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ω-ULTRADISTRIBUTIONS AND THEIR APPLICATION TO OPERATOR THEORY

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Introduction

In order to enlarge the family of L. Schwartz's distributions, I. M. Gelfand and G. Šilov considered in [25] the following problem: to define non-trivial locally convex topological vector spaces $\mathscr{X}$ of infinitely differentiable functions such that

(i) $\mathscr{X}$ is a Fréchet space or a countable inductive limit of Fréchet spaces;

(ii) the topology of $\mathscr{X}$ is stronger than the topology of pointwise convergence.

The elements of $\mathscr{X}$ are called *basic functions* and the elements of the dual $\mathscr{X}'$ of $\mathscr{X}$ are called *generalized functions*. If we "shrink" $\mathscr{X}$, then $\mathscr{X}'$ become larger.

The generalized functions $\mathscr{X}'$ are called *ultradistributions* if, roughly speaking, there exist partitions of the unity in $\mathscr{X}$. This establishes a "lower bound" for $\mathscr{X}$. Several ultradistribution theories, based on non-quasianalyticity are considered and developed by A. Beurling [6], G. Björk [7], H. Komatsu [34], J. L. Lions–E. Magenes [41], C. Roumieu [51], [52]. L. Schwartz's distributions and the above-mentioned ultradistributions can be imbedded in the large family of hyperfunctions considered by M. Sato in [54] (see [60], [61], [55], [34], [35]).

In all the above-mentioned ultradistribution theories it is difficult to handle differential operators of infinite order. For this reason, the structure of the ultradistributions with one-point support was clarified only in particular cases. Moreover, because of the restrictions required by the stability under different usual operations, the existing ultradistribution theories cannot be applied efficiently for example, in functional calculus problems and in the treatment of the abstract Cauchy problem.

In order to avoid these difficulties, we consider in this work a new ultradistribution theory, suggested to us by the function-theoretical tools from [43]. These ultradistributions are parametrized by entire functions ω of the form

$$\omega(z) = \prod_{k=1}^{\infty}\left(1+\frac{iz}{t_k}\right),$$

where $t_1, t_2, \ldots > 0$, $t_1 < +\infty$, $\sum_{k=1}^{\infty} 1/t_k < +\infty$.

For a fixed ω, we can define sufficiently many differential operators of infinite order to obtain a Peetre type theorem. This enables us to describe all ω-ultradistributions with one-point support.

Imposing on ω a quite general regularity condition, we characterize the ω-ultradistributions among the hyperfunctions. We give also an intrinsic characterization of the "union" of all ω-ultradistributions with "regular" ω.

The introduction of the ω-ultradistributions is justified also by the fact that they can be applied conveniently in the operator theory. Thus, using them, we explain the "Levinson condition" from [43] and we construct "abstract ω-spaces", similar to the abstract Gevrey spaces from [4].

Our main results were announced in Comptes Rendus Acad. Sci. Paris, série A, 285 (1977), 707–710, 753–756, 855–858.

1. Entire majorants

In this section of preliminary character we expose some results concerning functions with the module majorized by the module of some entire function of a particular type. These results are fundamental technical tools for the theory developed in the next sections.

Let f be an entire function on the angle $\{z \in \boldsymbol{C}\setminus\{0\};\ \alpha < \arg z < \beta\}$. For any $\theta \in (\alpha, \beta)$, we write

$$h_f(\theta) = \varlimsup_{r\to+\infty} \frac{\ln|f(re^{i\theta})|}{r}$$

and we call h_f the *indicator function* of f ([38], Ch. I, § 15).

We say that f is of exponential type $a < +\infty$ if a is the greatest lower bound of all $b \in \boldsymbol{R}$ for which there exists a $c_b > 0$ such that:

$$|f(z)| \leqslant c_b e^{b|z|} \quad \text{for all } z \in \boldsymbol{C}\setminus\{0\}, \quad \alpha < \arg z < \beta.$$

Obviously, if f is of exponential type $a < +\infty$, then $\sup_{\alpha<\theta<\beta} h_f(\theta) \leqslant a$.

Moreover, if f is of exponential type $a < +\infty$, then by [38], Ch. I, § 16, a), its indicator function is continuous and by [38], Ch. I, Th. 29, for any $\alpha < \alpha' < \beta'$ $< \beta$ the exponential type of the restriction of f to $\{z \in \boldsymbol{C}\setminus\{0\};\ \alpha' < \arg z < \beta'\}$ is equal to $\sup_{\alpha'<\theta<\beta'} h_f(\theta)$.

Using the "three line theorem" (see [20], Th. VI. 10.3) it is easy to see that if $\beta-\alpha = \pi$ and f is of exponential type $-\infty$ on $\{z \in \boldsymbol{C}\setminus\{0\};\ \alpha < \arg z < \beta\}$, then f vanishes identically.

We recall the following notation:

$$\ln_+ r = \max\{\ln r, 0\}, \quad r \geqslant 0.$$

LEMMA 1.1. *Let $\alpha, \beta \in \boldsymbol{R}$, $\beta-\alpha = \pi$ and consider an analytic function ϱ without zeros on $\{z \in \boldsymbol{C}\setminus\{0\};\ \alpha < \arg z < \beta\}$ such that*

$$a = \varliminf_{r\to+\infty} \frac{\ln|\varrho(re^{i(\alpha+\beta)/2})|}{r} > -\infty$$

and

$$\varliminf_{r\to+\infty} \sup_{\theta} \frac{\ln\ln_+|\varrho(re^{i\theta})|^{-1}}{\ln r} < 2.$$

Then, for any analytic function f of finite exponential type on $\{z \in \boldsymbol{C}\setminus\{0\};\ \alpha < \arg z < \beta\}$ with $h_f\left(\frac{\alpha+\beta}{2}\right) \leqslant a$ and such that $f\varrho^{-1}$ has a continuous extensions τ on $\{z \in \boldsymbol{C};\ \alpha \leqslant \arg z \leqslant \beta\}$ with $|\tau(z)| \leqslant 1$ on the line $\arg z = \alpha$ or β, we have:

(1.1) $$|f(z)| \leqslant |\varrho(z)|, \quad z \in \boldsymbol{C}\setminus\{0\},\ \alpha < \arg z < \beta.$$

Proof. Let $\varepsilon > 0$ be arbitrary; by the formula

$$\varphi(z) = \tau(z) e^{-\varepsilon z e^{-i(\alpha+\beta)/2}}$$

we define a continuous function φ on $\{z \in \boldsymbol{C};\ \alpha \leqslant \arg z \leqslant \beta\}$ which is analytical on $\{z \in \boldsymbol{C}\setminus\{0\};\ \alpha < \arg z < \beta\}$ and such that

$$|\varphi(z)| = |\tau(z)| \leqslant 1 \quad \text{for} \quad \arg z = \alpha \text{ or } \beta.$$

Since

$$\varlimsup_{r\to+\infty} \frac{\ln|\varphi(re^{i(\alpha+\beta)/2})|}{r} \leqslant h_f\left(\frac{\alpha+\beta}{2}\right) - \varliminf_{r\to+\infty} \frac{\ln|\varrho(re^{i(\alpha+\beta)/2})|}{r},$$

we have

$$\sup_{r\geqslant 0} |\varphi(re^{i(\alpha+\beta)/2})| < +\infty.$$

On the other hand, if we choose b such that

$$\max\left\{\varliminf_{r\to+\infty} \sup_{\alpha<\theta<\beta} \frac{\ln\ln_+|\varrho(re^{i\theta})|^{-1}}{\ln r}, 1\right\} < b < 2,$$

there exists a sequence $0 < r_1 < r_2 < \ldots, r_n \to +\infty$, such that

$$|\varrho(z)|^{-1} \leqslant e^{|z|^b}, \quad z \in \boldsymbol{C}\setminus\{0\}, \quad \alpha < \arg z < \beta, |z| = r_n \text{ for some } n.$$

Consequently there exists a constant $c > 0$ such that

$$|\varphi(z)| \leqslant ce^{|z|^b}, \quad z \in \boldsymbol{C}, \quad \alpha \leqslant \arg z \leqslant \beta, \quad |z| = r_n, \text{ for some } n.$$

Applying the Phragmén–Lindelöf principle (see [38], Ch. I, Th. 21) to φ restricted to $\{z \in \boldsymbol{C};\, \alpha \leqslant \arg z \leqslant (\alpha+\beta)/2\}$ respectively to $\{z \in \boldsymbol{C};\, (\alpha+\beta)/2 \leqslant \arg z \leqslant \beta\}$, we conclude that φ is bounded. Again, by the Phragmén–Lindelöf principle it follows that for any $z \in \boldsymbol{C}\setminus\{0\}$, $\alpha < \arg z < \beta$, we have successively

$$|\varphi(z)| \leqslant 1, \quad |\tau(z)| \leqslant e^{\varepsilon|z|}, \quad |f(z)| \leqslant e^{\varepsilon|z|}|\varrho(z)|.$$

Since $\varepsilon > 0$ is arbitrary, (1.1) results. ∎

We remark that there exist deeper results than the above theorem, for which we refer the reader to [37], Lemma 2, and [38], Ch. IX, § 4, Lemma 1.

By the classical Liouville theorem any entire function of exponential type < 0 vanishes identically.

Next we consider an entire function of exponential type zero, which plays a fundamental role in this work.

Definition I. Let $t_1, t_2, \ldots > 0$ be such that $\sum_{k=1}^{\infty} 1/t_k < +\infty$; we define the *entire function* $\omega_{\{t_k\}}$ by

$$\omega_{\{t_k\}}(z) = \prod_{k=1}^{\infty}\left(1+\frac{iz}{t_k}\right).$$

For any integer $n \geqslant 1$ we have

$$\varlimsup_{|z|\to+\infty} \frac{\ln|\omega_{\{t_k\}}(z)|}{|z|} \leqslant \varlimsup_{|z|\to+\infty} \frac{\sum_{k=1}^{n}\ln(1+|z|/t_k)}{|z|} + \varlimsup_{|z|\to+\infty} \frac{|z|\sum_{k=n+1}^{\infty} 1/t_k}{|z|} = \sum_{k=n+1}^{\infty}\frac{1}{t_k},$$

so that $\omega_{\{t_k\}}$ is of exponential type zero. Moreover,

$$\lim_{r\to+\infty}\frac{\ln|\omega_{\{t_k\}}(-ir)|}{r}$$

exists and is equal to 0. We also remark that

$$|\omega_{\{t_k\}}(z)| \geqslant |\omega_{\{t_k\}}(\operatorname{Re} z)| \geqslant 1, \quad z \in \boldsymbol{C}, \quad \operatorname{Im} z \leqslant 0.$$

We recall that for any entire function f we can define another entire function $\bar{f}$ by the formula

$$\bar{f}(z) = \overline{f(\bar{z})}.$$

Theorem 1.2. *Let* $t_1, t_2, \ldots > 0$ *be such that* $\sum_{k=1}^{\infty} 1/t_k < +\infty$, $a, b \in \boldsymbol{R}$ *and* $c > 0$. *If f is a non-identically zero entire function of finite exponential type such that* $h_f(-\pi/2) \leqslant a$, $h_f(\pi/2) \leqslant b$ *and*

$$|f(t)| \leqslant c|\omega_{\{t_k\}}(t)|, \quad t \in \boldsymbol{R},$$

then $a+b \geqslant 0$ *and*

$$|f(z)| \leqslant \begin{cases} ce^{-a\operatorname{Im} z}|\omega_{\{t_k\}}(z)|, & z \in \boldsymbol{C}, \operatorname{Im} z \leqslant 0,\\ ce^{b\operatorname{Im} z}|\omega_{\{t_k\}}(z)|, & z \in \boldsymbol{C}, \operatorname{Im} z \geqslant 0.\end{cases}$$

Proof. Define the entire function ϱ by

$$\varrho(z) = ce^{iaz}\omega_{\{t_k\}}(z).$$

Then

$$\lim_{r\to+\infty}\frac{\ln|\varrho(-ir)|}{r}$$

exists and is equal to a and

$$\varliminf_{r\to+\infty}\ \sup_{-\pi<\theta<0}\frac{\ln\ln_+|\varrho(re^{i\theta})|^{-1}}{\ln r}$$

exists and is equal to 1. By Lemma 1.1 it follows that

$$|f(z)| \leqslant |\varrho(z)| = ce^{-a\operatorname{Im} z}|\omega_{\{t_k\}}(z)|, \quad z \in \boldsymbol{C}, \operatorname{Im} z \leqslant 0.$$

Further we remark that $h_{\bar{f}}(-\pi/2) \leqslant b$ and that

$$|\bar{f}(t)| \leqslant c|\omega_{\{t_k\}}(t)|, \quad t \in \boldsymbol{R}.$$

By a similar reasoning to the above we get

$$|\bar{f}(z)| \leqslant ce^{-b\operatorname{Im} z}|\omega_{\{t_k\}}(z)|, \quad z \in \boldsymbol{C}, \operatorname{Im} z \leqslant 0,$$

that is

$$|f(z)| \leqslant ce^{b\operatorname{Im} z}|\bar{\omega}_{\{t_k\}}(z)|, \quad z \in \boldsymbol{C}, \operatorname{Im} z \geqslant 0.$$

Finally, we suppose that $a+b < 0$; then we can find a $d \in \boldsymbol{R}$ such that $a < d$ and $d+b < 0$. Consider the entire function g defined by

$$g(z) = e^{-idz}f(z).$$

Then

$$|g(z)| \leqslant \begin{cases} ce^{(d-a)\operatorname{Im} z}|\omega_{\{t_k\}}(z)|, & z \in \boldsymbol{C}, \operatorname{Im} z \leqslant 0,\\ ce^{(d+b)\operatorname{Im} z}|\bar{\omega}_{\{t_k\}}(z)|, & z \in \boldsymbol{C}, \operatorname{Im} z \geqslant 0.\end{cases}$$

Since $\omega_{\{t_k\}}$ is of exponential type zero and $d-a > 0$, $d+b < 0$, it follows that g is bounded on any set of the form

$$\{z \in \boldsymbol{C};\ \pi/2-\varepsilon \leqslant \arg z \leqslant \pi/2+\varepsilon\} \cup \{z \in \boldsymbol{C};\ -\pi/2-\varepsilon \leqslant \arg z \leqslant -\pi/2+\varepsilon\}$$

where $0 < \varepsilon < \pi/2$. By the Phragmén–Lindelöf principle it follows that g is bounded and by the classical Liouville theorem we infer that g is constant. Since $\lim_{r\to+\infty}|g(ir)| = 0$, g vanishes identically, in contradiction to our hypothesis on f. ∎

Having established how inequalities are preserved by analytical extension, we next look for situations in which inequalities are preserved by derivation. In this direction the strongest result is given by the following theorem of B. Ya. Levin (see [37], Th. 4.5 and Th. 4.6, or [38], Ch. IX, § 4, Th. 11 and Th. 12):

Theorem 1.3. *Let ϱ be an entire function of exponential type* $a < +\infty$ *without zeros in* $\{z \in \boldsymbol{C};\, \operatorname{Im} z < 0\}$ *and such that* $h_\varrho(\pi/2) \leqslant h_\varrho(-\pi/2)$.

If f is an entire function of exponential type $\leqslant a$ *such that*

$$|f(t)| \leqslant |\varrho(t)|, \quad t \in \boldsymbol{R},$$

then for each integer $n \geqslant 0$ *we have*

$$|f^{(n)}(t)| \leqslant |\varrho^{(n)}(t)|, \quad t \in \boldsymbol{R}.$$

A consequence of this theorem is the following result of S. N. Bernstein (see [5]):

COROLLARY 1.4. *Let f be an entire function of exponential type $a < +\infty$ and $c \geqslant 0$ such that*

$$|f(t)| \leqslant c, \quad t \in \mathbf{R}.$$

Then for any integer $n \geqslant 0$

$$|f^{(n)}(t)| \leqslant a^n c, \quad t \in \mathbf{R}.$$

Next we shall characterize situations in which the module of a function is majorized on the real line by the module of a certain $\omega_{\{t_k\}}$ or by a transform of the module of $\omega_{\{t_k\}}$, which was considered by É. Borel and which we call the Borel transform of $|\omega_{\{t_k\}}|$.

For this purpose we need the following result on convergent series (compare with [32], Satz 80 and [51], Ch. II, Section 1, Lemma 1):

LEMMA 1.5. (i) *If $a_1, a_2, \ldots \geqslant 0$, $\sum_{k=1}^{\infty} a_k < +\infty$, then $a_k \to 0$ and there exist $b_1, b_2, \ldots > 0$, $\sum_{k=1}^{\infty} b_k < +\infty$, with $a_k/b_k \to 0$.*

(ii) *If $a_1 \geqslant a_2 \geqslant \ldots \geqslant 0$, $\sum_{k=1}^{\infty} a_k < +\infty$, then $ka_k \to 0$ and there exist $b_1 \geqslant b_2 \geqslant \ldots > 0$, $\sum_{k=1}^{\infty} b_k < +\infty$, with $a_k/b_k \to 0$.*

(iii) *If $a_1 \geqslant 2a_2 \geqslant 3a_3 \geqslant \ldots \geqslant 0$, $\sum_{k=1}^{\infty} a_k < +\infty$, then $\left(1+\frac{1}{2}+\ldots+\frac{1}{k}\right)ka_k \to 0$ and there exist $b_1 \geqslant 2b_2 \geqslant 3b_3 \geqslant \ldots > 0$, $\sum_{k=1}^{\infty} b_k < +\infty$ with $a_k/b_k \to 0$.*

Proof. (i) This is well known; for example we can take

$$b_k = \begin{cases} (a_k+a_{k+1}+\ldots)^{1/2}-(a_{k+1}+a_{k+2}+\ldots)^{1/2} & \text{if} \quad a_k > 0, \\ 2^{-k} & \text{if} \quad a_k = 0. \end{cases}$$

(ii) Let $\varepsilon > 0$; then there exists an n_ε such that $\sum_{n>n_\varepsilon} a_n \leqslant \varepsilon/2$ and there exists a $k_\varepsilon > n_\varepsilon$ such that $n_\varepsilon a_k \leqslant \varepsilon/2$ whenever $k \geqslant k_\varepsilon$. For every $k \geqslant k_\varepsilon$, we have

$$(k-n_\varepsilon)a_k \leqslant \sum_{n=n_\varepsilon+1}^{k} a_n \leqslant \sum_{n>n_\varepsilon} a_n \leqslant \varepsilon/2;$$

so

$$ka_k \leqslant \varepsilon/2+n_\varepsilon a_k \leqslant \varepsilon.$$

Consequently, $ka_k \to 0$.

Define $c_1, c_2, \ldots \geqslant 0$, $\sum_{k=1}^{\infty} c_k < +\infty$, by

$$c_k = k(a_k-a_{k+1}).$$

Then

$$a_k = \sum_{n=k}^{\infty} c_n/n.$$

By (i) there exist $d_1, d_2, \ldots > 0$, $\sum_{k=1}^{\infty} d_k < +\infty$, and $c_k/d_k \to 0$. Taking

$$b_k = \sum_{n=k}^{\infty} d_n/n,$$

it is easy to verify that $b_1 \geqslant b_2 \geqslant \ldots > 0$, $\sum_{k=1}^{\infty} b_k < +\infty$ and $a_k/b_k \to 0$.

(iii) Let $\varepsilon > 0$; then there exists an n_ε such that $\sum_{n>n_\varepsilon} a_n \leqslant \varepsilon/2$ and by (ii) there exists a $k_\varepsilon > n_\varepsilon$ such that $(1+1/2+\ldots+1/n_\varepsilon)ka_k \leqslant \varepsilon/2$ whenever $k \geqslant k_\varepsilon$. On the other hand, for $k \geqslant k_\varepsilon$ we have

$$\left(\frac{1}{n_\varepsilon+1}+\ldots+\frac{1}{k}\right)ka_k \leqslant \sum_{n=n_\varepsilon+1}^{k} a_n \leqslant \sum_{n>n_\varepsilon} a_n \leqslant \frac{\varepsilon}{2};$$

hence

$$\left(1+\frac{1}{2}+\ldots+\frac{1}{k}\right)ka_k \leqslant \frac{\varepsilon}{2}+\left(1+\frac{1}{2}+\ldots+\frac{1}{n_\varepsilon}\right)ka_k \leqslant \varepsilon.$$

Consequently, $(1+1/2+\ldots+1/k)ka_k \to 0$.

