sigma(\underset{1}{\mathfrak{A}})$. Hence we have to show that the following two inequalities

$$\|(\mathfrak{A}-\lambda)u\| \geqslant c\|u\|, \quad u \in \text{A.P.}(\boldsymbol{R}^n), \quad c = \text{const}.$$

and

$$\|(\underset{1}{\mathfrak{A}}-\lambda)v\| \geqslant c_1\|v\|, \quad v \in \text{A.P.}^2_\varrho(\boldsymbol{R}^n), \quad c_1 = \text{const}.$$

are equivalent. But for $v = \varrho u$, $u \in \mathrm{A.P.}^2(\boldsymbol{R}^n)$, we have

$$\|(\underset{1}{\mathfrak{A}} - \lambda)v\| = \|\varrho(\mathfrak{A} - \lambda)u\| \quad \text{and} \quad \|v\| = \|\varrho u\|.$$

Hence the equivalence of the inequalities is a simple consequence of (6).

Let us now introduce a function $\varphi_T \in C_0^2(|x| < T)$ such that $0 \leqslant \varphi_T \leqslant 1$, $\varphi_T(x) = 1$ for $|x| < T-1$ and for $u \in \mathrm{A.P.}_\varrho^2(\boldsymbol{R}^n)$ the following equalities take place:

$$\|u\|^2 = \lim_{T\to\infty} \frac{1}{T^n} \|\varphi_T u\|_{L^2}^2, \qquad (\underset{1}{\mathfrak{A}_{\mathrm{cut}}} u, u) = \lim_{T\to\infty} \frac{1}{T^n} (A_{\mathrm{cut}}(\varphi_T u), \varphi_T u)_{L^2},$$

$$\|\underset{1}{\mathfrak{A}_{\mathrm{cut}}} u\|^2 = \lim_{T\to\infty} \frac{1}{T^n} \|A_{\mathrm{cut}}(\varphi_T u)\|_{L^2}^2.$$

Hence and from the inequality

$$\|(A_{\mathrm{cut}} - \lambda)v\|_{L^2} \geqslant c\|v\|_{L^2}, \quad v \in C_0^2(\boldsymbol{R}^n), \quad c = \mathrm{const},$$

it follows that

$$\|(\underset{1}{\mathfrak{A}_{\mathrm{cut}}} - \lambda)u\| \geqslant c\|u\|, \quad u \in \mathrm{A.P.}_\varrho^2(\boldsymbol{R}^n).$$

This gives the last inclusion of (9) and ends the proof of the cutting theorem. For more information on spectral analysis in nonseparable spaces see [3]–[6], [8]–[11].

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Presented to the semester
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PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1982

DILATIONS TO SYSTEMS OF MATRIX UNITS

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0. The context

The dilation theorem of M. A. Naĭmark ([9], Thm. I.8.2) concerns dilations to a spectral measure. The object dilated has some but not all of the defining properties of a spectral measure. The dilation theorem of W. F. Stinespring ([8], [2]), and that of the present paper, are of analogous nature. To show what we are about, we begin by restating Naĭmark's theorem in the case which gives a spectral measure of finite support.

THEOREM 0.1 (Naĭmark). *Assume the operators A_j (j running over a finite index set) satisfy $0 \leqslant A_j \in \mathscr{B}(\mathscr{H})$, $\sum_j A_j = 1$. Then there exists an isometric injection ι of $\mathscr{H}$ into a larger Hilbert space $\mathscr{K}$, and there exist commuting orthoprojectors $E_j \in \mathscr{B}(\mathscr{K})$ with $\sum_j E_j = 1$, such that $A_j = \iota^* E_j \iota$.*

The last equation holding for all j is what is meant by saying the E_j are a simultaneous dilation of the A_j; so the theorem may be phrased briefly thus: a finite family of positive operators adding to the identity have a simultaneous dilation to complementary orthoprojectors.

One way of proving Naĭmark's theorem [6] begins with a simple explicit construction for the case where there are only two A_j, and handles more numerous families by iterating this construction in nested fashion. Infinite families can be handled in the same way and the full force of Naĭmark's theorem recovered.

Stinespring's theorem concerns linear mappings on a C^*-algebra into $\mathscr{B}(\mathscr{H})$. We will restate it in the special case where the given algebra is that of all $n \times n$ complex matrices. This algebra is the linear span of the e_{jk} (this denotes the matrix having entry 1 in the j, k-place and all other entries zero), so any linear mapping of it is determined by the images A_{jk} of the e_{jk}. It is known ([4], Remark 1.8, or [5], Lemma 2.1) that the mapping is completely positive if and only if the A_{jk} form a positive operator-matrix; and that the mapping is a *-homomorphism if and only if, in addition, $A_{ij}A_{kl} = \delta_{jk}A_{il}$. The following is therefore a variant of this case of Stinespring's theorem (cf. [3], Lemma 3.2):