Define $c_1, c_2, \ldots \geqslant 0$, $\sum_{k=1}^{\infty} c_k < +\infty$ by

$$c_k = \left(1+\frac{1}{2}+\ldots+\frac{1}{k}\right)[ka_k-(k+1)a_{k+1}].$$

Then

$$a_k = \frac{1}{k}\sum_{n=k}^{\infty} \frac{c_n}{1+1/2+\ldots+1/n}.$$

By (i) there exist $d_1, d_2, \ldots > 0$, $\sum_{k=1}^{\infty} d_k < +\infty$ with $c_k/d_k \to 0$. If we put

$$b_k = \frac{1}{k}\sum_{n=k}^{\infty} \frac{d_n}{1+1/2+\ldots+1/n},$$

we get $b_1 \geqslant 2b_2 \geqslant 3b_3 \geqslant \ldots > 0$, $\sum_{k=1}^{\infty} b_k < +\infty$ and $a_k/b_k \to 0$. ∎

The following result is a combination of [30], Th. 1, and [51], Ch. II, Section 1, Lemma 2:

THEOREM 1.6. *For any f: $[1, +\infty) \to [1, +\infty)$ the following statements are equivalent:*

(i) *f is bounded on every compact subset of* $[1,+\infty)$ *and*

$$\int_1^\infty \frac{\ln(\sup_{1\leqslant s\leqslant t} f(s))}{t^2}\,dt < +\infty;$$

(ii) *there exist* $0<t_1\leqslant t_2\leqslant \ldots,\ \sum_{k=1}^{\infty} 1/t_k < +\infty$ *and* $c>0$ *such that*

$$f(t)\leqslant c\sup_{k\geqslant 1}\frac{t^k}{t_1 t_2\ldots t_k},\qquad t\in[1,+\infty);$$

(iii) *there exist* $t_1,t_2,\ldots>0,\ \sum_{k=1}^{\infty} 1/t_k<+\infty$ *and* $c>0$ *such that*

$$f(t)\leqslant c|\omega_{\{t_k\}}(t)|,\qquad t\in[1,+\infty).$$

Proof. (i) $\Rightarrow$ (ii). Let us define g: $[1,+\infty)\to[1,+\infty)$ by

$$g(t)=\Big(\int_1^2\big(\sup_{1\leqslant s\leqslant r} f(s)\big)dr\Big)^{-1} t\int_t^{t+1}\big(\sup_{1\leqslant s\leqslant r} f(s)\big)dr.$$

Then

$$f(t)\leqslant\Big(\int_1^2\big(\sup_{1\leqslant s\leqslant r} f(s)\big)dr\Big)g(t),\qquad t\in[1,+\infty).$$

Moreover, g is continuous, strictly increasing, $g(1)=1$, $\lim_{t\to+\infty} g(t)=+\infty$ and

$$\int_1^\infty \frac{\ln g(t)}{t^2}\,dt<+\infty.$$

For every integer $n\geqslant 1$ we define $a_n\in[1,+\infty)$ by the equation $g(a_n)=e^{n-1}$; then we have

$$\frac{n-1}{a_n}=(n-1)\int_{a_n}^{+\infty}\frac{1}{t^2}\,dt\leqslant\int_{a_n}^{+\infty}\frac{\ln g(t)}{t^2}\,dt,\qquad\text{for}\quad n\geqslant 1.$$

Since $a_n\to+\infty$, it follows that $n/a_n\to 0$. On the other hand, for each $n\geqslant 1$ we have

$$\sum_{k=1}^{n}\frac{1}{a_k}-\frac{n-1}{a_{n+1}}=\frac{1}{a_1}+\sum_{k=2}^{n}(k-1)\Big(\frac{1}{a_k}-\frac{1}{a_{k+1}}\Big)$$
$$=\frac{1}{a_1}+\sum_{k=2}^{n}(k-1)\int_{a_k}^{a_{k+1}}\frac{1}{t^2}\,dt\leqslant\frac{1}{a_1}+\int_1^{+\infty}\frac{\ln g(t)}{t^2}\,dt.$$

Hence

$$\sum_{k=1}^{\infty}\frac{1}{a_k}\leqslant\frac{1}{a_1}+\int_1^{+\infty}\frac{\ln g(t)}{t^2}\,dt<+\infty.$$

Since the sequence $\{1/a_k\}$ is decreasing, by Lemma 1.5 (ii) there exist $0<t_1\leqslant t_2\leqslant\ldots,\ \sum_{k=1}^{\infty}1/t_k<+\infty$, such that

$$\lim_{k\to\infty}\frac{t_k}{a_k}=0.$$

Let $t\in[1,+\infty)$; there exists a unique $n\geqslant 1$ such that $a_n\leqslant t<a_{n+1}$. We have

$$g(t)<g(a_{n+1})=e^n,$$

and

$$\sup_{k\geqslant 1}\frac{t^k}{t_1\ldots t_k}\geqslant\frac{t^n}{t_1\ldots t_n}\geqslant\frac{a_1\ldots a_n}{t_1\ldots t_n}.$$

Hence for each $t\in[1,+\infty)$

$$\frac{g(t)}{\sup_{k\geqslant 1}\frac{t^k}{t_1\ldots t_k}}\leqslant\sup_{n\geqslant 1}\prod_{k=1}^{n}\frac{et_k}{a_k}.$$

Since $t_k/a_k\to 0$, we have

$$\sup_{n\geqslant 1}\prod_{k=1}^{n}\frac{et_k}{a_k}<+\infty,$$

and so, for $t\in[1,+\infty)$ we get

$$f(t)\leqslant\Big(\int_1^2\big(\sup_{1\leqslant s\leqslant r} f(s)\big)dr\Big)\Big(\sup_{n\geqslant 1}\prod_{k=1}^{n}\frac{et_k}{a_k}\Big)\sup_{k\geqslant 1}\frac{t^k}{t_1\ldots t_k}.$$

(ii) $\Rightarrow$ (iii). This statement results from the obvious inequality

$$\sup_{k\geqslant 1}\frac{t^k}{t_1\ldots t_k}\leqslant|\omega_{\{t_k\}}(t)|,\qquad t\in[1,+\infty).$$

(iii) $\Rightarrow$ (i). We get this last implication from the computations

$$\int_1^\infty\frac{\ln|\omega_{\{t_k\}}(t)|}{t^2}\,dt=\frac{1}{2}\sum_{k=1}^{\infty}\int_1^\infty\frac{\ln\Big(1+\frac{t^2}{t_k}\Big)}{t^2}\,dt$$
$$=\frac{1}{2}\sum_{k=1}^{\infty}\Big[\ln\Big(1+\frac{1}{t_k^2}\Big)+2\int_1^\infty\frac{dt}{t_k^2+t^2}\Big]\leqslant\frac{1}{2}\sum_{k=1}^{\infty}\Big[\frac{1}{t_k^2}+\frac{\pi}{t_k}\Big]<+\infty.\ \blacksquare$$

Let $t_1, t_2, \ldots > 0$ be such that $\sum_{k=1}^{\infty} 1/t_k < +\infty$; then for f defined by

$$f(t) = \omega_{\{t_k\}}(-it) = \prod_{k=1}^{\infty}\left(1+\frac{t}{t_k}\right),$$

we have

$$\int_1^{+\infty} \frac{\ln f(t)}{t^2}\,dt = \sum_{k=1}^{\infty}\int_1^{+\infty}\frac{\ln\left(1+\frac{t}{t_k}\right)}{t^2}\,dt = \sum_{k=1}^{\infty}\left[\ln\left(1+\frac{1}{t_k}\right)+\frac{\ln(1+t_k)}{t_k}\right];$$

hence

$$\sum_{k=1}^{\infty}\frac{\ln t_k}{t_k} \leqslant \int_1^{+\infty}\frac{\ln f(t)}{t^2}\,dt \leqslant \sum_{k=1}^{\infty}\frac{1}{t_k}+\sum_{k=1}^{\infty}\frac{\ln 2}{t_k}+\sum_{t_k\geqslant 1}\frac{\ln t_k}{t_k}.$$

Consequently $t \to \omega_{\{t_k\}}(-it) = \prod_{k=1}^{\infty}(1+t/t_k)$ satisfies the equivalent conditions from Theorem 1.6 if and only if

$$\sum_{k=1}^{\infty}\frac{\ln t_k}{t_k} < +\infty.$$

Next we give a similar result to the above, concerning a refined estimation. For this purpose we need the following

LEMMA 1.7. *Let* $0 < t_1 \leqslant t_2/2 \leqslant t_3/3 \leqslant \ldots$, $\sum_{k=1}^{\infty} 1/t_k < +\infty$ *and consider the functions* α: $[0, +\infty) \to [1, +\infty)$, β: $(0, +\infty) \to [1, +\infty)$ *defined by*

$$\alpha(t) = 1+\sum_{k=1}^{\infty}\frac{t^k}{t_1\ldots t_k}, \qquad \beta(t) = \sup_{s\geqslant t}\alpha(s)^{t/s}.$$

Then α *is submultiplicative,* $(0, +\infty) \ni t \to \beta(t)^{1/t}$ *is decreasing and*

$$\max\left\{1, \sup_{k\geqslant 1}\frac{t^k}{t_1\ldots t_k}\right\} \leqslant \alpha(t) \leqslant \beta(t) \leqslant 4\max\left\{1, \left(\sup_{k\geqslant 1}\frac{(2t)^k}{t_1\ldots t_k}\right)^2\right\}, \qquad t > 0. \tag{1.2}$$

Moreover

$$\max\left\{1, \sup_{k\geqslant 1}\frac{t^k}{t_1\ldots t_k}\right\} \leqslant |\omega_{\{t_k\}}(t)| \leqslant 3\max\left\{1, \left(\sup_{k\geqslant 1}\frac{(4t)^k}{t_1\ldots t_k}\right)^2\right\}, \qquad t > 0. \tag{1.3}$$

Proof. Denote $a_0 = 1$ and $a_k = 1/t_1\ldots t_k$, $k \geqslant 1$. Since the sequence $\{t_k/k\}$ is increasing, for any $p, q \geqslant 0$ we have

$$\binom{p+q}{p}a_{p+q} \leqslant a_p a_q.$$

Hence for all $t, s \in [0, +\infty)$,

$$\alpha(t+s) = \sum_{k=0}^{\infty}\sum_{p+q=k}\binom{k}{p}a_k t^p s^q \leqslant \sum_{k=0}^{\infty}\sum_{p+q=k} a_p t^p a_q s^q = \alpha(t)\alpha(s),$$

that is α is submultiplicative.

It is clear that $(0, +\infty) \ni t \to \beta(t)^{1/t} = \sup_{s\geqslant t}\alpha(s)^{1/s}$ is decreasing. Let $t > 0$; then, for every $s \geqslant t$, there exist an integer $p \geqslant 1$ and $0 \leqslant r < t$ such that $s = pt+r$ and by the submultiplicativity of α it follows that

$$\alpha(s)^{t/s} = \alpha(pt+r)^{t/s} \leqslant \alpha(t)^{pt/s}\alpha(r)^{t/s} \leqslant \alpha(t)\alpha(r) \leqslant \alpha(t)^2.$$

Thus

$$\beta(t) \leqslant \alpha(t)^2,$$

and using the estimations

$$\max\left\{1, \sup_{k\geqslant 1}\frac{t^k}{t_1\ldots t_k}\right\} \leqslant \alpha(t) = 1+\sum_{k=1}^{\infty}2^{-k}\frac{(2t)^k}{t_1\ldots t_k} \leqslant 2\max\left\{1, \sup_{k\geqslant 1}\frac{(2t)^k}{t_1\ldots t_k}\right\},$$

we easily obtain (1.2).

Further, for each $\lambda > 0$ we denote by $n(\lambda)$ the number of all t_k with $t_k \leqslant \lambda$ and write

$$N(s) = \ln\max\left\{1, \sup_{k\geqslant 1}\frac{s^k}{t_1\ldots t_k}\right\}, \qquad s > 0.$$

Since $\{t_k\}$ is increasing, for $s > 0$ we have:

$$N(s) = \max\left\{0, \sup_{k\geqslant 1}\sum_{p=1}^{k}\ln\frac{s}{t_p}\right\} = \sum_{p=1}^{n(s)}\ln\frac{s}{t_p} = \int_0^s \ln\frac{s}{\lambda}\,dn(\lambda) = \int_0^s\frac{n(\lambda)}{\lambda}\,d\lambda.$$

By Theorem 1.6 $\int_1^{+\infty}\frac{N(s)}{s^2}\,ds < +\infty$ and by the above relation between $n(\lambda)$ and $N(\lambda)$,

$$\int_1^{+\infty}\frac{n(\lambda)}{\lambda^2}\,d\lambda = \int_1^{+\infty}\left(\int_\lambda^{+\infty}\frac{ds}{s^2}\right)\frac{n(\lambda)}{\lambda}\,d\lambda = \int_1^{+\infty}\left(\int_1^s\frac{n(\lambda)}{\lambda}\,d\lambda\right)\frac{ds}{s^2} \leqslant \int_1^{\infty}\frac{N(s)}{s^2}\,ds < +\infty.$$

Hence $\lim_{s\to+\infty}\frac{N(s)}{s^2} = 0$.

Indeed, otherwise there would exist $\varepsilon > 0$ and $1 \leqslant s_1 \leqslant s_2/2 \leqslant s_3/3 \leqslant \ldots$, with $N(s_j)/s_j \geqslant \varepsilon$ for all j, so that

$$\int_1^{+\infty}\frac{N(s)}{s^2}\,ds \geqslant \sum_{j=1}^{\infty}\int_{s_j}^{2s_j}\frac{N(s)}{s^2}\,ds \geqslant \sum_{j=1}^{\infty}\frac{N(s_j)}{2s_j} = +\infty.$$

Similarly, $\lim_{\lambda\to+\infty}\frac{n(\lambda)}{\lambda} = 0$.

(For the above facts concerning $n(\lambda)$ and $N(\lambda)$ we refer the reader to [45], 1.8; see also [34], §§ 3, 4.)

Consequently, by partial integration we obtain for all $t>0$:

$$\ln|\omega_{\{t_k\}}(t)| = \frac{1}{2}\int_0^{+\infty} \ln\left(1+\frac{t^2}{\lambda^2}\right) dn(\lambda)$$

$$= t^2 \int_0^{+\infty} \frac{n(\lambda)}{(t^2+\lambda^2)}\, d\lambda = 2t^2 \int_0^{+\infty} \frac{sN(s)}{(t^2+s^2)^2}\, ds$$

$$\leqslant 2t^2 N(t)\int_0^t \frac{s}{(t^2+s^2)^2}\, ds + 2t^2 \sup_{s\geqslant t}\frac{N(s)}{s}\int_t^{+\infty}\frac{s^2}{(t^2+s^2)^2}\, ds$$

$$= 2N(t)\int_0^1 \frac{r}{(1+r^2)^2}\, dr + 2t\sup_{s\geqslant t}\frac{N(s)}{s}\int_1^{+\infty}\frac{r^2}{(1+r^2)^2}\, dr$$

$$= \frac{1}{2}N(t) + \frac{\pi+2}{4}\, t\sup_{s\geqslant t}\frac{N(s)}{s}.$$

But using the submultiplicativity of α, we have

$$\sup_{s\geqslant t}\frac{N(s)}{s} \leqslant \sup_{s\geqslant t}\frac{\ln\alpha(s)}{s} = \sup_{t\leqslant s<2t}\frac{\ln\alpha(s)}{s} \leqslant \frac{\ln\alpha(2t)}{t} \leqslant \frac{\ln 2+N(4t)}{t},$$

so

$$\ln|\omega_{\{t_k\}}(t)| \leqslant \frac{1}{2}N(t) + \frac{\pi+2}{4}(\ln 2+N(4t)) \leqslant \ln 3 + 2N(4t), \qquad t>0,$$

that is

$$|\omega_{\{t_k\}}(t)| \leqslant 3\max\left\{1, \left(\sup_{k\geqslant 1}\frac{(4t)^k}{t_1\ldots t_k}\right)^2\right\}, \qquad t>0.$$

On the other hand, it is obvious that

$$\max\left\{1, \sup_{k\geqslant 1}\frac{t^k}{t_1\ldots t_k}\right\} \leqslant |\omega_{\{t_k\}}(t)|, \qquad t>0,$$

and thus (1.3) is proved. ■

The following result extends [30], Th. 2:

THEOREM 1.8. *For any $f\colon [1,+\infty)\to[1,+\infty)$ the following statements are equivalent*:

(i) $\sup\limits_{s\geqslant 1} f(s)^{1/s} < +\infty$ *and* $\displaystyle\int_1^{+\infty}\frac{\ln\left(\sup_{s\geqslant t} f(s)^{t/s}\right)}{t^2}\, dt < +\infty$;

(ii) *there exist* $0<t_1\leqslant t_2/2\leqslant t_3/3\leqslant\ldots$, $\sum\limits_{k=1}^{\infty} 1/t_k < +\infty$ *and* $c>0$ *such that*

$$f(t) \leqslant c\sup_{k\geqslant 1}\frac{t^k}{t_1\ldots t_k}, \qquad t\in[1,+\infty);$$

(iii) *there exists a Lebesgue measurable submultiplicative function* $\alpha\colon [0,+\infty)\to$ $\to[1,+\infty)$, *bounded on each compact subset of* $[0,+\infty)$ *and with* $\displaystyle\int_1^{+\infty}\frac{\ln\alpha(t)}{t^2}\,dt$ $<+\infty$, *and* $c>0$ *such that*

$$f(t)\leqslant c\alpha(t), \qquad t\in[1,+\infty);$$

(iv) *there exist* $0<t_1\leqslant t_2/2\leqslant t_3/3\leqslant\ldots$, $\sum\limits_{k=1}^{\infty} 1/t_k<+\infty$ *and* $c>0$ *such that*

$$f(t)\leqslant c|\omega_{\{t_k\}}(t)|, \qquad t\in[1,+\infty).$$

Proof. (i) ⇒ (ii). Define $g\colon [0,+\infty)\to[1,+\infty)$ by

$$g(t) = \sup_{s\geqslant\max\{t,1\}} f(s)^{t/s}, \qquad t\geqslant 0.$$

Then $(0,+\infty)\ni t\mapsto g(t)^{t/s}$ is decreasing; in particular it is Lebesgue measurable and hence g is also Lebesgue measurable.

For any $0<t\leqslant s<+\infty$ we have

$$g(t+s)^{1/(t+s)} \leqslant g(s)^{1/s} \leqslant g(t)^{1/t},$$

so that

$$g(t+s)\leqslant g(s)^{(t+s)/s} = g(s)^{t/s}g(s) \leqslant g(t)g(s).$$

Thus g is submultiplicative. Since $g(t)\leqslant\left(\sup\limits_{s\geqslant 1} f(s)^{1/s}\right)^t$, g is bounded on each compact subset of $[0,+\infty)$.

By our assumptions on f, we have

$$\int_1^{+\infty}\frac{\ln g(t)}{t^2}\,dt < +\infty,$$

and obviously $f(t)\leqslant g(t)$, $t\in[1,+\infty)$.

For every integer $k\geqslant 2$, we have

$$\frac{\ln g(k)}{k}\leqslant\frac{\ln g(t)}{t}, \qquad t\in[k-1,k);$$

hence

$$\sum_{k=1}^{\infty}\frac{\ln g(k)}{k^2} \leqslant \ln g(1) + \sum_{k=2}^{\infty}\int_{k-1}^{k}\frac{\ln g(t)}{t^2}\,dt = \ln g(1) + \int_1^{+\infty}\frac{\ln g(t)}{t^2} < +\infty.$$

It is easy to verify that the sequence $\left\{k\frac{\ln[(k+1)g(k)]}{k^2}\right\}$ decreases, so that by applying Lemma 1.5(iii) with

$$a_k = \frac{\ln[(k+1)g(k)]}{k^2},$$

it follows that there exist $0 < t_1 \leqslant t_2/2 \leqslant t_3/3 \leqslant \ldots, \sum_{k=1}^{\infty} 1/t_k < +\infty$, such that

$$\lim_{k\to\infty} t_k \frac{\ln[(k+1)g(k)]}{k^2} = 0.$$

Write $a = \sup_{0\leqslant s\leqslant 1}(s+1)g(s) \in (1, +\infty)$ and let $t \in [a^2, +\infty)$. Then there exists an integer $n \geqslant 1$ such that

$$n\ln a^2 \leqslant \ln[(t+1)g(t)] < (n+1)\ln a^2.$$

If $1 \leqslant k \leqslant n$ and $t = k(p+\theta)$, where $p \geqslant 0$ is an integer and $0 \leqslant \theta < 1$, then by the submultiplicativity of g

$$(t+1)g(t) \leqslant [(k+1)g(k)]^p[(\theta+1)g(\theta)]^k \leqslant [(k+1)g(k)]^{t/k}a^k.$$

So we have

$$\frac{\ln[(t+1)g(t)]}{t} \leqslant \frac{\ln[(k+1)g(k)]}{k} + \frac{k\ln a^2}{2t} \leqslant \frac{\ln[(k+1)g(k)]}{k} + \frac{\ln[(t+1)g(t)]}{2t},$$

and we obtain

$$\frac{\ln[(t+1)g(t)]}{2t} \leqslant \frac{\ln[(k+1)g(k)]}{k}.$$

It follows that

$$\prod_{k=1}^{n} \frac{\ln[(k+1)g(k)]}{k} \geqslant \left(\frac{\ln[(t+1)g(t)]}{2t}\right)^n \geqslant \left(\frac{n\ln a}{t}\right)^n.$$

Thus

$$\sup_{k\geqslant 1}\frac{t^k}{t_1\ldots t_k} \geqslant \frac{t^n}{t_1\ldots t_n}$$
$$= \left(\prod_{k=1}^{n}\frac{k^2}{t_k\ln[(k+1)g(t)]}\right)\left(\prod_{k=1}^{n}\frac{\ln[(k+1)g(t)]}{k}\right)\frac{t^n}{n!}$$
$$\geqslant \left(\prod_{k=1}^{n}\frac{k^2}{t_k\ln[(k+1)g(t)]}\right)\frac{(n\ln a)^n}{n!}.$$

Consequently

$$\frac{g(t)}{\sup_{k\geqslant 1}\frac{t^k}{t_1\ldots t_k}} \leqslant \left(\prod_{k=1}^{n}\frac{t_k\ln[(k+1)g(k)]}{k^2}\right)\frac{n!}{(n\ln a)^n}a^{2n+2}$$
$$\leqslant a^2\prod_{k=1}^{n}\frac{a^2 t_k\ln[(k+1)g(k)]}{k^2\ln a}.$$

We conclude that writing

$$c = \max\left\{t_1 \sup_{1\leqslant s\leqslant a^2} g(s),\, a^2\sup_{n\geqslant 1}\prod_{k=1}^{n}\frac{a^2 t_k\ln[(k+1)g(k)]}{k^2\ln a}\right\} < +\infty,$$

we have

$$f(t) \leqslant g(t) \leqslant c\sup_{k\geqslant 1}\frac{t^k}{t_1\ldots t_k}, \qquad t\in[1, +\infty).$$

(ii) $\Rightarrow$ (iii). This follows immediately by using Lemma 1.7 and Theorem 1.6.

(iii) $\Rightarrow$ (i). Fix $t \geqslant 1$; for each $s \geqslant t$, there exist an integer $p \geqslant 1$ and $0 \leqslant r < t$ such that $s = pt+r$ and we have

$$f(s)^{t/s} \leqslant (1+c)\,\alpha(pt+r)^{t/s} \leqslant (1+c)\,\alpha(t)^{pt/s}\alpha(r)^{t/s} \leqslant (1+c)\,\alpha(t)^2.$$

Hence

$$\sup_{s\geqslant 1} f(s)^{t/s} \leqslant (1+c)\,\alpha(s)^2 < +\infty$$

and

$$\int_1^{+\infty}\frac{\ln(\sup_{s\geqslant t} f(s)^{t/s})}{t^2}\,dt \leqslant \int_1^{+\infty}\frac{\ln(1+c)}{t^2}\,dt + 2\int_1^{+\infty}\frac{\ln\alpha(t)}{t^2}\,dt.$$

Clearly (ii) $\Rightarrow$ (iv).

Finally, assume that (iv) is satisfied. Then by Lemma 1.7 there exists a $d > 0$ such that

$$f(t) \leqslant d\left(\sup_{k\geqslant 1}\frac{(4t)^k}{t_1\ldots t_k}\right)^2, \qquad t\in[1, +\infty).$$

Consider the sequence $0 < s_1 \leqslant s_2/2 \leqslant s_3/3 \leqslant \ldots, \sum_{k=1}^{\infty} 1/s_k < +\infty$ defined by

$$s_k = \frac{k}{p}\cdot\frac{t_p}{8}, \qquad 2(p-1) < k \leqslant 2p.$$

Then

$$f(t) \leqslant d\sup_{k\geqslant 1}\frac{t^k}{s_1\ldots s_k}, \qquad t\in[1, +\infty).$$

Hence (iv) $\Rightarrow$ (ii). ∎

Corollary 1.9. *Let $t_1, t_2, \ldots > 0$, $\sum_{k=1}^{\infty} 1/t_k < +\infty$; then the following statements are equivalent:*

(i) $\sum_{k=1}^{\infty}\frac{\ln t_k}{t_k} < +\infty$;

(ii) *$[1, +\infty) \ni t \to \omega_{\{t_k\}}(-it)$ satisfies the equivalent conditions from Theorem* 1.6;

(iii) *$[1, +\infty) \ni t \to \omega_{\{t_k\}}(-it)$ satisfies the equivalent conditions from Theorem* 1.8.

Proof. (i) $\Leftrightarrow$ (ii). This equivalence is proved in the remark after Theorem 1.6.

(ii) $\Leftrightarrow$ (iii). Since for any k the function $s \to \left(1+\frac{s}{t_k}\right)^{1/s}$ is decreasing, we have

$$\sup_{s>t} \omega_{\{t_k\}}(-is)^{t/s} = \omega_{\{t_k\}}(-it), \quad t > 0.$$

Hence the conditions (i) of Theorem 1.6 and Theorem 1.8 are equivalent. ∎

Let us now assume additionally that $t_1 \leqslant t_2 \leqslant \ldots$ By Lemma 1.5(ii), $\lim_{k\to\infty} (k/t_k) = 0$, so that the set $\{k; k > t_k\}$ is finite and we have

$$\sum_{k=1}^{\infty} \frac{\ln k}{t_k} \leqslant \sum_{k>t_k} \frac{\ln k}{t_k} + \sum_{t_k \geqslant 1} \frac{\ln t_k}{t_k}.$$

On the other hand, for $k \geqslant 2$ we have either $t_k < k^2$, that is

$$\frac{\ln t_k}{t_k} < 2\frac{\ln k}{t_k},$$

or $t_k \geqslant k^2$, that is

$$\frac{\ln t_k}{t_k} \leqslant 2\frac{\ln k}{k^2};$$

hence

$$\sum_{k=2}^{\infty} \frac{\ln t_k}{t_k} \leqslant 2\sum_{k=2}^{\infty} \frac{\ln k}{t_k} + 2\sum_{k=2}^{\infty} \frac{\ln k}{k^2}.$$

Hence condition (i) from the above corollary is equivalent to

$$\sum_{k=1}^{\infty} \frac{\ln k}{t_k} < +\infty.$$

DEFINITION II. Let τ: $[0, +\infty) \to [1, +\infty)$ be a Lebesgue measurable function, bounded on each compact subset of $[0, +\infty)$ and such that $\lim_{t\to+\infty} \frac{\ln \tau(t)}{t} = 0$. We call the *Borel transform of* τ the function τ_{Borel}: $(0, +\infty) \to [1, +\infty)$ defined by

$$\tau_{\mathrm{Borel}}(t) = t\int_0^{+\infty} \tau(ts)e^{-s}ds = \int_0^{+\infty} \tau(s)e^{-s/t}ds.$$

We remark that if τ can be extended to an entire function $z \to \sum_{k=0}^{\infty} c_k z^k$ of exponential type zero, then τ_{Borel} can be extended to the entire function $z \to \sum_{k=0}^{\infty} k!\, c_k z^{k+1}$ (for more details see [38], Ch. I, § 20 and [18], 5 Kapitel, § 5).

Concerning the Borel transform we next give a result extending [27], Lemma 1:

THEOREM 1.10. *For any f: $[1, +\infty) \to [e, +\infty)$ the following statements are equivalent*:

(i) *f is bounded on each compact subset of $[1, +\infty)$ and*

$$\int_1^{+\infty} \frac{\ln\ln(\sup_{1\leqslant s\leqslant t} f(s))}{t^2} dt < +\infty;$$

(ii) *there exist* $0 < t_1 \leqslant t_2/2 \leqslant t_3/3 \leqslant \ldots, \sum_{k=1}^{\infty} 1/t_k < +\infty$ *and* $c > 0$ *such that*

$$f(t) \leqslant c \sup_{k\geqslant 1} \frac{k!\,t^{k+1}}{t_1 \ldots t_k}, \quad t \in [1, +\infty);$$

(iii) *there exists a Lebesgue measurable submultiplicative function* α: $[0, +\infty) \to [e, +\infty)$, *bounded on each compact subset of* $[0, +\infty)$ *and with* $\int_1^{+\infty} \frac{\ln \alpha(t)}{t^2} dt < \infty$ *and* $c > 0$ *such that*

$$f(t) \leqslant c\alpha_{\mathrm{Borel}}, \quad t \in [1, +\infty);$$

(iv) *there exist* $0 < t_1 \leqslant t_2/2 \leqslant t_3/3 \leqslant \ldots, \sum_{k=1}^{\infty} 1/t_k < +\infty$ *and* $c > 0$ *such that*

$$f(t) \leqslant c|\omega_{\{t_k\}}|_{\mathrm{Borel}}(t), \quad t \in [1, +\infty).$$

Proof. (i) $\Rightarrow$ (ii). Define g: $[1, +\infty) \to [e, +\infty)$ by

$$g(t) = e\left(\int_1^2 (\sup_{1\leqslant s\leqslant r} f(s))dr\right)^{-1} t\int_t^{t+1} (\sup_{1\leqslant s\leqslant r} f(s))dr.$$

Then

$$f(t) \leqslant \left(\int_1^2 (\sup_{1\leqslant s\leqslant r} f(s))dr\right) g(t), \quad t \in [1, +\infty).$$

Moreover, g is continuous, strictly increasing, $g(s) = e$, $\lim_{t\to+\infty} g(t) = +\infty$ and

$$\int_1^{+\infty} \frac{\ln\ln g(t)}{t^2} dt < +\infty.$$

For each integer $n \geqslant 1$ we define $a_n \in [1, +\infty)$ by $g(a_n) = e^n$; then

$$\frac{\ln n}{a_n} = \ln n \int_{a_n}^{+\infty} \frac{1}{t^2} dt \leqslant \int_{a_n}^{+\infty} \frac{\ln\ln g(t)}{t^2} dt;$$

hence $\frac{\ln n}{a_n} \to 0$. On the other hand,

$$\left(\sum_{k=2}^{n} \frac{\ln(1+1/(k-1))}{a_k}\right) - \frac{\ln n}{a_{n+1}} = \sum_{k=2}^{n}\left(\frac{1}{a_k}-\frac{1}{a_{k+1}}\right)\ln k \leqslant \int_1^{+\infty} \frac{\ln\ln g(t)}{t^2}\,dt;$$

hence

$$\sum_{k=2}^{\infty} \frac{\ln(1+1/(k-1))}{a_k} < +\infty.$$

Since the sequence $\left\{k\frac{\ln(1+1/(k-1))}{a_k}\right\}_{k\geqslant 2}$ is decreasing, by Lemma 1.5(iii), there exist $0 < t_1 \leqslant t_2/2 \leqslant t_3/3 \leqslant \ldots, \sum_{k=1}^{\infty} 1/t_k < +\infty$, such that

$$\lim_{k\to+\infty} \frac{t_k \ln(1+1/(k-1))}{a_k} = 0.$$

Let $t \in [1, +\infty)$; there exists an $n \geqslant 1$ such that $a_n \leqslant t \leqslant a_{n+1}$. Then $g(t) < g(a_{n+1}) = e^{n+1}$ and

$$\sup_{k\geqslant 1} \frac{k!t^{k+1}}{t_1 \ldots t_k} \geqslant \frac{n!t^n}{t_1 \ldots t_n} \geqslant \frac{n!a_1 \ldots a_n}{t_1 \ldots t_n}.$$

Consequently

$$\frac{g(t)}{\sup\limits_{k\geqslant 1} \frac{k!t^{k+1}}{t_1 \ldots t_k}} \leqslant e\prod_{k=1}^{n} \frac{et_k}{ka_k}.$$

Since

$$\frac{et_k}{ka_k} = e\frac{t_k\ln(1+1/(k-1))}{a_k}\cdot\frac{1}{\ln(1+1/(k-1))^k} \to 0,$$

we have

$$\sup_{n\geqslant 1}\prod_{k=1}^{n}\frac{et_k}{ka_k} < +\infty.$$

Hence writing

$$c = \left(\int_1^2 \left(\sup_{1\leqslant s\leqslant r} f(s)\right)dr\right) e\sup_{n\geqslant 1}\prod_{k=1}^{n}\frac{et_k}{ka_k},$$

we conclude that

$$f(t) \leqslant c\sup_{k\geqslant 1}\frac{k!t^{k+1}}{t_1 \ldots t_k}, \qquad t \in [1, +\infty).$$

(ii) ⇒ (i). For any $\lambda > 0$ we denote by $n(\lambda)$ the number of all t_k/k with $t_k/k \leqslant \lambda$. Then for each $t \geqslant t_1$

$$\ln\left(\sup_{k\geqslant 1}\frac{k!t^{k+1}}{t_1 \ldots t_k}\right) = \sup_{k\geqslant 1}\left(\sum_{p=1}^{k}\ln\frac{pt}{t_p}\right)+\ln t$$
$$= \left(\sum_{p=1}^{n(t)}\ln\frac{pt}{t_p}\right)+\ln t \leqslant n(t)\ln\frac{t}{t_1}+\ln t;$$

hence it is enough to prove that

$$\int_{t_1}^{+\infty}\frac{\ln n(t)}{t^2}\,dt < +\infty.$$

But by Lemma 1.5(iii), $\frac{k\ln k}{t_k} \to 0$, so that

$$\int_{t_1}^{+\infty}\frac{\ln n(t)}{t^2}\,dt = \sum_{k=1}^{\infty}\int_{t_k/k}^{t_{k+1}/k}\frac{\ln n(t)}{t^2}\,dt$$
$$= \sum_{k=1}^{\infty}\left(\frac{k}{t_k}-\frac{k+1}{t_{k+1}}\right)\ln k = \sum_{k=2}^{\infty}\frac{\ln(1+1/(k-1))^k}{t_k} \leqslant \sum_{k=2}^{\infty}\frac{\ln 4}{t_k}.$$

(ii) ⇒ (iii). By Theorem 1.8 there exists an α satisfying the assumptions from (iii) and $b > 0$ such that

$$\sup_{k\geqslant 1}\frac{r^k}{t_1 \ldots t_k} \leqslant b\alpha(r), \qquad r \in [0, +\infty).$$

Let $t \in [1, +\infty)$; then for each $k \geqslant 1$

$$\frac{k!t^{k+1}}{t_1 \ldots t_k} = t\int_0^{+\infty}\frac{(ts)^k}{t_1 \ldots t_k}e^{-s}ds \leqslant bt\int_0^{+\infty}\alpha(ts)e^{-s}ds = b\alpha_{\mathrm{Borel}}(t);$$

hence

$$\sup_{k\geqslant 1}\frac{k!t^{k+1}}{t_1 \ldots t_k} \leqslant b\alpha_{\mathrm{Borel}}(t).$$

The implication (iii) ⇒ (iv) is an immediate consequence of the corresponding implication from Theorem 1.8.

(iv) ⇒ (ii). By Theorem 1.8 there exist $0 < s_1 \leqslant s_2/2 \leqslant s_3/3 \leqslant \ldots, \sum_{k=1}^{\infty} 1/s_k < +\infty$ and $b > 0$ such that

$$|\omega_{\{t_k\}}(r)| \leqslant b\max\left\{1, \sup_{k\geqslant 1}\frac{(r/2)^k}{s_1 \ldots s_k}\right\}, \qquad r \in [0, +\infty).$$

Then for each $t \in [1, +\infty)$,

$$|\omega_{\{t_k\}}|_{\mathrm{Borel}}(t) = t \int_0^{+\infty} |\omega_{\{t_k\}}(ts)|\, e^{-s} ds$$

$$\leqslant bt \int_0^{+\infty} \left(1 + \sum_{k=1}^{\infty} \frac{(ts/2)^k}{s_1 \ldots s_k}\right) e^{-s} ds$$

$$= bt \left(1 + \sum_{k=1}^{\infty} 2^{-k} \frac{k!t^k}{s_1 \ldots s_k}\right)$$

$$\leqslant b \max\left\{t, \sup_{k \geqslant 1} \frac{k!t^{k+1}}{s_1 \ldots s_k}\right\} \leqslant bs_1 \sup_{k \geqslant 1} \frac{k!t^{k+1}}{s_1 \ldots s_k}. \;\blacksquare$$

We remark that condition (i) from Theorem 1.10 appears in several topics of the theory of analytic functions, in the harmonic analysis (see [39], [27]) and in the operator theory (see [43]); in [43] it is called the "Levinson condition".

COROLLARY 1.11. *Let $t_1, t_2, \ldots > 0$, $\sum_{k=1}^{\infty} 1/t_k < +\infty$; then the following statements are equivalent*:

(i) $\sum_{k=1}^{\infty} \frac{\ln t_k}{t_k} < +\infty$;

(ii) *the Borel transform of* $[0, +\infty) \ni t \to \omega_{\{t_k\}}(-it)$ *satisfies the equivalent conditions from Theorem* 1.10.

Proof. Let f be the Borel transform of the function $[0, +\infty) \ni t \to \omega_{\{t_k\}}(-it)$.

Assume that f satisfies the equivalent conditions from Theorem 1.10; then there exist $0 < s_1 \leqslant s_2/2 \leqslant s_3/3 \leqslant \ldots$, $\sum_{k=1}^{\infty} 1/s_k < +\infty$ and $c > 0$ such that

$$f(t) \leqslant c \sup_{k \geqslant 1} \frac{k!t^{k+1}}{s_1 \ldots s_k}, \quad t \in [s_1, +\infty).$$

Let $\omega_{\{t_k\}}(-iz) = \sum_{n=0}^{\infty} c_n z^n$; then $f(z) = \sum_{n=0}^{\infty} n!\, c_n z^{n+1}$ and thus for each $n \geqslant 1$ we get

$$n!c_n \leqslant \frac{f(s_n/n)}{(s_n/n)^{n+1}} \leqslant c \frac{\sup_{k \geqslant 1} \frac{k!(s_n/n)^k}{s_1 \ldots s_k}}{(s_n/n)^n} = c \frac{n!}{s_1 \ldots s_n},$$

that is

$$c_n \leqslant c \frac{1}{s_1 \ldots s_n}.$$

Consequently, for all $t \in [0, +\infty)$

$$\omega_{\{t_k\}}(-it) = \sum_{k=0}^{\infty} 2^{-k} c_k (2t)^k \leqslant 2 \sup_{k \geqslant 0} c_k (2t)^k$$

$$\leqslant 2(1+c) \max\left\{1, \sup_{k \geqslant 1} \frac{(2t)^k}{s_1 \ldots s_k}\right\} \leqslant 2(1+c)|\omega_{\{t_k\}}(2t)|.$$

By Corollary 1.9 it follows that $\sum_{k=0}^{\infty} \frac{\ln t_k}{t_k} < +\infty$.

Conversely, if the above condition is satisfied, then by Corollary 1.9, there exist $0 < s_1 \leqslant s_2/2 \leqslant s_3/3 \leqslant \ldots$, $\sum_{k=1}^{\infty} 1/s_k < +\infty$, and $c > 0$ such that

$$\omega_{\{t_k\}}(-ir) \leqslant c|\omega_{\{s_k\}}(r)|, \quad r \in [0, +\infty),$$

hence

$$f(t) \leqslant c|\omega_{\{s_k\}}|_{\mathrm{Borel}}(t), \quad t > 0. \;\blacksquare$$

We shall give further a new version of a result of N. Sjöberg ([59], Th. III); in the proof we shall use the techniques of Y. Domar ([19]) as well as the idea of O. I. Inozemčev and V. A. Marčenko ([30]), used also in the proof of Theorem 1.6.

Let f: $[1, +\infty) \to [1, +\infty)$ be a strictly increasing continuous function such that $f(1) = 1$, $\lim_{t \to +\infty} f(t) = +\infty$ and

$$\int_1^{+\infty} \frac{\ln f(t)}{t^2} dt < +\infty.$$

We define the sequence $1 = \alpha_1^f < \alpha_2^f < \ldots$, by

$$f(\alpha_k^f) = e^{k-1}, \quad k \geqslant 1.$$

Then, by a reasoning similar to that used in the proof of Theorem 1.6, we have

$$\sum_{k=1}^{\infty} \frac{1}{\alpha_k^f} < +\infty.$$

Let $\Omega \subset \boldsymbol{C}$ be open; we say that u: $\Omega \to \boldsymbol{R}$ is *subharmonic* if

(i) u is a Borel function;

(ii) u is bounded on every compact subset of Ω;

(iii) if $z \in \Omega$ and $r > 0$ are such that $\{\xi \in \boldsymbol{C};\, |\xi - z| \leqslant r\} \subset \Omega$, then

$$u(z) \leqslant \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta.$$

We note that in the usual definition of subharmonic functions, u is assumed to be upper semi-continuous (see [49]).

If $u\colon \Omega\to \boldsymbol{R}$ is subharmonic and z, r are as in (iii), then

$$u(z)\leqslant \frac{1}{\pi r^2}\int_0^r\int_0^{2\pi} u(z+\varrho e^{i\theta})\varrho\, d\theta d\varrho = \frac{1}{\pi r^2}\int_{|\xi-z|\leqslant r} u(\xi)d\xi,$$

where $d\xi$ is the Lebesgue measure on $\boldsymbol{C}$.

THEOREM 1.12. *Let $f\colon [1,+\infty)\to[1,+\infty)$ be a strictly increasing continuous function such that $f(1)=1$, $\lim\limits_{t\to+\infty} f(t)=+\infty$ and*

$$\int_1^{+\infty}\frac{\ln f(t)}{t^2}\,dt<+\infty.$$

Then for any $a\in(0,+\infty]$, any subharmonic function $u\colon \{z\in \boldsymbol{C};\ |\mathrm{Re}\, z|<a,\ |\mathrm{Im}\, z|<1\}\mapsto \boldsymbol{R}$ with

$$u(z)\leqslant f\left(\frac{1}{|\mathrm{Im}\, z|}\right),\quad z\in\boldsymbol{C},\ |\mathrm{Re}\, z|<a,\ 0\neq|\mathrm{Im}\, z|<1$$

and any integer $k\geqslant 1$ with

$$\frac{8e}{\pi}\sum_{p=k}^{\infty}\frac{1}{\alpha_k^f}\leqslant\frac{1}{2},$$

we have

$$u(z)\leqslant\max\{e^k, f(2)\}$$

for $z\in\boldsymbol{C}$, with $|\mathrm{Re}\, z|<a-\dfrac{8e}{\pi}\sum\limits_{p=k}^{\infty}\dfrac{1}{\alpha_p^f}$, $|\mathrm{Im}\, z|<1$.

Proof. If $z\in\boldsymbol{C}$, $|\mathrm{Re}\, z|<a$, $1/2<|\mathrm{Im}\, z|<1$, then

$$u(z)\leqslant f\left(\frac{1}{|\mathrm{Im}\, z|}\right)\leqslant f(2).$$

Hence, it is enough to show that for $z\in\boldsymbol{C}$, $|\mathrm{Re}\, z|<a-\dfrac{8e}{\pi}\sum\limits_{p=k}^{+\infty}\dfrac{1}{\alpha_p^f}$, $|\mathrm{Im}\, z|<\frac{1}{2}$, we have

$$u(z)\leqslant e^k.$$

Assuming the contrary, there exists a $z_0\in\boldsymbol{C}$, with $|\mathrm{Re}\, z_0|<a-\dfrac{8e}{\pi}\sum\limits_{p=k}^{+\infty}\dfrac{1}{\alpha_p^f}$, $|\mathrm{Im}\, z_0|<\frac{1}{2}$ and such that

$$u(z_0)>e^k.$$

We note that the set $\boldsymbol{K}=\left\{\xi\in\boldsymbol{C};\ |\xi-z_0|\leqslant\dfrac{8e}{\pi}\sum\limits_{p=k}^{+\infty}\dfrac{1}{\alpha_p^f}\right\}$ is compact and contained in $\{\xi\in\boldsymbol{C};\ |\mathrm{Re}\,\xi|<a,\ |\mathrm{Im}\,\xi|<1\}$.

Next we shall prove that there exists a $z_1\in\boldsymbol{C}$, $|z_1-z_0|\leqslant\dfrac{8e}{\pi}\cdot\dfrac{1}{\alpha_k^f}$ such that

$$u(z_1)>e^{k+1}.$$

Indeed, assuming the contrary and denoting

$$D_1=\left\{\xi\in\boldsymbol{C};|\xi-z_0|\leqslant\frac{8e}{\pi}\cdot\frac{1}{\alpha_k^f}\right\}\cap\left\{\xi\in\boldsymbol{C};\ |\mathrm{Re}\,\xi|<a,\ |\mathrm{Im}\,\xi|<\frac{1}{\alpha_k^f}\right\},$$

$$D_2=\left\{\xi\in\boldsymbol{C};|\xi-z_0|\leqslant\frac{8e}{\pi}\cdot\frac{1}{\alpha_k^f}\right\}\setminus D_1,$$

we should have

$$u(\xi)\leqslant e^{k+1},\quad \xi\in D_1,$$

$$u(\xi)\leqslant f\left(\frac{1}{|\mathrm{Im}\,\xi|}\right)\leqslant f(\alpha_k^f)=e^{k-1},\quad \xi\in D_2,$$

which implies

$$u(z_0)\leqslant\pi^{-1}\left(\frac{8e}{\pi}\cdot\frac{1}{\alpha_k^f}\right)^{-2}\int_{|\xi-z_0|\leqslant\frac{8e}{\pi}\cdot\frac{1}{\alpha_k^f}} u(\xi)d\xi$$

$$=\pi^{-1}\left(\frac{8e}{\pi}\cdot\frac{1}{\alpha_k^f}\right)^{-2}\left(\int_{D_1}u(\xi)d\xi+\int_{D_2}u(\xi)d\xi\right)$$

$$\leqslant\pi^{-1}\left(\frac{8e}{\pi}\cdot\frac{1}{\alpha_k^f}\right)^{-2}\left(e^{k+1}\mathrm{meas}\,(D_1)+e^{k-1}\mathrm{meas}(D_2)\right)$$

$$\leqslant e^{k+1}\pi^{-1}\left(2\cdot\frac{8e}{\pi}\cdot\frac{1}{\alpha_k^f}\right)\left(2\frac{1}{\alpha_k^f}\right)\left(\frac{8e}{\pi}\frac{1}{\alpha_k^f}\right)^{-2}+e^{k-1}$$

$$=e^k\left(\frac{1}{2}+\frac{1}{e}\right)<e^k,$$

in contradiction to the fact that $u(z_0)>e^k$.

By induction, we find a sequence $z_0, z_1, z_2, \ldots$ in $\boldsymbol{C}$, such that for any integer $q\geqslant 0$

$$|z_{q+1}-z_q|\leqslant\frac{8e}{\pi}\cdot\frac{1}{\alpha_{k+q}^f},\qquad u(z_{q+1})>e^{k+q+1}.$$

Since all z_q, $q\geqslant 0$, belong to the compact set $\boldsymbol{K}$, it follows that u is not bounded on $\boldsymbol{K}$, in contradiction to the subharmonicity of u. ∎

Using the above theorem, we can prove a result of N. Levinson ([39], Th. XLIII):

COROLLARY 1.13. *Let $g\colon [1,+\infty)\to[e,+\infty)$ be an increasing function such that*

$$\int_1^{+\infty}\frac{\ln\ln g(t)}{t^2}\,dt<+\infty.$$

Then for every $\varepsilon>0$ there exists a $c_{g,\varepsilon}>0$ such that for any $a\in(0,+\infty]$ and any analytic function $\Phi\colon \{z\in\boldsymbol{C};\,|\mathrm{Re}z|<a,\,|\mathrm{Im}z|<1\}\to\boldsymbol{C}$ with

$$|\Phi(z)|\leqslant g\left(\frac{1}{|\mathrm{Im}z|}\right),\quad z\in\boldsymbol{C},\ |\mathrm{Re}z|<a,\ 0\neq|\mathrm{Im}z|<1,$$

we have

$$|\Phi(z)|\leqslant c_{g,\varepsilon},\quad z\in\boldsymbol{C},\quad |\mathrm{Re}z|<a-\varepsilon,\quad |\mathrm{Im}z|<1.$$

Proof. By Theorem 1.6 there exist $0<t_1\leqslant t_2\leqslant\ldots<+\infty$, $\sum_{k=1}^{\infty}1/t_k<+\infty$ and $c>0$ such that

$$\ln(1+g(t)^2)\leqslant c|\omega_{\{t_k\}}(t)|,\quad t\in[1,+\infty).$$

Let us write $f=|\omega_{\{t_k\}}|$ and choose an integral $k\geqslant 1$ such that

$$\frac{8e}{\pi}\sum_{p=k}^{+\infty}\frac{1}{\alpha_p^f}\leqslant\min\left\{\frac{1}{2},\varepsilon\right\}.$$

Next we define $u\colon \{z\in\boldsymbol{C};\,|\mathrm{Re}z|<a,\,|\mathrm{Im}z|<1\}\to\boldsymbol{R}$ by

$$u(z)=c^{-1}\ln(1+|\Phi(z)|^2).$$

Since u is subharmonic (see for example [1], Ch. 6.4.1, Exercise 2), by Theorem 1.11 we have

$$u(z)\leqslant\max\{e^k,f(2)\},\quad z\in\boldsymbol{C},\ |\mathrm{Re}z|<a-\varepsilon,\ |\mathrm{Im}z|<1.$$

Hence our statement holds for

$$c_{g,\varepsilon}=e^{\frac{1}{2}(c\max\{e^k,f(2)\})}.\ \blacksquare$$

We end this section with an improved version of J. Körner's "Zerlegungsatz" ([35], p. 19).

THEOREM 1.14. *Let $a_0,a_1>0,\ a_2,a_3,\ldots\geqslant 0$ be such that the sequence $\{a_k/a_{k+1}\}_{k\geqslant 1}$ is decreasing and converges to 0 and let X be a Banach space. Then for each X-valued analytic function on $\{z\in\boldsymbol{C};\,|z|<1\}$ satisfying*

$$\|f(z)\|\leqslant\sum_{k=1}^{\infty}a_k(1-|z|)^{-k},\quad |z|<1$$

there exists a sequence $f_1,f_2,\ldots$, of X-valued analytic functions on $\{z\in\boldsymbol{C};\,|z|<1\}$ such that

$$f(z)=\sum_{k=1}^{\infty}f_k(z),\quad |z|<1$$

and

$$\|f_k(z)\|\leqslant\frac{ea_1}{a_0}\,a_k\left(4e\frac{a_0+a_1}{a_1}\right)^k(1-|z|)^{-k-1},\quad |z|<1,\ k\geqslant 1.$$

Proof. Let us consider a decreasing continuous function $\varrho\colon[0,+\infty)\to\to[0,+\infty)$ such that

$$\varrho(k)=\frac{a_0+a_1}{a_1}\cdot\frac{a_k}{a_{k-1}},\quad k\geqslant 1.$$

Fix an integer $n\geqslant 1$. Since the function $[0,+\infty)\ni s\to\varrho(s)-s/(n+s)$ is continuous,

$$\varrho(1)-\frac{1}{n+1}\geqslant 1-\frac{1}{n+1}>0\quad\text{and}\quad\lim_{s\to+\infty}\left(\varrho(s)-\frac{s}{n+s}\right)=-1,$$

there exists an $1\leqslant s_n<+\infty$ such that

$$\varrho(s_n)=\frac{s_n}{n+s_n}.$$

Let k_n be the integer part of s_n, that is $k_n\leqslant s_n\leqslant k_n+1$, and let $r_n=1-\varrho(s_n)=n/(n+s_n)$. Then $k_n\geqslant 1$, $0<r_n<1$ and

$$\frac{a_0+a_1}{a_1}\cdot\frac{a_{k_n}}{a_{k_n-1}}=\varrho(k_n)\geqslant\varrho(s_n)\geqslant\varrho(k_n+1)=\frac{a_0+a_1}{a_1}\cdot\frac{a_{k_n+1}}{a_{k_n}},$$

that is

$$\frac{a_1}{a_0+a_1}\cdot\frac{a_{k_n-1}}{a_{k_n}}\leqslant(1-r_n)^{-1}\leqslant\frac{a_1}{a_0+a_1}\cdot\frac{a_{k_n}}{a_{k_n+1}}.$$

Consequently

$$\begin{aligned}\sum_{k=1}^{\infty}a_k(1-r_n)^{-k}&=\sum_{k=1}^{\infty}\left(\frac{a_1}{a_0+a_1}\right)^k\cdot\left(\frac{a_0+a_1}{a_1}\right)^k a_k(1-r_n)^{-k}\\&\leqslant\frac{a_1}{a_0}\sup_{k\geqslant 1}\left(\frac{a_0+a_1}{a_1}\right)^k a_k(1-r_n)^{-k}\\&=a_1\sup_{k\geqslant 1}\prod_{l=1}^{k}\left(\frac{a_0+a_1}{a_1}\cdot\frac{a_l}{a_{l-1}}(1-r_n)^{-1}\right)\\&=a_1\prod_{l=1}^{k_n}\left(\frac{a_0+a_1}{a_1}\cdot\frac{a_l}{a_{l-1}}(1-r_n)^{-1}\right)\\&=\frac{a_1}{a_0}\cdot a_{k_n}\left(\frac{a_0+a_1}{a_1}\right)^{k_n}(1-r_n)^{-k_n}.\end{aligned}$$

Using the Stirling formula $m! = \sqrt{2\pi m}\cdot\left(\frac{m}{e}\right)^m e^{\theta_m/12m}$, $0<\theta_m<1$, we obtain

$$r_n^{-n}\sum_{k=1}^{\infty}a_k(1-r_n)^{-k} \leqslant \left(1+\frac{s_n}{n}\right)^n \frac{a_1}{a_0}\cdot a_{k_n}\left(\frac{a_0+a_1}{a_1}\right)^{k_n}\left(\frac{n+s_n}{s_n}\right)^{k_n}$$
$$\leqslant \left(1+\frac{k_n+1}{n}\right)^n\cdot\frac{a_1}{a_0}\cdot a_{k_n}\left(\frac{a_0+a_1}{a_1}\right)^{k_n}\left(\frac{n+k_n}{k_n}\right)^{k_n}$$
$$\leqslant e^{k_n+1}\cdot\frac{a_1}{a_0}\cdot a_{k_n}\left(\frac{a_0+a_1}{a_1}\right)^{k_n}\sqrt{2\pi k_n}\, e^{1/6}\binom{n+k_n}{n}$$
$$\leqslant \frac{ea_1}{a_0}a_{k_n}\left(4e\,\frac{a_0+a_1}{a_1}\right)^{k_n}\binom{n+k_n}{n}$$
$$\leqslant \frac{ea_1}{a_0}\sum_{k=1}^{\infty}a_k\left(4e\,\frac{a_0+a_1}{a_1}\right)^k\binom{n+k}{n}.$$

Let $f(z) = \sum_{n=0}^{\infty}c_n z^n$, with $c_0, c_1, \ldots \in X$, $|z|<1$. By the Cauchy integral formula, for $n\geqslant 1$ we get

$$\|c_n\| \leqslant \inf_{0<r<1} r^{-n}\sum_{k=1}^{\infty}a_k(1-r)^{-k} \leqslant r_n^{-n}\sum_{k=1}^{\infty}a_k(1-r_n)^{-k}$$
$$\leqslant \frac{ea_1}{a_0}\sum_{k=1}^{\infty}a_k\left(4e\,\frac{a_0+a_1}{a_1}\right)^k\binom{n+k}{n}.$$

As $\|c_0\| = \|f(0)\| \leqslant \sum_{k=1}^{\infty}a_k$, we conclude that for any $n\geqslant 0$ there exists a $d_n\in X$, $\|d_n\|\leqslant 1$, such that

$$c_n = \frac{ea_1}{a_0}\sum_{k=1}^{\infty}a_k\left(4e\,\frac{a_0+a_1}{a_1}\right)^k\binom{n+k}{n}d_n.$$

Finally, for each $k\geqslant 1$ we define

$$f_k(z) = \frac{ea_1}{a_0}\cdot a_k\left(4e\frac{a_0+a_1}{a_1}\right)^k\sum_{n=0}^{\infty}\binom{n+k}{n}d_n z^n, \quad |z|<1.$$

Using the equality

$$\sum_{n=0}^{\infty}\binom{n+k}{n}|z|^n = (1-|z|)^{-k-1},$$

it is easy to verify that the sequence $f_1, f_2, \ldots$, satisfies the statements of the theorem. ∎

2. ω-ultradifferentiable functions

Let $t_1, t_2, \ldots > 0$, $t_1 < +\infty$ be such that $\sum_{k=1}^{\infty}1/t_k < +\infty$. In the whole of this section we shall denote by ω the entire function $\omega_{\{t_k\}}$.

For each $\varphi\in L^1(\boldsymbol{R})$ we denote by $\hat{\varphi}$ its Fourier transform, defined by

$$\hat{\varphi}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi(t)e^{-its}ds.$$

We remark that if $\varphi\in L^1(\boldsymbol{R})$ then the inversion formula holds, that is, φ is almost everywhere equal to the inverse Fourier transform of $\hat{\varphi}$:

$$\varphi(s) = \int_{-\infty}^{+\infty}\hat{\varphi}(t)e^{its}dt.$$

We remark moreover that if φ has compact support and if we put $d = \sup_{t\in\operatorname{supp}\varphi}|t|$, then $\hat{\varphi}$ can be extended to an entire function of exponential type $\leqslant d$, which we also denote by $\hat{\varphi}$.

Definition III. We say that a continuous complex function φ on $\boldsymbol{R}$ with compact support is *ω-ultradifferentiable* if for any $L>0$ and any integer $n\geqslant 1$

$$p_{L,n}^{\omega}(\varphi) = \sup_{t\in\boldsymbol{R}}|\hat{\varphi}(t)\omega(Lt)^n| < +\infty.$$

For $-\infty<a<b<+\infty$, we denote by $\mathcal{D}_\omega[a,b]$ the vector space of all ω-ultradifferentiable functions with the support contained in $[a,b]$ and we put

$$\mathcal{D}_\omega = \bigcup_{-\infty<a<b<+\infty}\mathcal{D}_\omega[a,b].$$

We consider on every $\mathcal{D}_\omega[a,b]$ the locally convex topology defined by the seminorms $p_{L,n}^{\omega}$, $L>0$, $n\geqslant 1$, integer, and we endow $\mathcal{D}_\omega$ with the inductive limit topology of the space $\mathcal{D}_\omega[a,b]$.

If φ is a continuous complex function on $\boldsymbol{R}$ with compact support, $L>0$ and $n\geqslant 1$ an integer, then we write

$$q_{L,n}^{\omega}(\varphi) = \int_{-\infty}^{+\infty}|\hat{\varphi}(t)\omega(Lt)^n|dt.$$

Proposition 2.1. *Let φ be a continuous complex function on $\boldsymbol{R}$ with compact support. Then for any $L>0$ and any integer $n\geqslant 1$ we have*

$$q_{L,n}^{\omega}(\varphi) \leqslant \frac{\pi t_1}{L}p_{L,n+2}^{\omega}(\varphi), \tag{2.1}$$

$$p_{L,n}^{\omega}(\varphi) \leqslant \sup_{t\in\operatorname{supp}\varphi}|t|\,q_{L,n}^{\omega}(\varphi). \tag{2.2}$$

Proof. Indeed

$$q^{\omega}_{L,n}(\varphi) \leqslant \int_{-\infty}^{+\infty} \left|\hat{\varphi}(t)\left(1+\frac{L^2t^2}{t_1^2}\right)^{-1} \omega(Lt)^{n+2}\right| dt \leqslant \int_{-\infty}^{+\infty} \left(1+\frac{L^2t^2}{t_1^2}\right)^{-1} dt \cdot p^{\omega}_{L,n+2}(\varphi);$$

hence (2.1) results.

If $q^{\omega}_{L,n}(\varphi) = +\infty$, then (2.2) is obvious; so let us suppose that $q^{\omega}_{L,n}(\varphi) < +\infty$. Then f defined by

$$|f(z)| = \int_0^z \hat{\varphi}(\lambda)\omega(L\lambda)^n d\lambda$$

is an entire function of exponential type $\leqslant d = \sup\limits_{t \in \operatorname{supp}\varphi} |t|$ and

$$f(t) \leqslant q^{\omega}_{L,n}(\varphi), \quad t \in \boldsymbol{R}.$$

Using Corollary 1.4, we obtain

$$|\hat{\varphi}(t)\omega(Lt)^n| = |f'(t)| \leqslant d \cdot q^{\omega}_{L,n}(\varphi), \quad t \in \boldsymbol{R}. \blacksquare$$

We remark that this proposition is essentially proved in [43], Ch. I, § 1.

By Proposition 2.1 a continuous complex function φ on $\boldsymbol{R}$ with a compact support belongs to $\mathscr{D}_\omega$ if and only if $q^{\omega}_{L,n}(\varphi) < +\infty$ for all L, n, and the topology of each $\mathscr{D}_\omega[a, b]$ is defined by the seminorms $q^{\omega}_{L,n}$.

COROLLARY 2.2. *If $\varphi, \psi \in \mathscr{D}_\omega$ then $\varphi\psi \in \mathscr{D}_\omega$ and*

$$q^{\omega}_{L,n}(\varphi\psi) \leqslant q^{\omega}_{\sqrt{2}L,n}(\varphi)q^{\omega}_{\sqrt{2}L,n}(\psi), \quad L > 0, n \geqslant 1 \text{ integer}.$$

If $\varphi \in \mathscr{D}_\omega$ then $\overline{\varphi} \in \mathscr{D}_\omega$ and

$$p^{\omega}_{L,n}(\overline{\varphi}) = p^{\omega}_{L,n}(\varphi), \quad L > 0,\ n \geqslant 1 \text{ integer}.$$

Proof. Let $\varphi, \psi \in \mathscr{D}_\omega$. Then $\varphi\psi$ is continuous, has a compact support and $\widehat{\varphi\psi} = \hat{\varphi} * \hat{\psi}$. Since

$$1+\frac{(u+v)^2}{t_k^2} \leqslant \left(1+\frac{2u^2}{t_k^2}\right)\left(1+\frac{2v^2}{t_k^2}\right), \quad u, v \in \boldsymbol{R}, k \geqslant 1,$$

we have

$$|\omega(u+v)| \leqslant |\omega(\sqrt{2}u)|\,|\omega(\sqrt{2}v)|, \quad u, v \in \boldsymbol{R}.$$

Hence

$$q^{\omega}_{L,n}(\varphi\psi) = \int_{-\infty}^{+\infty} \left|\left(\int_{-\infty}^{+\infty} \hat{\varphi}(t-r)\hat{\psi}(r)dr\right)\omega(Lt)^n\right| dt$$

$$\leqslant \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\hat{\varphi}(t-r)\hat{\psi}(r)\omega(\sqrt{2}L(t-r))^n\omega(\sqrt{2}Lr)^n| dr\,dt$$

$$= q^{\omega}_{\sqrt{2}L,n}(\varphi)\cdot q^{\omega}_{\sqrt{2}L,n}(\psi).$$

Further, since $\hat{\overline{\varphi}}(t) = \overline{\hat{\varphi}(-t)}$, $t \in \boldsymbol{R}$ and $|\omega(u)| = |\omega(-u)|$, $u \in \boldsymbol{R}$, for every $\varphi \in \mathscr{D}_\omega$ we have:

$$p^{\omega}_{L,n}(\overline{\varphi}) = \sup_{t\in\boldsymbol{R}} |\overline{\hat{\varphi}(-t)\omega(Lt)^n}| = \sup_{t\in\boldsymbol{R}} |\hat{\varphi}(-t)\omega(-Lt)^n| = p^{\omega}_{L,n}(\varphi). \blacksquare$$

By Corollary 2.2, for any $-\infty < a < b < +\infty$, *$\mathscr{D}_\omega[a, b]$ is an involutive algebra for pointwise multiplication and complex conjugation. Moreover, the multiplication and the conjugation are continuous.*

Similar statements are true also for $\mathscr{D}_\omega$.

An important tool to handle ω-ultradifferentiable functions consists in the following Paley–Wiener type theorem (compare with [15], Proposition 1.3):

THEOREM 2.3. *Let $-\infty < a < b < +\infty$ and let f be an entire function. Then the following statements are equivalent:*

(i) *the restriction of f to $\boldsymbol{R}$ is the Fourier transform of some function from $\mathscr{D}_\omega[a, b]$;*

(ii) *f is of finite exponential type, $h_f(-\pi/2) \leqslant -a$, $h_f(\pi/2) \leqslant b$ and for each $L > 0$ and each integer $n \geqslant 1$*

$$\int_{-\infty}^{+\infty} |f(t)\omega(Lt)^n|\, dt < +\infty;$$

(iii) *f is of finite exponential type, $h_f(-\pi/2) \leqslant -a$, $h_f(\pi/2) \leqslant b$ and for each $L > 0$ and integer $n \geqslant 1$*

$$\sup_{t\in\boldsymbol{R}} |f(t)\omega(Lt)^n| < +\infty;$$

(iv) *for each $L > 0$ and integer $n \geqslant 1$*

$$\alpha_{L,n} = \sup_{t\in\boldsymbol{R}} |f(t)\omega(Lt)^n| < +\infty$$

and

$$|f(z)| \leqslant \begin{cases} \alpha_{L,n}|\omega(Lz)|^{-n}e^{a\operatorname{Im}z}, & \operatorname{Im}z \leqslant 0, \\ \alpha_{L,n}|\overline{\omega}(Lz)|^{-n}e^{b\operatorname{Im}z}, & \operatorname{Im}z \geqslant 0. \end{cases}$$

Proof. (i) ⇒ (ii). Direct verification shows that f is of finite exponential typ and $h_f(-\pi/2) \leqslant -a$, $h_f(\pi/2) \leqslant b$. By Proposition 2.1, for every $L > 0$ and integer $n \geqslant 1$, we have

$$\int_{-\infty}^{+\infty} |f(t)\omega(Lt)^n| dt < +\infty.$$

(ii) ⇒ (iii). Let $L > 0$ and let $n \geqslant 1$ be an integer. It is clear that

$$g(z) = \int_0^z f(\zeta)\omega(L\zeta)^n d\zeta, \quad z \in \boldsymbol{C}$$

is an entire function of finite exponential type and

$$\sup_{t\in R}|g(t)| \leqslant \int_{-\infty}^{+\infty} |f(t)\omega(Lt)^n|dt < +\infty .$$

By Corollary 1.4 it follows that

$$\sup_{t\in R}|f(t)\omega(Lt)^n| = \sup_{t\in R}|g'(t)| < +\infty .$$

(iii) $\Rightarrow$ (iv). This follows by applying Theorem 1.2.

(iv) $\Rightarrow$ (i). For any $t\in R$

$$\left|f(t)\left(1+\frac{t^2}{t_1^2}\right)\right| \leqslant \alpha_{1,2} < +\infty,$$

hence $t\mapsto f(t)$ belongs to $L^1(R)$. So we can define a continuous complex function φ on R by

$$\varphi(s) = \int_{-\infty}^{+\infty} f(t)e^{its}dt .$$

Suppose that $s<a$; for any $t\in R$ and $r>0$

$$f(t-ir)\left(1+\frac{t^2}{t_1^2}\right) \leqslant \alpha_{1,2}e^{-ar};$$

hence, applying the Cauchy integral theorem, we have for each $r>0$

$$\varphi(s) = \int_{-\infty}^{+\infty} f(t-ir)e^{i(t-ir)s}dt = e^{rs}\int_{-\infty}^{+\infty} f(t-ir)e^{its}dt;$$

hence

$$|\varphi(s)| \leqslant e^{rs}\alpha_{1,2}e^{-ar}\int_{-\infty}^{+\infty}\left(1+\frac{t^2}{t_1^2}\right)^{-1}dt = \alpha_{1,2}\cdot\pi\cdot t_1\cdot e^{(s-a)r}.$$

Letting $r\to+\infty$ we get $\varphi(s)=0$; hence $\operatorname{supp}\varphi\subset[a,+\infty)$. Similarly we can prove that $\operatorname{supp}\varphi\subset(-\infty;b]$, so that finally we get $\operatorname{supp}\varphi\subset[a,b]$. ■

As usual, we define the support function H_K of a compact set $K\subset R$ by

$$H_K(t) = \sup_{r\in K}(rt), \qquad t\in R.$$

Then the equivalent conditions from Theorem 2.3 are also equivalent to

(v) *for each $L>0$ and integer $n\geqslant 1$, there is a constant $c>0$ such that*

$$|f(z)| \leqslant c|\omega(L\operatorname{Re}z)|^{-n}e^{H_{[a,b]}(\operatorname{Im}z)}, \qquad z\in C,$$

namely (iii) $\Rightarrow$ (v) $\Rightarrow$ (ii).

LEMMA 2.4. *$\mathcal{D}_\omega$ contains a non-identically zero function.*

Proof. Consider the enitre function τ defined by

$$\tau(z) = \prod_{m=1}^{\infty}\omega\left(\frac{z}{4^m}\right)^{2^m} = \prod_{k,m=1}^{\infty}\left(1+\frac{iz}{4^m t_k}\right)^{2^m}.$$

We have

$$|\tau(t)|^2 = \left|\prod_{m=1}^{\infty}\omega\left(\frac{t}{4^m}\right)^{2^{m+1}}\right| = \left|\prod_{m=2}^{\infty}\omega\left(\frac{4t}{4^m}\right)^{2^m}\right| \leqslant |\tau(4t)|, \qquad t\in R$$

and

$$|\omega(t)| \leqslant |\tau(4t)|, \qquad t\in R.$$

By Lemma 1.5(i), there exist $r_1, r_2, \ldots > 0$, $\sum_{k=1}^{\infty}1/r_k < +\infty$, such that

$$\lim_{k\to\infty}\frac{r_k}{t_k} = 0.$$

Let us remark that

$$z\to\frac{\sin(\pi/2)\sqrt{1+z^2}}{\sqrt{1+z^2}} = \frac{\pi}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(\pi/2)^{2n}(1+z^2)^n}{(2n+1)!}$$

is an entire function of exponential type $\pi/2$.

We define the entire function f by the formula

$$f(z) = \prod_{k,m=1}^{\infty}\left(\frac{\sin\frac{\pi}{2}\sqrt{1+\frac{z^2}{3^{2m}r_k^2}}}{\sqrt{1+\frac{z^2}{3^{2m}r_k^2}}}\right)^{2^m}.$$

Then f is of entire exponential type $\leqslant \frac{\pi}{2}\sum_{k,m=1}^{\infty}\frac{2^m}{3^m r_k} = \pi\sum_{k=1}^{\infty}\frac{1}{r_k}$ and for each $L>0$ and integer $n\geqslant 1$

$$\sup_{t\in R}|f(t)\omega(Lt)^n| \leqslant \sup_{t\in R}|f(t)\tau(4Lt)^{2^{n-1}}| \leqslant \sup_{t\in R}|f(t)\tau(4^nLt)|$$

$$= \sup_{t\in R}\prod_{k,m=1}^{\infty}\left(\sqrt{\frac{1+\frac{4^{2n}L^2t^2}{4^{2m}t_k^2}}{1+\frac{t^2}{3^{2m}t_k^2}}}\cdot\sin\frac{\pi}{2}\sqrt{1+\frac{t^2}{3^{2m}t_k^2}}\right)^{2^m} < +\infty .$$

By Theorem 2.3, the restriction of f to R is the Fourier transform of some function $\varphi\in\mathcal{D}_\omega$. Since $f(0)=1$, φ is non-identically zero. ■

In the proof of Lemma 2.4 the construction of τ is inspired by an argument from [35], p. 38, and for the construction of f we adapted the remark at the end of [30], attributed to S. N. Bernstein.

THEOREM 2.5. *If $K\subset D\subset R$, K compact, D open, then there exists a $\varphi\in\mathcal{D}_\omega$ such that*

$$0\leqslant\varphi\leqslant 1, \quad \varphi(s)=1 \quad \text{for} \quad s\in K, \quad \operatorname{supp}\varphi\subset D.$$

Proof. Let $0<\varepsilon<\inf\{|\lambda-\mu|:\ \lambda\in K, \mu\notin D\}$ and $S=\{\mu\in R; |\lambda-\mu|\leqslant\varepsilon/2$ for some $\lambda\in K\}$. Denote by χ_S the characteristic function of S.

By Lemma 2.4 and Corollary 2.2 there exists a $\theta \in \mathscr{D}_\omega, \theta \geqslant 0, \int\limits_{-\infty}^{+\infty} \theta(s)ds = 1.$ Let $a = \sup\limits_{\lambda \in \operatorname{supp}\theta} |\lambda|$ and define a function ψ by

$$\psi(s) = \frac{2a}{\varepsilon}\,\theta\left(\frac{2a}{\varepsilon}\,s\right).$$

It is easy to verify that $\psi \in \mathscr{D}_\omega[-\varepsilon/2, \varepsilon/2]$, $\psi \geqslant 0$, $\int\limits_{-\infty}^{+\infty} \psi(s)\,ds = 1$. Put

$$\varphi = \chi_S * \psi.$$

Then φ is continuous and for any $s \in \boldsymbol{R}$

$$\varphi(s) = \int\limits_{-\infty}^{+\infty} \chi_S(s-r)\psi(r)\,dr = \int\limits_{s-S} \psi(r)\,dr.$$

Hence $\varphi(s) \geqslant 0$ and $\varphi(s) \leqslant \int\limits_{-\infty}^{+\infty} \psi(r)\,dr = 1$. Since $s-S \supset [-\varepsilon/2, \varepsilon/2]$ for $s \in \boldsymbol{K}$ and $(s-S) \cap [-\varepsilon/2, \varepsilon/2] = 0$ for $s \notin \{\mu \in \boldsymbol{R};\, |\lambda-\mu| \leqslant \varepsilon$ for a certain $\lambda \in \boldsymbol{K}\}$, it follows that $\varphi(s) = 1$ for $s \in \boldsymbol{K}$ and $\operatorname{supp}\varphi \subset \boldsymbol{D}$.

Finally, for each $L > 0$ and integer $n \geqslant 1$, we have

$$p^\omega_{L,n}(\varphi) = 2\pi \sup_{t\in R} |\hat{\chi}_S(t)\hat{\psi}(t)\omega(Lt)^n| \leqslant 2\pi \operatorname{meas}(S)\, p^\omega_{L,n}(\psi) < +\infty,$$

so that $\varphi \in \mathscr{D}_\omega$. ■

Using Theorem 2.5 and Corollary 2.2, we show by standard arguments:

COROLLARY 2.6. *If $\boldsymbol{K} \subset \boldsymbol{R}$ is compact, $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_m \subset \boldsymbol{R}$ are open and $\boldsymbol{K} \subset \bigcup\limits_{k=1}^{m} \boldsymbol{D}_k$, then there exist $\varphi_1, \ldots, \varphi_m \in \mathscr{D}_\omega$ such that*

$$\varphi_k \geqslant 0, \quad 1 \leqslant k \leqslant m,$$
$$\operatorname{supp}\varphi_k \subset \boldsymbol{D}_k, \quad 1 \leqslant k \leqslant m,$$
$$\sum_{k=1}^{m} \varphi_k(s) = 1 \textit{ for } s \in \boldsymbol{K}.$$

We are looking further for a description of $\mathscr{D}_\omega$ without using the Fourier transform.

DEFINITION IV. For every integers $n \geqslant 1$ and $k \geqslant 0$, $a^{\omega,n}_k$ is the *square root* of the coefficient of z^{2k} in the power series expansion of $\prod\limits_{j=1}^{\infty}\left(1+\frac{z^2}{t_j^2}\right)^n$.

Hence

$$a^{\omega,1}_0 = 1,$$

$$a^{\omega,1}_k = \Bigg(\sum_{j_1<\ldots<j_k} \frac{1}{t_{j_1}^2 \ldots t_{j_k}^2}\Bigg)^{1/2}, \quad k \geqslant 1,$$

and

$$a^{\omega,n}_k = \Bigg(\sum_{k_1+\ldots+k_n=k} (a^{\omega,1}_{k_1})^2 \ldots (a^{\omega,1}_{k_n})^2\Bigg)^{1/2}, \quad n > 1, k \geqslant 0.$$

Clearly

$$a^{\omega,n}_k \leqslant a^{\omega,n+1}_k, \quad n \geqslant 1, k \geqslant 0.$$

It is easy to see that for each integer $n \geqslant 1$

$$\sup_{k\geqslant 0} a^{\omega,n}_k |t|^k \leqslant |\omega(t)^n| \leqslant \sqrt{2} \sup_{k\geqslant 0} a^{\omega,n}_k \left|\sqrt{2}\,t\right|^k, \quad t \in \boldsymbol{R}.$$

If φ is an infinitely differentiable complex function on $\boldsymbol{R}$ with a compact support, $L > 0$ and $n \geqslant 1$ is an integer, then we write

$$r^\omega_{L,n}(\varphi) = \sup_{k\geqslant 0}\Big(L^k a^{\omega,n}_k \sup_{s\in R} |\varphi^{(k)}(s)|\Big).$$

PROPOSITION 2.7. *For any infinitely differentiable complex function φ on $\boldsymbol{R}$ with compact support and for any $L > 0$ and integer $n \geqslant 1$ we have*

$$r^\omega_{L,n}(\varphi) \leqslant q^\omega_{L,n}(\varphi), \tag{2.3}$$

$$p^\omega_{L,n}(\varphi) \leqslant \frac{1}{\pi} \sup_{\lambda,\mu \in \operatorname{supp}\varphi} (\lambda-\mu)\, r^\omega_{\sqrt{2}L,n}(\varphi). \tag{2.4}$$

Proof. If $q^\omega_{L,n}(\varphi) < +\infty$, then for any $k \geqslant 0$ and $s \in \boldsymbol{R}$ we have

$$L^k a^{\omega,n}_k |\varphi^{(k)}(s)| = L^k a^{\omega,n}_k \left|\int\limits_{-\infty}^{+\infty} (it)^k \hat{\varphi}(t) e^{its} dt\right|$$

$$\leqslant \int\limits_{-\infty}^{+\infty} |\hat{\varphi}(t)(a^{\omega,n}_k |Lt|^k)|\,dt \leqslant \int\limits_{-\infty}^{+\infty} |\hat{\varphi}(t)\omega(Lt)^n|\,dt = q^\omega_{L,n}(\varphi),$$

which proves (2.3).

Further, let $t \in \boldsymbol{R}$; for any $k \geqslant 0$ we get

$$|\hat{\varphi}(t)t^k| = |\varphi^{(k)}(t)| \leqslant \frac{1}{2\pi} \sup_{\lambda,\mu\in\operatorname{supp}\varphi} (\lambda-\mu) \sup_{s\in R} |\varphi^{(k)}(s)|,$$

hence

$$|\hat{\varphi}(t)\omega(Lt)^n| \leqslant |\hat{\varphi}(t)|\, 2 \sup_{k\geqslant 0} a^{\omega,n}_k \left|\sqrt{2}\,Lt\right|^k$$
$$= 2 \sup_{k\geqslant 0} (\sqrt{2}L)^k a^{\omega,n}_k |\hat{\varphi}(t)t^k|$$
$$\leqslant \frac{1}{\pi} \sup_{\lambda,\mu\in\operatorname{supp}\varphi} (\lambda-\mu)\, r^\omega_{\sqrt{2}L,n}(\varphi).$$

Thus also the second inequality of (2.4) is proved. ■

Let $\varphi \in \mathscr{D}_\omega$; since $t \to it\hat{\varphi}(t)$ belongs to $L^1(\boldsymbol{R})$ and for $s_1, s_2, s_3 \in \boldsymbol{R}$, $s_1 < s_2$ we have

$$\left|\frac{\varphi(s_2)-\varphi(s_1)}{s_2-s_1} - \int\limits_{-\infty}^{+\infty} it\hat{\varphi}(t)e^{its_3}\,ds\right| \leqslant \int\limits_{-\infty}^{+\infty} |t\hat{\varphi}(t)|\big(\sup_{s_1\leqslant s\leqslant s_2} |e^{its}-e^{its_3}|\big)\,dt,$$

i tfollows that φ is differentiable. Moreover, it is easy to verify that $\varphi' \in \mathscr{D}_\omega$.

Inductively, it follows that φ is infinitely differentiable and that $\varphi^{(k)} \in \mathscr{D}_\omega$ for all $k \geqslant 1$.

By Proposition 2.7, *an infinitely differentiable complex function φ on $\boldsymbol{R}$ with compact support belongs to $\mathscr{D}_\omega$ if and only if $r^\omega_{L,n}(\varphi) < +\infty$ for all L, n, and the topology of each $\mathscr{D}_\omega[a,b]$ is defined by the semi-norms $r^\omega_{L,n}$.*

This remark enables us to define ω-ultradifferentiability for functions with arbitrary support.

DEFINITION V. Let $-\infty \leqslant a < b \leqslant +\infty$; we say that an infinitely differentiable complex function φ on (a,b) is *ω-ultradifferentiable* if for any compact $\boldsymbol{K} \subset (a,b)$, any $L > 0$ and any integer $n \geqslant 1$

$$r^{\omega,K}_{L,n}(\varphi) = \sup_{k\geqslant 0}\big(L^k a^{\omega,n}_k \sup_{s\in K}|\varphi^{(k)}(s)|\big) < +\infty.$$

We denote the vector space of all ω-ultradifferentiable functions on (a,b) by $\mathscr{E}_\omega(a,b)$ and we consider on $\mathscr{E}_\omega(a,b)$ the locally convex topology defined by the semi-norms $r^{\omega,K}_{L,n}$, compact $\boldsymbol{K} \subset (a,b)$, $L > 0$, and integer $n \geqslant 1$.

Denote $\mathscr{E}_\omega(-\infty,+\infty)$ simply by $\mathscr{E}_\omega$.

By Proposition 2.7, $\mathscr{D}_\omega \subset \mathscr{E}_\omega$ and if φ belongs to a certain $\mathscr{E}_\omega(a,b)$ and has compact support, then $\varphi \in \mathscr{D}_\omega$.

In order to study the spaces $\mathscr{E}_\omega(a,b)$, we need the following result, essentially due to I. Schur (see [56], § 1, Hilfssatz I, or [38], Ch. VIII, § 2, Lemma 3):

PROPOSITION 2.8. *If $b_1, b_2, \ldots \in \boldsymbol{R}$, $\sum_{k=1}^{\infty}|b_k| < +\infty$ and*

$$c_0 = 1,\quad c_k = \sum_{j_1<\ldots<j_k} b_{j_1}\ldots b_{j_k},\quad k\geqslant 1,$$

then

$$(k!c_k)^2 \geqslant \big((k-1)!c_{k-1}\big)\big((k+1)!c_{k+1}\big),\quad k\geqslant 1.$$

Proof. Fix an integer $n \geqslant 1$ and $\varepsilon > 0$, write for each $1 \leqslant k \leqslant n$

$$b^{n,\varepsilon}_k = \begin{cases} b_k & \text{if}\quad b_k \neq 0,\\ \varepsilon & \text{if}\quad b_k = 0\end{cases}$$

and put

$$c^{n,\varepsilon}_0 = 1,\quad c^{n,\varepsilon}_k = \sum_{j_1<\ldots<j_k\leqslant n} b^{n,\varepsilon}_{j_1}\ldots b^{n,\varepsilon}_{j_k},\quad 1\leqslant k\leqslant n.$$

Then the polynomial

$$P_{n,\varepsilon}(z) = \sum_{m=0}^{n} c^{n,\varepsilon}_m z^m = \prod_{m=1}^{n}(1+b^{n,\varepsilon}_m z)$$

has only real zeros. Hence, by Rolle's theorem, for each integer $1 \leqslant k \leqslant n-1$, the polynomial

$$P^{(k-1)}_{n,\varepsilon}(z) = \sum_{m=k-1}^{n}\frac{m!}{(m-k+1)!}c^{n,\varepsilon}_m z^{m-k+1}$$

has only real zeros, say $t^k_1, \ldots, t^k_{n-k+1}$. But it is easy to verify that this implies for all $1 \leqslant k \leqslant n-1$

$$(k!c^{n,\varepsilon}_k)^2 - \big((k-1)!c^{n,\varepsilon}_{k-1}\big)\big((k+1)!c^{n,\varepsilon}_{k+1}\big) = \big((k-1)!c^{n,\varepsilon}_{k-1}\big)^2\sum_{m=1}^{n-k+1}(t^k_m)^{-2} > 0.$$

By letting $\varepsilon \to 0$ and then $n \to \infty$, our statement follows. ∎

COROLLARY 2.9. *For each integer $n \geqslant 1$,*

$$(a^{\omega,n}_k)^2 \geqslant a^{\omega,n}_{k-1}\cdot a^{\omega,n}_{k+1},\quad k\geqslant 1; \tag{2.5}$$

$$\sum_{k=1}^{\infty}\frac{a^{\omega,n}_k}{a^{\omega,n}_{k-1}} < +\infty. \tag{2.6}$$

Proof. Inequality (2.5) results immediately from the above proposition, taking $b_k = t_p^{-2}$, $n(p-1) < k \leqslant np$.

In order to prove (2.6), we remark that by Theorem 1.6 there exist $0 < s_1 \leqslant s_2 \leqslant \ldots$, $\sum_{m=1}^{\infty} 1/s_m < +\infty$ and $c > 0$ such that

$$|\omega^n(t)| \leqslant c\sup_{m\geqslant 1}\frac{t^m}{s_1\ldots s_m},\quad t\in[s,+\infty).$$

For every integer $k \geqslant 1$ we have

$$a^{\omega,n}_k \leqslant \frac{|\omega^n(s_k)|}{(s_k)^k} \leqslant \frac{c}{(s_k)^k}\sup_{m\geqslant 1}\frac{(s_k)^m}{s_1\ldots s_m} = \frac{c}{s_1\ldots s_k}.$$

Using the classical inequality of T. Carleman (see [45], Lemma 1.8.VI, or [48], Ch. XVI, §§ 4, 5, 6), we get

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{a^{\omega,n}_k}{a^{\omega,n}_{k-1}} &\leqslant \sum_{k=1}^{\infty}\left(\frac{a^{\omega,n}_1}{a^{\omega,n}_0}\ldots\frac{a^{\omega,n}_k}{a^{\omega,n}_{k-1}}\right)^{1/k}\\ &= \sum_{k=1}^{\infty}(a^{\omega,n}_k)^{1/k} \leqslant c\sum_{k=1}^{\infty}(s_1\ldots s_k)^{-1/k}\\ &\leqslant ce\sum_{k=1}^{\infty}s_k^{-1} < +\infty.\ \blacksquare\end{aligned}$$

We are now able to prove

PROPOSITION 2.10. *Let $-\infty \leqslant a < b \leqslant +\infty$. If $\varphi, \psi \in \mathscr{E}_\omega(a,b)$, then $\varphi\psi \in \mathscr{E}_\omega(a,b)$ and*

$$r^{\omega,K}_{L,n}(\varphi\psi) \leqslant r^{\omega,K}_{2L,n}(\varphi)r^{\omega,K}_{2L,n}(\psi),\quad K \subset (a,b)\ \text{compact},\ L>0, n\geqslant 1. \tag{2.7}$$

If $\varphi \in \mathscr{E}_\omega^-(a, b)$, *then* $\overline{\varphi} \in \mathscr{E}_\omega(a, b)$ *and*

$$r_{L,n}^{\omega;K}(\overline{\varphi}) = r_{L,n}^{\omega;K}(\varphi), \quad K \subset (a, b) \text{ compact}, \ L > 0, \text{ integer } n \geqslant 1. \tag{2.8}$$

Proof. Let $\varphi, \psi \in \mathscr{E}_\omega(a, b)$; then $\varphi\psi$ is infinitely differentiable. Using Corollary 2.9, it is easy to see that

$$a_m^{\omega,n} \cdot a_{k-m}^{\omega,n} \geqslant a_0^{\omega,n} \cdot a_k^{\omega,n} = a_k^{\omega,n}, \quad m \leqslant k,$$

so that for $s \in K$,

$$|(\varphi\psi)^{(k)}(s)| \leqslant \sum_{m=0}^{k} \binom{k}{m} |\varphi^{(m)}(s)||\psi^{(k-m)}(s)|$$

$$\leqslant \sum_{m=0}^{k} \binom{k}{m} \frac{1}{(2L)^k a_k^{\omega,n}} r_{2L,n}^{\omega;K}(\varphi) r_{2L,n}^{\omega;K}(\psi)$$

$$= \frac{1}{L^k a_k^{\omega,n}} r_{2L,n}^{\omega;K}(\varphi) r_{2L,n}^{\omega;K}(\psi).$$

Consequently

$$r_{L,n}^{\omega;K}(\varphi\psi) \leqslant r_{2L,n}^{\omega;K}(\varphi) r_{2L,n}^{\omega;K}(\psi).$$

The proof of (2.8) is immediate. ∎

By this proposition, *$\mathscr{E}_\omega(a, b)$ is an involutive algebra for pointwise multiplication and complex conjugation. Moreover, the multiplication and the conjugation are continuous.*

By Proposition 2.10, Proposition 2.7 and Theorem 2.5, *a complex function φ on (a, b) belongs to $\mathscr{E}_\omega(a, b)$ if and only if $\varphi\psi \in \mathscr{D}_\omega$ for every $\psi \in \mathscr{D}_\omega$ with* $\operatorname{supp}\psi$ *$\subset (a, b)$ and the topology of $\mathscr{E}_\omega(a, b)$ is the weakest locally convex topology for which all mappings*

$$\mathscr{E}_\omega(a, b) \ni \varphi \to \varphi\psi \in \mathscr{D}_\omega, \quad \psi \in \mathscr{D}_\omega, \quad \operatorname{supp}\psi \subset (a, b)$$

are continuous.

Moreover, it is easy to see that $\bigcup_{a<c<d<b} \mathscr{D}_\omega[c, d]$ *is dense in* $\mathscr{E}_\omega(a, b)$.

Since for $\varphi \in \mathscr{E}_\omega(a, b)$ and $\psi \in \mathscr{D}_\omega$ with $\operatorname{supp}\psi \subset (a, b)$, we have $\varphi'\psi = (\varphi\psi)' - \varphi\psi'$, it follows that $\mathscr{E}_\omega(a, b)$ is stable under the derivation operator.

DEFINITION VI. Let φ be a complex function on (a, b). We say that φ is *ω-ultradifferentiable in* $s \in (a, b)$ if for a certain $\varepsilon > 0$, $(s-\varepsilon, s+\varepsilon) \subset (a, b)$ and the restriction of φ to $(s-\varepsilon, s+\varepsilon)$ belongs to $\mathscr{E}_\omega(s-\varepsilon, s+\varepsilon)$.

It is obvious that $\varphi \in \mathscr{E}_\omega(a, b)$ if and only if φ is ω-ultradifferentiable in each $s \in (a, b)$.

Denote by $\mathscr{A}(a, b)$ the vector space of all complex functions on (a, b) which have an analytical extension on a complex neighborhood of (a, b).

LEMMA 2.11. *For* $-\infty \leqslant a < b \leqslant +\infty$ *we have*

$$\mathscr{A}(a, b) \subset \mathscr{E}_\omega(a, b).$$

Proof. Let $\varphi \in \mathscr{A}(a, b)$ and $K \subset (a, b)$ compact, $L > 0$, integer $n \geqslant 1$. By the Cauchy integral formula, there exists a $d > 0$ such that

$$|\varphi^{(k)}(s)| \leqslant d^k k!, \quad k \geqslant 1, s \in K.$$

By Corollary 2.9 and Lemma 1.5(ii)

$$\lim_{m\to+\infty} m \frac{a_m^{\omega,n}}{a_{m-1}^{\omega,n}} = 0;$$

thus

$$c = \sup_{k\geqslant 1} d^k k! L^k a_k^{\omega,n} = \sup_{k\geqslant 1} \prod_{m=1}^{k} \left(dLm \frac{a_m^{\omega,n}}{a_{m-1}^{\omega,n}}\right) < +\infty.$$

We have

$$|\varphi^{(k)}(s)| \leqslant \frac{c}{L^k a_k^{\omega,n}}, \quad k \geqslant 1, s \in K;$$

hence

$$r_{L,n}^{\omega;K}(\varphi) \leqslant \max\{\sup_{s\in K} |\varphi(s)|, c\} < +\infty. ∎$$

In particular, for $z \in C$, the function $s \to e^{zs}$ belongs to $\mathscr{E}_\omega$.

LEMMA 2.12. *Let* $-\infty < a < b < +\infty$ *and* $\varphi \in \mathscr{D}_\omega[a, b]$. *Writing for any integer* $p \geqslant 0$

$$\varphi_p(s) = \frac{2\pi}{b-a} \sum_{m=-p}^{p} \hat{\varphi}\left(\frac{2\pi}{b-a} m\right) e^{i\frac{2\pi}{b-a}ms}, \quad s \in R,$$

we have

$$\lim_{p\to+\infty} r_{L,n}^{\omega;[a,b]}(\varphi-\varphi_p) = 0, \quad L > 0, \text{ integer } n \geqslant 1.$$

Proof. Let $\psi \in \mathscr{D}_\omega[a, b]$. Define the function θ on R by

$$\theta(s) = \psi\left(\frac{b-a}{2\pi} s + \frac{b+a}{2}\right).$$

Then $\theta \in \mathscr{D}_\omega[-\pi, \pi]$. By a classical result on Fourier series (see for example [63], Section 13.25) we have

$$\theta(s) = \sum_{m=-\infty}^{+\infty} c_m e^{ims}, \quad s \in [-\pi, \pi],$$

where $c_m = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \theta(u) e^{-imu} du$ and the series on the right side of the equality converges uniformly on $[-\pi, \pi]$. Hence $\psi_p \to \psi$ uniformly on $[a, b]$ when $p \to +\infty$.

Further, let $k \geqslant 0$ be an integer; it is easy to see that for each integer p we have $(\varphi^{(k)})_p = (\varphi_p)^{(k)}$, and applying the above reasoning to $\psi = \varphi^{(k)}$, we get $(\varphi_p)^{(k)} \to \varphi^{(k)}$ uniformly on $[a, b]$, when $p \to +\infty$.

Finally, let $L > 0$, $n \geqslant 1$ and $p > 0$ integers; for every integer $k \geqslant 0$, $p' > p$ and for $s \in [a, b]$, we have

$$L^k a_k^{\omega,n} |(\varphi_{p'}^{(k)} - \varphi_p^{(k)})(s)| \leqslant \frac{2}{b-a} \sum_{|m|>p} \left|\hat{\varphi}\left(\frac{2}{b-a} m\right) a_k^{\omega,n}\left(L \frac{2}{b-a} m\right)^k\right|$$

$$\leqslant \frac{2}{b-a} \sum_{|m|>p} \left|\hat{\varphi}\left(\frac{2}{b-a} m\right) \omega\left(L \frac{2}{b-a} m\right)^n\right|$$

$$\leqslant \frac{2}{b-a} p_{L,n+2}^{\omega}(\varphi) \sum_{|m|>p} \left|\omega\left(L \frac{2}{b-a} m\right)^{-2}\right|$$

$$\leqslant \frac{2}{b-a} p_{L,n+2}^{\omega}(\varphi) \sum_{|m|>p} \frac{t_1^2(b-a)^2}{t_1^2(b-a)^2 + 4\pi^2 L^2 m^2}.$$

Letting $p' \to +\infty$, we obtain

$$r_{L,n}^{\omega,[a,b]}(\varphi - \varphi_p) \leqslant \frac{2\pi}{b-a} p_{L,n+2}^{\omega}(\varphi) \sum_{|m|>p} \frac{t_1^2(b-a)^2}{t_1^2(b-a)^2 + 4\pi^2 L^2 m^2}$$

and this implies our statement. ∎

THEOREM 2.13. *Let $-\infty \leqslant a < b \leqslant +\infty$. Then the linear combinations of the functions $(a, b) \ni s \to e^{i\lambda s}$, $\lambda \in \boldsymbol{R}$, are dense in $\mathscr{E}_\omega(a, b)$.*

Proof. Let $\varphi \in \mathscr{E}_\omega(a, b)$, $K \subset (a, b)$ compact, $n \geqslant 1$ an integer and choose $a < c < d < b$ such that $K \subset (c, d)$.

By Theorem 2.5 there exists a $\psi \in \mathscr{D}_\omega[c, d]$ such that $\psi(s) = 1$ for s in a neighbourhood of K and by the above lemma there are $c_j \in \boldsymbol{C}$, $\lambda_j \in \boldsymbol{R}$, $j = -p, \ldots, 0, \ldots, p$, such that

$$r_{L,n}^{\omega,[c,d]}\left(\varphi\psi - \sum_{|j|\leqslant p} c_j e^{i\lambda_j \cdot}\right) \leqslant \varepsilon.$$

So

$$r_{L,n}^{\omega,K}\left(\varphi - \sum_{|j|\leqslant p} c_j e^{i\lambda_j \cdot}\right) \leqslant r_{L,n}^{\omega,K}(\varphi - \varphi\psi) + r_{L,n}^{\omega,[c,d]}\left(\varphi\psi - \sum_{|j|\leqslant p} c_j e^{i\lambda_j \cdot}\right) \leqslant \varepsilon. \ ∎$$

Next we shall study the spaces $\mathscr{D}_\omega$ and $\mathscr{E}_\omega$ as topological spaces.

LEMMA 2.14. *Let $-\infty < a < b < +\infty$. Then the topology of $\mathscr{D}_\omega[a, b]$ is defined by the semi-norms*

$$\varphi \to \sum_{m=-\infty}^{+\infty} \left|\hat{\varphi}\left(\frac{2\pi}{b-a} m\right) \omega\left(L \frac{2\pi}{b-a} m\right)^n\right|, \quad L > 0,\ n \geqslant 1 \text{ integer}.$$

Proof. Using the notation from Lemma 2.12, we have

$$r_{L,n}^{\omega}(\varphi) \leqslant r_{L,n}^{\omega,[a,b]}(\varphi - \varphi_p) + r_{L,n}^{\omega,[a,b]}(\varphi_p)$$

$$\leqslant r_{L,n}^{\omega,[a,b]}(\varphi - \varphi_p) + \frac{2\pi}{b-a} \sum_{m=-\infty}^{+\infty} \left|\hat{\varphi}\left(\frac{2\pi}{b-a} m\right) \omega\left(L \frac{2\pi}{b-a} m\right)^n\right|.$$

Letting $p \to +\infty$ and using Lemma 2.11, we obtain

$$r_{L,n}^{\omega}(\varphi) \leqslant \frac{2\pi}{b-a} \sum_{m=-\infty}^{+\infty} \left|\hat{\varphi}\left(\frac{2\pi}{b-a} m\right) \omega\left(L \frac{2\pi}{b-a} m\right)^n\right|.$$

On the other hand,

$$\sum_{m=-\infty}^{+\infty} \left|\hat{\varphi}\left(\frac{2\pi}{b-a} m\right) \omega\left(L \frac{2\pi}{b-a} m\right)^n\right|$$

$$\leqslant \left(\sum_{m=-\infty}^{+\infty} \frac{t_1^2(b-a)^2}{t_1^2(b-a)^2 + 4\pi^2 L^2 m^2}\right) p_{L,n+2}^{\omega}(\varphi). \ ∎$$

THEOREM 2.15. *The spaces*

$$\mathscr{D}_\omega[a, b], \quad -\infty < a < b < +\infty,$$
$$\mathscr{E}_\omega(a, b), \quad -\infty \leqslant a < b \leqslant +\infty,$$

are nuclear Fréchet spaces.

Proof. The verification of the fact that the above spaces are Fréchet uses standard arguments, so we restrict ourselves to the verification of nuclearity.

Let $-\infty < a < b < +\infty$. It is easy to see that the semi-norms on $\mathscr{D}_\omega[a, b]$ defined in Lemma 2.14 are quasi-nuclear in the sense of the terminology of [22], § 27, 1.10. By Lemma 2.14 and [22], § 27, 1.10 follows the nuclearity of $\mathscr{D}_\omega[a, b]$.

Next let $-\infty \leqslant a < b \leqslant +\infty$. Since the topology on $\mathscr{E}_\omega(a, b)$ is the weakest locally convex topology such that all the mappings

$$\mathscr{E}_\omega(a, b) \ni \varphi \to \varphi\psi \in \mathscr{D}_\omega[c, d], \quad \psi \in \mathscr{D}_\omega[c, d], \quad a < c < d < b,$$

are continuous and since $\mathscr{D}_\omega[c, d]$ are nuclear, by [22], § 27, 1.10, it follows that $\mathscr{E}_\omega(a, b)$ is nuclear. ∎

We remark that by the general theory of locally convex vector spaces the fundamental properties from Theorem 2.15 imply further topological properties for the spaces $\mathscr{D}_\omega[a, b]$, $\mathscr{E}_\omega(a, b)$ and also for $\mathscr{D}_\omega$.

For example, by [22], § 24, 3.1, $\mathscr{D}_\omega$ is *complete*; by [22], § 23, 5.3, $\mathscr{D}_\omega$ is *barrelled* and *bornologic*; by [22], § 27, 2.8, $\mathscr{D}_\omega$ is *nuclear*; by [22], § 27, 3.1 and [22], § 24, 2.2, $\mathscr{D}_\omega$ is *Montel*.

$\mathscr{E}_\omega$ is also *complete*; by [22], § 10, 1.2, $\mathscr{E}_\omega$ is *barrelled*; by [22], § 11, 4.1, $\mathscr{E}_\omega$ is *bornologic*; $\mathscr{E}_\omega$ is *nuclear*; by [22], § 27, 3.1, $\mathscr{E}_\omega$ is *Montel*.

We end this section with a treatment of ω-ultradifferential operators.

DEFINITION VII. By an *ω-ultradifferential operator on $\mathscr{D}_\omega$* we mean any linear operator $T: \mathscr{D}_\omega \to \mathscr{D}_\omega$ such that

$$\operatorname{supp}(T\varphi) \subset \operatorname{supp}\varphi, \quad \varphi \in \mathscr{D}_\omega.$$

THEOREM 2.16. *Any ω-ultradifferential operator on $\mathscr{D}_\omega$ is continuous and can be extended to a continuous linear operator from $\mathscr{E}_\omega$ in $\mathscr{E}_\omega$.*

Proof. Let T be an ω-ultradifferential operator on $\mathscr{D}_\omega$. We shall verify first that, for each bounded complex Borel function ψ on $\boldsymbol{R}$, the linear functional T_ψ defined on $\mathscr{D}_\omega$ by

$$T_\psi(\varphi) = \int_{-\infty}^{+\infty} (T\varphi)(s)\psi(s)ds$$

is continuous. We say that T_ψ is continuous in $r \in \boldsymbol{R}$ if there exists an $\varepsilon > 0$ such that the restriction of T_ψ to $\mathscr{D}_\omega[r-\varepsilon, r+\varepsilon]$ is continuous. By Corollaries 2.6 and 2.2, T_ψ is continuous if and only if it is continuous in each $r \in \boldsymbol{R}$.

Let us write

$$\Lambda = \{r \in \boldsymbol{R};\ T_\psi \text{ is not continuous in } r \text{ for a certain } \psi\}.$$

Suppose that the intersection of Λ with a certain compact subset of $\boldsymbol{R}$ is infinite. Then there exists a sequence $\{[a_n, b_n]\}_{n\geqslant 1}$ of mutually disjoint compact intervals $\bigcup_{n=1}^{\infty} [a_n, b_n] \subset [a, b]$ for a certain $-\infty < a < b < +\infty$ and a sequence $\{\psi_n\}$ of complex Borel functions on $\boldsymbol{R}$, $|\psi_n| \leqslant 1$ for all n, such that for each n the functional T_{ψ_n} is non-continuous on $\mathscr{D}_\omega[a_n, b_n]$.

Hence there exist $\varphi_n \in \mathscr{D}_\omega[a_n, b_n]$ such that

$$T_{\psi_n}(\varphi_n) = 2^n, \quad p^{\omega}_{n,n}(\varphi_n) \leqslant 2^{-n}.$$

Denote by χ_S the characteristic function of $S \subset \boldsymbol{R}$. The series $\sum_{n=1}^{\infty} \varphi_n$ converges to a certain φ in $\mathscr{D}_\omega[a, b]$, and if we put $\psi = \sum_{n=1}^{\infty} \chi_{[a_n,b_n]}\psi_n$, we get

$$|T_\psi(\varphi)| \geqslant \Big|\int_{[a_n,b_n]} (T\varphi)(s)\psi(s)ds\Big| - \Big|\int_{\boldsymbol{R}\setminus[a_n,b_n]} (T\varphi)(s)\psi(s)ds\Big|$$

$$\geqslant 2^n - \int_{-\infty}^{+\infty} |(T\varphi)(s)|ds,$$

which is impossible. Thus the intersection of Λ with each compact subset of $\boldsymbol{R}$ is finite.

Let $r \in \boldsymbol{R}$. Then there exists an $\varepsilon > 0$ such that $[r-\varepsilon, r+\varepsilon]$ contains at most an element from Λ. Suppose that there exists an $r_0 \in [r-\varepsilon, r+\varepsilon] \cap \Lambda$ and let ψ be an arbitrary bounded complex Borel function on $\boldsymbol{R}$.

The linear functionals $T_{(1-\chi_{[r_0-1/n, r_0+1/n]})\psi}$ are continuous on $\mathscr{D}_\omega[r-\varepsilon, r+\varepsilon]$ and

$$T_{(1-\chi_{[r_0-1/n, r_0+1/n]})\psi}(\varphi) \underset{n\to\infty}{\to} T_\psi(\varphi), \quad \varphi \in \mathscr{D}_\omega[r-\varepsilon, r+\varepsilon].$$

By Theorem 2.14 and by the Banach–Steinhaus theorem (see [22], § 10, 2.2) it follows that T_ψ is continuous on $\mathscr{D}_\omega[r-\varepsilon, r+\varepsilon]$, in contradiction with the fact that $[r-\varepsilon, r+\varepsilon] \cap \Lambda \neq 0$.

We conclude that each T_ψ is continuous. Hence if $-\infty < a < b < +\infty$, $\varphi_\iota \to \varphi$ in $\mathscr{D}_\omega[a, b]$ and $T(\varphi_\iota) \to \theta$ in $\mathscr{D}_\omega[a, b]$, then for all bounded complex Borel functions ψ on $\boldsymbol{R}$ we have

$$\int_{-\infty}^{+\infty} \theta(s)\psi(s)ds = \lim_\iota \int_{-\infty}^{+\infty} T(\varphi_\iota)(s)\psi(s)ds = \int_{-\infty}^{+\infty} T(\varphi)(s)\psi(s)ds$$

and we get $\theta = T(\varphi)$.

Further, by the closed graph theorem (see [22], § 7, 2.3) it follows that $\mathscr{D}_\omega[a, b] \ni \varphi \to T(\varphi) \in \mathscr{D}_\omega[a, b]$ is continuous; since $-\infty < a < b < +\infty$ are arbitrary, we deduce that T is continuous.

Finally, if $\varphi, \psi \in \mathscr{D}_\omega$ and $\varphi(s) = \psi(s)$ for s in a certain open set $D \subset \boldsymbol{R}$, then $(T\varphi)(s) = (T\psi)(s)$ for $s \in D$. Hence if $\eta \in \mathscr{E}_\omega$ and $\{\varphi_n\}$ is a sequence in $\mathscr{D}_\omega$ such that $\varphi_n(s) = 1$ for $s \in (-n, n)$, then the sequence $\{T(\varphi_n\eta)\}$ converges pointwise to a certain $P(\eta) \in \mathscr{E}_\omega$ which is independent of the choice of $\{\varphi_n\}$.

It is easy to verify that $P: \mathscr{E}_\omega \to \mathscr{E}_\omega$ is a continuous linear operator which obviously extends T. ∎

We remark that in the first part of the proof of the present theorem we used an idea from [47].

By Theorem 2.16 we have a single notion of an ω-ultradifferential operator which acts on $\mathscr{D}_\omega$ or on $\mathscr{E}_\omega$.

COROLLARY 2.17. *Let T be an ω-ultradifferential operator on $\mathscr{E}_\omega$, $s_0 \in \boldsymbol{R}$ and $\varphi \in \mathscr{E}_\omega$ such that $\varphi^{(k)}(s_0) = 0$ for all $k \geqslant 0$. Then $(T\varphi)^{(k)}(s_0) = 0$ for all $k \geqslant 0$.*

Proof. Define the functions φ_1, φ_2 on $\boldsymbol{R}$ by

$$\varphi_1(s) = \begin{cases} \varphi(s), & s \geqslant s_0, \\ 0, & s \leqslant s_0; \end{cases} \qquad \varphi_2(s) = \begin{cases} 0, & s \geqslant s_0, \\ \varphi(s), & s \leqslant s_0. \end{cases}$$

By our hypothesis on $\varphi, \varphi_1, \varphi_2 \in \mathscr{E}_\omega$. For each $\varepsilon > 0$ and $\varphi \in \mathscr{E}_\omega$, we put $\tau_\varepsilon\varphi = \varphi(\cdot - \varepsilon)$; then $\tau_\varepsilon\varphi_1 \in \mathscr{E}_\omega$ and $\operatorname{supp}\tau_\varepsilon\varphi_1 \subset [s_0+\varepsilon, +\infty)$, so that $\operatorname{supp}T(\tau_\varepsilon\varphi_1) \subset [s_0+\varepsilon, +\infty)$. Hence $(T(\tau_\varepsilon\varphi_1))^{(k)}(s_0) = 0$ for all $\varepsilon > 0$ and $k \geqslant 0$. Since $\lim_{\varepsilon\to 0}\tau_\varepsilon\varphi_1 = \varphi_1$ in $\mathscr{E}_\omega$, by Theorem 2.16 it follows that $\lim_{\varepsilon\to 0} T(\tau_\varepsilon\varphi_1) = T\varphi_1$ in $\mathscr{E}_\omega$. Consequently, $(T\varphi_1)^{(k)}(s_0) = 0$ for all $k \geqslant 0$.

Similarly, $(T\varphi_2)^{(k)}(s_0) = 0$ for all $k \geqslant 0$. Thus

$$(T\varphi)^{(k)}(s_0) = (T\varphi_1)^{(k)}(s_0) + (T\varphi_2)^{(k)}(s_0) = 0, \quad k \geqslant 0. \ ∎$$

By Corollary 2.17, if T is an ω-ultradifferential operator, then $(T\varphi)(s_0)$, $(T\varphi)'(s_0), \ldots,$ depend only on $\varphi(s_0), \varphi'(s_0), \ldots,$ for all $\varphi \in \mathscr{E}_\omega$ and $s_0 \in \boldsymbol{R}$.

Next we give a method of constructing ω-ultradifferential operators.

Let f be an entire function of exponential type 0 such that there exist $L_0 > 0$, an integer $n_0 \geq 1$ and $c_0 > 0$ with

$$|f(it)| \leq c_0 |\omega(L_0 t)^{n_0}|, \quad t \in \boldsymbol{R}.$$

If $-\infty < a < b < +\infty$ and $\varphi \in \mathcal{D}_\omega[a, b]$, then by Theorem 2.3, $\boldsymbol{R} \ni t \to f(it)\hat{\varphi}(t)$ is the Fourier transform of some function from $\mathcal{D}_\omega[a, b]$, which we denote by $f(D)\varphi$. Then $f(D)$: $\mathcal{D}_\omega \mapsto \mathcal{D}_\omega$ defined by $(\widehat{f(D)\varphi})(t) = f(it)\hat{\varphi}(t)$ is a linear operator.

If $\varphi \in \mathcal{D}_\omega$ and $s \notin \operatorname{supp}\varphi$, then by Corollaries 2.6 and 2.2, there exist $-\infty < a_1 < b_1 < s < a_2 < b_2 < +\infty$ and $\varphi_1 \in \mathcal{D}_\omega[a_1, b_1]$, $\varphi_2 \in \mathcal{D}_\omega[a_2, b_2]$ such that $\varphi = \varphi_1 + \varphi_2$. Hence the support of $f(D)\varphi = f(D)\varphi_1 + f(D)\varphi_2$ does not contain s.

It follows that $f(D)$ is an ω-ultradifferential operator. We remark moreover that the continuity of $f(D)$ results directly from the definition, without the use of Theorem 2.16.

The following result shows how f is determined by the action of $f(D)$ on $\mathcal{E}_\omega$:

PROPOSITION 2.18. *Let f be an entire function of exponential type* 0 *such that for a certain $L_0 > 0$, an integer $n_0 \geq 1$ and a $c_0 > 0$*

$$|f(it)| \leq c_0 |\omega(L_0 t)^{n_0}|, \quad t \in \boldsymbol{R}.$$

Then

$$f(z) = e^{-zs}\big(f(D)e^{z\cdot}\big)(s), \quad z \in \boldsymbol{C}, s \in \boldsymbol{R}.$$

Proof. Let $z \in \boldsymbol{C}$ and $s \in \boldsymbol{R}$ be arbitrary.

Using Theorem 1.2, it is easy to see that for every $t \in \boldsymbol{R}$

$$|f(it+z)| \leq c_0 \omega(-iL_0|\operatorname{Re} z|)^{n_0} |\omega(L_0(t+\operatorname{Im} z))^{n_0}|$$
$$\leq c_0 \omega(-iL_0|\operatorname{Re} z|)^{n_0} |\omega(\sqrt{2}L_0 \operatorname{Im} z)^{n_0}| \, |\omega(\sqrt{2}L_0 t)^{n_0}|.$$

Let us define the entire function g_z by the formula

$$g_z(\zeta) = \frac{f(\zeta+z)-f(z)}{\zeta}, \quad 0 \neq \zeta \in \boldsymbol{C}.$$

By the above estimation we can consider the ω-ultradifferential operator $g_z(D)$.

By Theorem 2.5 we can choose a $\varphi \in \mathcal{D}_\omega$ such that $\varphi(r) = 1$ for r in some neighbourhood of s. Then $\varphi'(r) = 0$ for r in some neighbourhood of s; hence $(g_z(D)\varphi')(s) = 0$. Thus, using the inversion formula, we get

$$\begin{aligned} f(z) &= (g_z(D)\varphi')(s) + f(z)\varphi(s) \\ &= \int_{-\infty}^{+\infty} g_z(it)\hat{\varphi}'(t)e^{its}dt + f(z)\int_{-\infty}^{+\infty} \hat{\varphi}(t)e^{its}dt \\ &= \int_{-\infty}^{+\infty} \big(f(it+z)-f(z)\big)\hat{\varphi}(t)e^{its}dt + \int_{-\infty}^{+\infty} f(z)\hat{\varphi}(t)e^{its}dt \\ &= \int_{-\infty}^{+\infty} f(it+z)\hat{\varphi}(t)e^{its}dt. \end{aligned}$$

By the Cauchy integral theorem we obtain

$$\begin{aligned} f(z) &= e^{-zs}\int_{-\infty}^{+\infty} f(it)\hat{\varphi}(t+iz)e^{its}dt \\ &= e^{-zs}\int_{-\infty}^{+\infty} f(it)\widehat{(\varphi e^{z\cdot})}(t)e^{its}dt \\ &= e^{-zs}[f(D)(\varphi e^{z\cdot})](s). \end{aligned}$$

Since $\varphi(r)e^{zr} = e^{zr}$ for r in some neighbourhood of s, we conclude that

$$f(z) = e^{-zs}\big(f(D)e^{z\cdot}\big)(s). \quad \blacksquare$$

DEFINITION VIII. We define $\mathcal{G}_\omega$ as the vector space of all functions g: $\boldsymbol{R} \times \boldsymbol{C} \to \boldsymbol{C}$ which satisfy the following conditions:

(i) g is continuous;

(ii) for each $s \in \boldsymbol{R}$ the function $g(s, \cdot)$ is entire and of exponential type 0;

(iii) for each $t \in \boldsymbol{R}$, $g(\cdot, it) \in \mathcal{E}_\omega$ and for every compact $K \subset \boldsymbol{R}$, $L > 0$ and integer $n \geq 1$, there exist an $L_0 > 0$, an integer $n_0 \geq 1$ and a $c_0 > 0$ with

$$r_{L,n}^{\omega,K}\big(g(\cdot, it)\big) \leq c_0 |\omega(L_0 t)^{n_0}|, \quad t \in \boldsymbol{R}.$$

If $g \in \mathcal{G}_\omega$, then, by the above considerations, for every $s \in \boldsymbol{R}$ we can define the ω-ultradifferential operator $g(s, D)$. Further we remark that, for every $\varphi \in \mathcal{E}_\omega$, we can define a function $g(\cdot, D)\varphi$ by the formula:

$$\big(g(\cdot, D)\varphi\big)(s) = \big(g(s, D)\varphi\big)(s), \quad s \in \boldsymbol{R}.$$

We shall prove that $g(\cdot, D)$ is an ω-ultradifferential operator and that every ω-ultradifferential operator is of this form. For this purpose we need some lemmas.

LEMMA 2.19. *For any $L > 0$, integer $n \geq 1$ and $a > 0$, there exists an $c_{L,n,a} > 0$ such that for every $s \in \boldsymbol{R}$ and integer $k \geq 0$, we have*

$$r_{L,n}^{\omega,[s-a,s+a]}\big((\cdot - s)^k\big) \leq c_{L,n,a}(ae)^k.$$

In particular, for every $s \in \boldsymbol{R}$ and $z \in \boldsymbol{C}$

$$e^{z(\cdot - s)} = \sum_{k=0}^{\infty} \frac{z^k}{k!}(\cdot - s)^k,$$

where the series on the right side converges in $\mathcal{E}_\omega$.

Proof. Since ω is an entire function of exponential type 0, for any $L > 0$, integer $n \geq 1$ and $a > 0$, there exists a $c_{L,n,a} > 0$ such that

$$\left|\omega\left(\frac{Lz}{a}\right)^n\right| \leq c_{L,n,a}e^{|z|}, \quad z \in \boldsymbol{C}.$$

So for every $s \in \boldsymbol{R}$ and integer $k > 0$, we have

$$r_{L,n}^{\omega,[s-a,s+a]}((\cdot - s)^k) \leqslant \sup_{0 \leqslant p \leqslant k} L^p a_p^{\omega,n} \frac{k!}{(k-p)!} a^{k-p}$$

$$\leqslant a^k \sup_{0 \leqslant p \leqslant k} a_p^{\omega,n} \left(\frac{Lk}{a}\right)^p \leqslant a^k \left|\omega\left(\frac{Lk}{a}\right)^n\right| \leqslant c_{L,n,a}(ae)^k. \blacksquare$$

LEMMA 2.20. *Let T be an ω-ultradifferential operator. Then for any $L > 0$, integer $n \geqslant 1$, $-\infty < a < b < +\infty$ and $\varepsilon > 0$ there exist an $L_0 > 0$, an integer $n_0 \geqslant 1$ and a $c_0 > 0$ such that*

$$r_{L,n}^{\omega,[a,b]}(T\varphi) \leqslant c_0 r_{L_0,n_0}^{\omega,[a-\varepsilon,b+\varepsilon]}(\varphi), \quad \varphi \in \mathscr{E}_\omega.$$

Proof. By Theorem 2.16, there exist an $L' > 0$, an integer $n' \geqslant 1$, a compact $K' \subset \boldsymbol{R}$ and a $c' > 0$ such that

$$r_{L,n}^{\omega,[a,b]}(T\varphi) \leqslant c' r_{L',n'}^{\omega,K'}(\varphi), \quad \varphi \in \mathscr{E}_\omega.$$

On the other hand, by Theorem 2.5 there exists a $\varphi_0 \in \mathscr{D}_\omega[a-\varepsilon, b+\varepsilon]$ such that $\varphi_0(r) = 1$ for r in some neighbourhood of $[a, b]$.

Let $\varphi \in \mathscr{E}_\omega$ be arbitrary. Since φ and $\varphi_0\varphi$ coincide on some neighbourhood of $[a, b]$, the same holds also for $T\varphi$ and for $T(\varphi_0\varphi)$, so that

$$r_{L,n}^{\omega,[a,b]}(T\varphi) = r_{L,n}^{\omega,[a,b]}(T(\varphi_0\varphi)) \leqslant c' r_{L',n'}^{\omega,K'}(\varphi_0\varphi).$$

But $\operatorname{supp}(\varphi_0\varphi) \subset [a-\varepsilon, b+\varepsilon]$, so that using Proposition 2.9, we get

$$r_{L,n}^{\omega,[a,b]}(T\varphi) \leqslant c' r_{L',n'}^{\omega,[a-\varepsilon,b+\varepsilon]}(\varphi_0\varphi) \leqslant c' r_{2L',n'}^{\omega}(\varphi_0) r_{2L',n'}^{\omega,[a-\varepsilon,b+\varepsilon]}(\varphi).$$

We conclude that, denoting $L_0 = 2L'$, $n_0 = n'$ and $c_0 = c' r_{2L',n'}(\varphi_0)$, we have

$$r_{L,n}^{\omega,[a,b]}(T\varphi) \leqslant c_0 r_{L_0,n_0}^{\omega,[a-\varepsilon,b+\varepsilon]}(\varphi), \quad \varphi \in \mathscr{E}_\omega. \blacksquare$$

We shall next prove our main result concerning ω-ultradifferential operators, extending some results of J. Peetre ([47]) and E. Albrecht ([2]).

THEOREM 2.21. *If $g \in \mathscr{G}_\omega$ then the formula*

$$(g(\cdot, D)\varphi)(s) = (g(s, D)\varphi)(s), \quad \varphi \in \mathscr{E}_\omega, s \in \boldsymbol{R},$$

defines an ω-ultradifferential operator $g(\cdot, D)$ and we have

$$g(s, z) = e^{-zs}(g(\cdot, D)e^{z\cdot})(s), \quad s \in \boldsymbol{R}, z \in \boldsymbol{C}.$$

Conversely, if T is an ω-ultradifferential operator, then the formula

$$g(s, z) = e^{-zs}(Te^{z\cdot})(s), \quad s \in \boldsymbol{R}, z \in \boldsymbol{C},$$

defines a function $g \in \mathscr{G}_\omega$ and we have

$$T = g(\cdot, D).$$

Proof. Let $g \in \mathscr{G}_\omega$ and fix a certain $\varphi \in \mathscr{D}_\omega$. By the inversion formula, for every $s \in \boldsymbol{R}$

$$(g(\cdot, D)\varphi)(s) = \int_{-\infty}^{+\infty} \widehat{g(s, D)\varphi}(t) e^{its} dt = \int_{-\infty}^{+\infty} g(s, it)\hat{\varphi}(t) e^{its} dt;$$

so $g(\cdot, D)\varphi$ is a continuous function. Clearly,

$$\operatorname{supp}(g(\cdot, D)\varphi) \subset \operatorname{supp}\varphi.$$

By Theorem 2.5 there exists a $\varphi_0 \in \mathscr{D}_\omega$ such that $\varphi_0(s) = 1$ for $s \in \operatorname{supp}\varphi$, so that for every $s \in \boldsymbol{R}$ we have

$$\widehat{g(\cdot, D)\varphi}(r) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varphi_0(s)(g(\cdot, D)\varphi)(s) e^{-irs} ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi_0(s) g(s, it)\hat{\varphi}(t) e^{-i(r-t)} dt ds$$

$$= \int_{-\infty}^{+\infty} \widehat{\varphi_0 g(\cdot, it)}(r-t)\hat{\varphi}(t) dt.$$

Further, let $L > 0$ and integer $n \geqslant 1$ be arbitrary; since $g \in \mathscr{G}_\omega$, there exist an $L_0 > 0$, an integer $n_0 \geqslant 1$ and a $c_0 > 0$ such that

$$r_{4L,n}^{\omega,\operatorname{supp}\varphi_0}(g(\cdot, it)) \leqslant c_0 |\omega(L_0 t)^{n_0}|, \quad t \in \boldsymbol{R}.$$

Using Propositions 2.7 and 2.9, we get for every $r, t \in \boldsymbol{R}$

$$|\widehat{\varphi_0 g(\cdot, it)}(r-t)\hat{\varphi}(t)\omega(Lr)^n|$$

$$\leqslant p_{\sqrt{2}L,n}(\varphi_0 g(\cdot, it)) \left|\hat{\varphi}(t) \frac{\omega(Lr)^n}{\omega(\sqrt{2}L(r-t))^n}\right|$$

$$\leqslant \frac{1}{\pi} \sup_{\lambda,\mu \in \operatorname{supp}\varphi_0} (\lambda-\mu) r_{2L,n}^{\omega}(\varphi_0 g(\cdot, it)) |\hat{\varphi}(t)\omega(\sqrt{2}Lt)^n|$$

$$\leqslant \frac{1}{\pi} \sup_{\lambda,\mu \in \operatorname{supp}\varphi_0} (\lambda-\mu) r_{4L,n}^{\omega}(\varphi_0) r_{4L,n}^{\omega,\operatorname{supp}\varphi_0}(g(\cdot, it)) |\hat{\varphi}(t)\omega(\sqrt{2}Lt)^n|$$

$$\leqslant \frac{1}{\pi} \sup_{\lambda,\mu \in \operatorname{supp}\varphi_0} (\lambda-\mu) r_{4L,n}^{\omega}(\varphi_0) |\hat{\varphi}(t)\omega(\max\{L_0, \sqrt{2}L\} t)^{n_0+n}|.$$

Denoting

$$L_1 = \max\{L_0, \sqrt{2}L\}, \quad n_1 = n_0 + n \quad \text{and} \quad c_1 = \frac{c_0}{\pi} \sup_{\lambda,\mu \in \operatorname{supp}\varphi_0} (\lambda-\mu) r_{4L,n}^{\omega}(\varphi_0),$$

we infer that

$$p_{L,n}^{\omega}(g(\cdot, D)\varphi) \leqslant \sup_{r \in R} \int_{-\infty}^{+\infty} |\widehat{\varphi_0 g(\cdot, it)}(r-t)\hat{\varphi}(t)\omega(Lr)^n| dt \leqslant c_1 \cdot q_{L_1,n_1}^{\omega}(\varphi) < +\infty.$$

We conclude that $g(\cdot, D)\varphi \in \mathscr{D}_\omega$. Thus $g(\cdot, D)$ is an ω-ultradifferential operator and the formula

$$g(s, z) = e^{-zs}(g(\cdot, D)e^{z\cdot})(s), \quad s \in \boldsymbol{R}, z \in \boldsymbol{C},$$

is an immediate consequence of Propostion 2.18.

Now let T be an ω-ultradifferential operator and consider the function g: $\boldsymbol{R}\times\boldsymbol{C}\mapsto\boldsymbol{C}$ defined by

$$g(s,z)=e^{-zs}(Te^{z\cdot})(s).$$

Let $b>0$ be arbitrary; by Theorem 2.16 there exist $a>0$, $L>0$, integer $n\geqslant 1$ and $c>0$ such that

$$\sup_{s\in[-b,b]}|(T\varphi)(s)|\leqslant cr_{L,n}^{\omega;[-a,a]}(\varphi),\qquad \varphi\in\mathscr{E}_\omega.$$

If $c_{L,n,a}>0$ is as in Lemma 2.19, then for every $s\in[-b,b]$ and $z\in\boldsymbol{C}$ we have

$$\big|\big(T((\cdot)^k)\big)(s)\big|\leqslant cr_{L,n}^{\omega;[-a,a]}\big((\cdot)^k\big)\leqslant cc_{L,n,a}(ae)^k,\qquad k\geqslant 0 \text{ integer},$$

$$(Te^{z\cdot})(s)=\sum_{k=0}^{\infty}\frac{z^k}{k!}\big(T((\cdot)^k)\big)(s).$$

So for every $s_0, s\in[-b,b]$ and $z_0, z\in\boldsymbol{C}$

$$\begin{aligned}&|(Te^{z_0\cdot})(s_0)-(Te^{z\cdot})(s)|\\ &\quad\leqslant|(Te^{z_0\cdot})(s_0)-(Te^{z_0\cdot})(s)|+|(Te^{z_0\cdot})(s)-(Te^{z\cdot})(s)|\\ &\quad\leqslant|(Te^{z_0\cdot})(s_0)-(Te^{z_0\cdot})(s)|+\sum_{k=0}^{\infty}\frac{|z_0^k-z^k|}{k!}\big|\big(T((\cdot)^k)\big)(s)\big|\\ &\quad\leqslant|(Te^{z_0\cdot})(s_0)-(Te^{z_0\cdot})(s)|+cc_{L,n,a}\sum_{k=0}^{\infty}\frac{|z_0^k-z^k|(ae)^k}{k!}.\end{aligned}$$

We conclude that $\boldsymbol{R}\times\boldsymbol{C}\ni(s,z)\to(Te^{z\cdot})(s)$ is continuous, and hence g is continuous.

Let $s\in\boldsymbol{R}$ be fixed; by Lemma 2.19 we have for every $z\in\boldsymbol{C}$

$$g(s,z)=\sum_{k=0}^{\infty}\frac{z^k}{k!}\big(T((\cdot-s)^k)\big)(s),$$

where the series on the right side converges. By Lemma 2.20, for every $\varepsilon>0$ there exist $L_\varepsilon>0$, integer $n_\varepsilon>1$ and $c_\varepsilon>0$ such that

$$|(T\varphi)(s)|\leqslant c_\varepsilon r_{L,n}^{\omega;[s-\varepsilon,s+\varepsilon]}(\varphi),\qquad \varphi\in\mathscr{E}_\omega.$$

Choosing $c_{L_\varepsilon,n_\varepsilon,\varepsilon}>0$ as in Lemma 2.19, we get for every integer $k\geqslant 0$

$$\big|\big(T((\cdot-s))\big)(s)\big|\leqslant c_\varepsilon r_{L,n}^{\omega;[s-\varepsilon,s+\varepsilon]}\big((\cdot-s)^k\big)\leqslant c_\varepsilon c_{L_\varepsilon,n,\varepsilon}(\varepsilon e)^k.$$

Hence $\boldsymbol{C}\ni z\mapsto g(s,z)$ is an entire function and for every $z\in\boldsymbol{C}$ we have

$$|g(s,z)|\leqslant\sum_{k=0}^{\infty}\frac{|z|^k}{k!}c_\varepsilon c_{L_\varepsilon,n_\varepsilon,\varepsilon}(\varepsilon e)^k=c_\varepsilon c_{L_\varepsilon,n_\varepsilon,\varepsilon}e^{\varepsilon e|z|}.$$

Since $\varepsilon>0$ is arbitrary, we conclude that $\boldsymbol{C}\ni z\mapsto g(s,z)$ is of exponential type 0.

It is clear that $g(\cdot,it)\in\mathscr{E}_\omega$, for every $t\in\boldsymbol{R}$. Finally, let $\boldsymbol{K}\subset\boldsymbol{R}$ be compact, $L>0$ and integer $n\geqslant 1$. By Theorem 2.16 there exist compact $\boldsymbol{K}'\subset\boldsymbol{R}$, $L'>0$, an integer $n'\geqslant 1$ and a $c'>0$ such that

$$r_{2L,n}^{\omega;K}(T\psi)\leqslant c'r_{L',n'}^{\omega;K'}(\psi),\qquad \psi\in\mathscr{E}_\omega.$$

By Proposition 2.9, for every $t\in\boldsymbol{R}$ we have

$$\begin{aligned}r_{L,n}^{\omega;K}\big(g(\cdot,it)\big)&\leqslant r_{2L,n}^{\omega;K}(e^{-it\cdot})r_{2L,n}^{\omega;K}(Te^{it\cdot})\\ &\leqslant r_{2L,n}^{\omega;K}(e^{-it\cdot})c'r_{L',n'}^{\omega;K'}(e^{it\cdot})\\ &=c'\big(\sup_{k\geqslant 0}(2L)^k a_k^{\omega,n}|t|^k\big)\big(\sup_{k\geqslant 0}(L')^k a_k^{\omega,n'}|t|^k\big)\\ &\leqslant c'|\omega(2Lt)^n\omega(L't)^{n'}|.\end{aligned}$$

Denoting $L_0=\max\{2L,L'\}$, $n_0=n+n'$ and $c_0=c'$, we conclude that

$$r_{L,n}^{\omega;K}\big(g(\cdot,it)\big)\leqslant c_0|\omega(L_0t)^{n_0}|,\qquad t\in\boldsymbol{R}.$$

In conclusion, $g\in\mathscr{G}_\omega$. By the definition of g and by Proposition 2.18, for every $z\in\boldsymbol{C}$ and $s\in\boldsymbol{R}$ we have

$$(Te^{z\cdot})(s)=e^{zs}g(s,z)=\big(g(s,D)e^{z\cdot}\big)(s)=\big(g(\cdot,D)e^{z\cdot}\big)(s),$$

so that

$$Te^{z\cdot}=g(\cdot,D)e^{z\cdot},\qquad z\in\boldsymbol{C}.$$

Now using Theorems 2.12 and 2.16, we obtain

$$T=g(\cdot,D).\ \blacksquare$$

By the above theorem, the mapping

$$\mathscr{G}_\omega\ni g\mapsto g(\cdot,D)$$

is an isomorphism between the vector space $\mathscr{G}_\omega$ and the vector space of all ω-ultradifferential operators.

COROLLARY 2.22. *If $g\in\mathscr{G}_\omega$ then*

$$\frac{\partial g}{\partial s},\ \frac{\partial g}{\partial z}\in\mathscr{G}_\omega.$$

Proof. We define the ω-ultradifferential operator T by

$$T\varphi=\big(g(\cdot,D)\varphi\big)',\qquad \varphi\in\mathscr{D}_\omega.$$

By Theorem 2.21,

$$g(s,z)=e^{-zs}\big(g(\cdot,D)e^{z\cdot}\big)(s),\qquad s\in\boldsymbol{R},\ z\in\boldsymbol{C},$$

and so

$$\frac{\partial g}{\partial s}(s,z)=e^{-zs}(Te^{z\cdot})(s)-zg(s,z),\qquad s\in\boldsymbol{R},\ z\in\boldsymbol{C}.$$

Again by Theorem 2.21, $\partial g/\partial s\in\mathscr{G}_\omega$.

Next we consider the ω-ultradifferential operator $S=g(\cdot,D)\,\mathrm{Mult}$, where Mult is defined by

$$(\mathrm{Mult}\,\varphi)(s)=s\varphi(s),\qquad \varphi\in\mathscr{D}_\omega,\ s\in\boldsymbol{R}.$$

Since

$$\frac{\partial g}{\partial s}(s,z) = e^{-zs}(Se^{z\cdot})(s) - sg(s,z), \qquad s \in \boldsymbol{R},\ z \in \boldsymbol{C},$$

by Theorem 2.21, $\partial g/\partial s \in \mathscr{G}_\omega$. ■

DEFINITION IX. Let $g \in \mathscr{G}_\omega$. The functions $c_k^g\colon \boldsymbol{R} \to \boldsymbol{C}$, $k \geqslant 0$, defined by

$$g(s,z) = \sum_{k=0}^{\infty} c_k^g(s) z^k, \qquad s \in \boldsymbol{R},\ z \in \boldsymbol{C},$$

are called the *coefficients of the ω-ultradifferential operator* $g(\cdot, D)$.

Using Corollary 2.22, we get

$$c_k^g = \frac{1}{k!}\frac{\partial^k g}{\partial z^k}(\cdot, 0) \in \mathscr{E}_\omega, \qquad k \geqslant 0.$$

It is easy to see that the ω-ultradifferential operators with constant coefficients are exactly those of the form $f(D)$, where f is an entire function of exponential type zero such that for a certain $L_0 > 0$ an integer $n_0 \geqslant 1$ and a $c_0 > 0$

$$|f(it)| \leqslant c_0|\omega(L_0 t)^{n_0}|, \qquad t \in \boldsymbol{R}.$$

If f is as above and $g \in \mathscr{G}_\omega$, then $\boldsymbol{R}\times\boldsymbol{C} \ni (s,z) \to g(s,z)f(z)$ belongs to $\mathscr{G}_\omega$ and

$$(gf)(\cdot, D) = g(\cdot, D)\circ f(D).$$

Indeed, by Theorem 2.21, $(gf)(\cdot, D)$ and $g(\cdot, D)\circ f(D)$ coincide on the functions $e^{z\cdot}$, $z \in \boldsymbol{C}$; hence by applying Theorems 2.12 and 2.16, the above assertion follows.

In particular, if $f_1(D)$ and $f_2(D)$ are ω-ultradifferential operators with constant coefficients, then

$$(f_1 f_2)(D) = f_1(D)\circ f_2(D) = f_2(D)\circ f_1(D).$$

DEFINITION X. For integer $n \geqslant 1$ and $k \geqslant 0$ we define $c_k^{\omega,n}$ as the *coefficient of z^k in the power series expansion of $\omega(-iz)^n$.*

Hence

$$c_0^{\omega,1} = 1,$$

$$c_k^{\omega,1} = \sum_{j_1<\ldots<j_k} \frac{1}{t_{j_1}\ldots t_{j_k}}, \qquad k \geqslant 1,$$

and

$$c_k^{\omega,n} = \sum_{k_1+\ldots+k_n=k} c_{k_1}^{\omega,1}\ldots c_{k_n}^{\omega,1}, \qquad n \geqslant 1,\ k \geqslant 0.$$

It is clear that we have

$$a_k^{\omega,n} \leqslant c_k^{\omega,n} \leqslant c_k^{\omega,n+1}, \qquad n \geqslant 1,\ k \geqslant 0. \tag{2.9}$$

PROPOSITION 2.23. *Let $g \in \mathscr{G}_\omega$ and let c_k^g, $k \geqslant 0$, be the coefficients of $g(\cdot, D)$. Then for any compact $\boldsymbol{K} \subset \boldsymbol{R}$, any $L > 0$ and any integer $n \geqslant 1$ there exist an $L_0 > 0$, an integer $n_0 \geqslant 1$ and a $c_0 > 0$ such that*

$$r_{L,n}^{\omega,K}(c_k^g) \leqslant c_0 L_0 c_k^{\omega,n}, \qquad k \geqslant 0. \tag{2.10}$$

Proof. Let $p \geqslant 0$ be an integer and let $s \in \boldsymbol{K}$. Then

$$L^p a_p^{\omega,n}\left|\frac{\partial^p g}{\partial s^p}(s, it)\right| \leqslant c_0|\omega(L_0 t)^{n_0}|, \qquad t \in \boldsymbol{R}.$$

By Corollary 2.22, $\dfrac{\partial^p g}{\partial s^p}(s, \cdot)$ is an entire function of exponential type 0, so that by Theorem 1.3, for every integer $k \geqslant 0$ we have

$$L^p a_p^{\omega,n}\left|\frac{\partial^{p+k} g}{\partial s^p \partial z^k}(s, it)\right| \leqslant c_0 L_0^k|(\omega^{n_0})^{(k)}(L_0 t)|, \qquad t \in \boldsymbol{R}.$$

Hence for all $k \geqslant 0$ we have

$$L^p a_p^{\omega,n}|(c_k^g)^{(p)}(s)| = L^p a_p^{\omega,n}\frac{1}{k!}\left|\frac{\partial^{p+k} g}{\partial s^p \partial t^k}(s, 0)\right|$$

$$\leqslant c_0 L_0^k \frac{1}{k!}|(\omega^{n_0})^{(k)}(0)| = c_0 L_0^k c_k^{\omega,n}.$$

Since $p \geqslant 0$ and $s \in \boldsymbol{K}$ are arbitrary, (2.10) results. ■

Proposition 2.23 gives a necessary condition in order that a sequence $\{c_k\}_{k\geqslant 0} \subset \mathscr{E}_\omega$ form the coefficient of some ω-ultradifferential operator. Next we shall give also a sufficient condition.

PROPOSITION 2.24. *Let $\{c_k\}_{k\geqslant 0} \subset \mathscr{E}_\omega$ be a sequence such that for every compact $\boldsymbol{K} \subset \boldsymbol{R}$, every $L > 0$ and integer $n \geqslant 1$, there are an $L_0 > 0$, an integer $n_0 \geqslant 1$ and a $c_0 > 0$ such that*

$$r_{L,n}^{\omega,K}(c_k) \leqslant c_0 L_0^k a_k^{\omega,n}, \qquad k \geqslant 0.$$

Then $g\colon \boldsymbol{R}\times\boldsymbol{C} \to \boldsymbol{C}$ defined by

$$g(s,z) = \sum_{k=0}^{\infty} c_k(s) z^k, \qquad s \in \boldsymbol{R},\ z \in \boldsymbol{C},$$

belongs to $\mathscr{G}_\omega$ and

$$g(\cdot, D) = \sum_{k=0}^{\infty} c_k D^k,$$

where the series on the right side converge in the space $\mathscr{L}(\mathscr{E}_\omega)$, endowed with the topology of the uniform convergence on the bounded subsets of $\mathscr{E}_\omega$.

Proof. Let $\boldsymbol{K} \subset \boldsymbol{R}$ be compact, let $L > 0$ and let $n \geqslant 1$ be an integer. By our assumptions, there exist an $L_0 > 0$, an integer $n_0 \geqslant 1$ and a $c_0 > 0$ with

$$r_{2L,n}^{\omega,K}(c_k) \leqslant c_0 L_0^k a_k^{\omega,n}, \qquad k \geqslant 0.$$

Denoting $L_1 = \max\{2L_0, 2L\}$, $n_1 = n_0 + n$, and using Proposition 2.9 and the quite evident inequality

$$a_{k'}^{\omega,n'} a_{k''}^{\omega,n''} \leqslant a_{k'+k''}^{\omega,n'+n''}, \qquad n', n'' \geqslant 1,\ k', k'' \geqslant 0$$

we have for every $\varphi \in \mathscr{E}_\omega$ and integer $k \geqslant 0$

$$\begin{aligned} r^{\omega,K}_{L,n}(c_k D^k \varphi) &\leqslant r^{\omega,K}_{2L,n}(c_k) r^{\omega,K}_{2L,n}(\varphi^{(k)}) \\ &\leqslant c_0 L_0^k a_k^{\omega,n} \sup_{p\geqslant 0} [(2L)^p a_p^{\omega,n} \sup_{s\in K} |\varphi^{(k+p)}(s)|] \\ &\leqslant c_0 2^{-k} \sup_{p\geqslant 0} [L_1^{k+p} a_{k+p}^{\omega,n} \sup_{s\in K} |\varphi^{(k+p)}(s)|] \leqslant c_0 2^{-k} r^{\omega,K}_{L_1,n_1}(\varphi). \end{aligned}$$

It follows that for every $\varphi \in \mathscr{E}_\omega$ the series $\sum_{k=0}^{\infty} c_k D^k \varphi$ converges in $\mathscr{E}_\omega$ to a certain $T\varphi \in \mathscr{E}_\omega$. Clearly, T is an ω-ultradifferential operator, and so by Theorem 2.15 it is continuous. For every compact $K \subset \boldsymbol{R}$, $L > 0$ and integer $n \geqslant 1$, we have

$$r^{\omega,K}_{L,n}\Big(T\varphi - \sum_{k=0}^{p} c_k D^k \varphi\Big) \leqslant c_0 2^{-p+1} r^{\omega,K}_{L_1,n_1}(\varphi), \qquad p \geqslant 0,\ \varphi \in \mathscr{E}_\omega,$$

where $L_1 > 0$, $n_1 \geqslant 1$ and $c_0 > 0$ are as above.

So

$$T = \sum_{k=0}^{\infty} c_k D^k$$

where the series on the right side converges in the linear space of all continuous linear mappings on $\mathscr{E}_\omega$, $\mathscr{L}(\mathscr{E}_\omega)$, endowed with the topology of the uniform convergence on the bounded subsets of $\mathscr{E}_\omega$.

Moreover, for every $z \in \boldsymbol{C}$ and $s \in \boldsymbol{R}$ we have

$$(Te^{z\cdot})(s) = \sum_{k=0}^{\infty} c_k(s) z^k e^{zs} = g(s,z) e^{zs},$$

that is,

$$g(s,z) = e^{-zs}(Te^{z\cdot})(s).$$

Now by Theorem 2.20, we get $g \in \mathscr{G}_\omega$ and $T = g(\cdot, D)$. ∎

In particular, if $\{c_k\}_{k\geqslant 0} \subset \mathscr{E}_\omega$ is such that for every compact $K \subset \boldsymbol{R}$ the functions c_k vanish on K for k sufficiently large, then $(s,z) \mapsto \sum_{k=0}^{\infty} c_k(s) z^k$ belongs to $\mathscr{G}_\omega$ and the corresponding ω-ultradifferential operator is $\sum_{k=0}^{\infty} c_k D^k$.

Finally we shall characterize the situation in which all ω-ultradifferential operators can be approximated in a suitable way by "ω-ultradifferential operators of finite degree".

THEOREM 2.25. *The following statements concerning ω are equivalent*:

(i) *there exist an $L_0 > 0$, an integer $n_0 \geqslant 1$ and a $c_0 > 0$ such that*

$$\omega(-it) \leqslant c|\omega(L_0 t)^{n_0}|, \qquad t > 0;$$

(ii) *there exist an $L_1 > 0$, an integer $n_1 \geqslant 1$ and a $c_1 > 0$ such that*

$$c_k^{\omega,1} \leqslant c_1 L_1 a_k^{\omega,n}, \qquad k \geqslant 0;$$

(iii) *there exist an $L_2 > 0$, an integer $n_2 \geqslant 1$ and a $c_2 > 0$ such that*

$$c_k^{\omega,n} \leqslant c_2^n L_2^k a_k^{\omega,nn_2}, \qquad k \geqslant 0,\ n \geqslant 1;$$

(iv) *for every ω-ultradifferential operator T, denoting by c_k the coefficients of T, we have*

$$T = \sum_{k=0}^{\infty} c_k D^k$$

where the series on the right side converges in the space $\mathscr{L}(\mathscr{E}_\omega)$, endowed with the topology of the uniform convergence on the bounded subsets of $\mathscr{E}_\omega$.

Proof. (i) $\Rightarrow$ (iii). For all integers $n \geqslant 1$, $k \geqslant 1$, we have

$$\begin{aligned} c_k^{\omega,n} &\leqslant \left(\frac{\sqrt{2}L_0 a_k^{\omega,nn_0}}{a_{k-1}^{\omega,nn_0}}\right)^k \omega\left(-i\frac{a_{k-1}^{\omega,nn_0}}{\sqrt{2}L_0 a_k^{\omega,nn_0}}\right)^n \\ &\leqslant c_0^n \left(\frac{\sqrt{2}L_0 a_k^{\omega,nn_0}}{a_{k-1}^{\omega,nn_0}}\right)^k \left|\omega\left(\frac{a_{k-1}^{\omega,nn_0}}{\sqrt{2} a_k^{\omega,nn_0}}\right)^{nn_0}\right| \\ &\leqslant \sqrt{2} c_0^n \left(\frac{\sqrt{2}L_0 a_k^{\omega,nn_0}}{a_{k-1}^{\omega,nn_0}}\right)^k \sup_{p\geqslant 0} a_p^{\omega,nn_0} \left(\frac{a_{k-1}^{\omega,nn_0}}{a_k^{\omega,nn_0}}\right)^p \\ &= \sqrt{2} c_0^n (\sqrt{2}L_0)^k \left(\frac{a_k^{\omega,nn_0}}{a_{k-1}^{\omega,nn_0}}\right)^k \max\left\{1, \sup_{p\geqslant 1} \prod_{q=1}^{p} \left(\frac{a_q^{\omega,nn_0}}{a_{q-1}^{\omega,nn_0}} \cdot \frac{a_{k-1}^{\omega,nn_0}}{a_k^{\omega,nn_0}}\right)\right\}. \end{aligned}$$

By Corollary 2.9,

$$\max\left\{1, \sup_{p\geqslant 1} \prod_{q=1}^{p} \left(\frac{a_q^{\omega,nn_0}}{a_k^{\omega,nn_0}} \cdot \frac{a_{k-1}^{\omega,nn_0}}{a_k^{\omega,nn_0}}\right)\right\} = \prod_{q=1}^{k} \frac{a_q^{\omega,nn_0}}{a_{q-1}^{\omega,nn_0}} \cdot \frac{a_{k-1}^{\omega,nn_0}}{a_k^{\omega,nn_0}} = a_k^{\omega,nn_0} \left(\frac{a_{k-1}^{\omega,nn_0}}{a_k^{\omega,nn_0}}\right)^k;$$

hence

$$c_k^{\omega,n} \leqslant \sqrt{2} c_0^n (\sqrt{2}L_0)^k a_k^{\omega,nn_0}.$$

So (iii) is satisfied for $L_2 = \sqrt{2}L_0$, $n_2 = n_0$ and $c_2 = \sqrt{2}c_0$. The implication (iii) $\Rightarrow$ (iv) is an immediate consequence of Propositions 2.22 and 2.23.

(iv) $\Rightarrow$ (ii). By (iv) we have

$$\omega(-iD) = \sum_{k=0}^{\infty} c_k^{\omega,1} D^k,$$

where the series on the right side converges in the topology of the uniform convergence on the bounded subsets of $\mathscr{E}_\omega$. Hence the sequence $\{c_k^{\omega,1} D^k\}_{k\geqslant 0}$ converges to 0 in the same topology.

It follows that, if we denote by F_k the linear functional $\mathscr{E}_\omega \ni \varphi \mapsto c_k^{\omega,1}\varphi^{(k)}(0)$, the sequence $\{F_k\}_{k\geqslant 0}$ converges to 0 in the strong topology of the dual space of $\mathscr{E}_\omega$. Then, by Theorem 2.15 and by [9], Ch. IV, § 2, Theorem 1, there exist compact $K \subset \boldsymbol{R}$, an $L_1 > 0$, an integer $n_1 \geqslant 1$ and a $c_1 > 0$ such that

$$c_k^{\omega,1}|\varphi^{(k)}(0)| = |F_k(\varphi)| \leqslant c_1 r^{\omega,K}_{L_1,n_1}(\varphi), \qquad k \geqslant 0,\ \varphi \in \mathscr{E}_\omega.$$

Applying this inequality for $\varphi = e^{it\cdot}$, we get for every $t \in \boldsymbol{R}$, $k > 0$

$$c_k^{\omega,1}|t|^k \leqslant c_1 \sup_{p \geqslant 0} L_1^p a_p^{\omega,n_1}|t|^p.$$

For $k = 0$ and $t = 0$ the above inequality gives

$$c_0^{\omega,1} \leqslant c_1 L_1^0 a_0^{\omega,n_1},$$

and for $k \geqslant 1$, by Corollary 2.9, we have:

$$c_k^{\omega,1}\left(\frac{a_{k-1}^{\omega,n_1}}{L_1 a_k^{\omega,n_1}}\right)^k \leqslant c_1 \sup_{p \geqslant 0} a_p^{\omega,n}\left(\frac{a_{k-1}^{\omega,n_1}}{a_k^{\omega,n_1}}\right)^p$$

$$= c_1 \max\left\{1, \sup_{p \geqslant 1} \prod_{q=1}^{p}\left(\frac{a_q^{\omega,n_1}}{a_{q-1}^{\omega,n_1}} \cdot \frac{a_{k-1}^{\omega,n_1}}{a_k^{\omega,n_1}}\right)\right\} = c_1 a_k^{\omega,n_1}\left(\frac{a_{k-1}^{\omega,n_1}}{a_k^{\omega,n_1}}\right)^k,$$

so that

$$c_k^{\omega,1} \leqslant c_1 L_1^k a_k^{\omega,n}.$$

(ii) $\Rightarrow$ (i). For every $t > 0$ we have

$$\omega(-it) = \sum_{k=0}^{\infty} 2^{-k} c_k^{\omega,1}(2t)^k \leqslant 2 \sup_{k \geqslant 0} c_k^{\omega,1}(2t)^k$$
$$\leqslant 2c_1 \sup_{k \geqslant 0} a_k^{\omega,n}(2L_1 t)^k \leqslant 2c_1|\omega(2L_1 t)^n|;$$

so (i) is satisfied for $L_0 = 2L_1$, $n_0 = n_1$ and $c_0 = 2c_1$. ∎

DEFINITION XI. We say that ω *satisfies the strong non-quasianalyticity condition* if the equivalent conditions from Theorem 2.25 are fulfilled.

If ω satisfies the strong non-quasianalyticity condition, then by Corollary 1.9

$$\sum_{k=1}^{\infty} \frac{\ln k}{t_k} < +\infty. \tag{2.11}$$

So if $t_1 = 1$ and $t_k = k(\ln k)^\varepsilon$, $k \geqslant 2$, $1 < \varepsilon \leqslant 2$, then $\sum_{k=1}^{\infty} \frac{\ln k}{t_k} = +\infty$; hence $\omega_{\{t_k\}}$ does not satisfy the strong non-quasianalyticity condition.

In spite of the fact that not every ω satisfies the strong non-quasianalyticity condition, we shall prove in the next section that every ω-ultradifferential operator for arbitrary ω can be approximated by "ω-ultradifferential operators of finite degree" in the topology of the uniform convergence on the bounded subsets of $\mathscr{E}_\omega$.

3. ω-ultradistributions

Let $t_1, t_2, \ldots > 0$, $t_1 < +\infty$ be such that $\sum_{k=1}^{\infty} 1/t_k < +\infty$; also in this section we shall denote $\omega_{\{t_k\}}$ simply by ω.

For $-\infty \leqslant a < b \leqslant +\infty$ we put

$$\mathscr{D}_\omega(a, b) = \bigcup_{a<c<d<b} \mathscr{D}_\omega[c, d]$$

and endow $\mathscr{D}_\omega(a, b)$ with the *inductive limit topology*.

DEFINITION XII. By *ω-ultradistributions on* (a, b) we mean the elements of the strong dual of $\mathscr{D}_\omega(a, b)$ denoted by $\mathscr{D}'_\omega(a, b)$.

By [22], § 24, 2.2, and [22], § 26, 2.1, $\mathscr{D}'_\omega(a, b)$ is the projective limit of the strong dual spaces $\mathscr{D}'_\omega[c, d]$, $a < c < d < b$.

By Theorem 2.15 and [22], § 23, 5.3, $\mathscr{D}_\omega(a, b)$ is *bornologic*, and hence by [22], § 13, 2.2, $\mathscr{D}'_\omega(a, b)$ is *complete*; moreover, by Theorem 2.15, [22], § 27, 3.1, [22], § 23, 5.3, and [22], § 24, 2.2, $\mathscr{D}_\omega(a, b)$ is *Montel*, so that by [22], § 22, 2.7, $\mathscr{D}'_\omega(a, b)$ is also *Montel*; in particular, $\mathscr{D}'_\omega(a, b)$ is *barrelled*.

Further, by Theorem 2.15, [22], § 24, 3.1, and [22], § 27, 2.8, $\mathscr{D}_\omega(a, b)$ is *complete and nuclear*, so that by a theorem of L. Schwartz (see [57] or [29], Ch. III, § 1) $\mathscr{D}'_\omega(a, b)$ is *bornologic*; finally, by Theorem 2.15 and [22], § 27.3.3, for any $a < c < d < b$ the strong dual space of $\mathscr{D}_\omega[c, d]$ is *nuclear*, so that by [22], § 27, 2.2, $\mathscr{D}'_\omega(a, b)$ is *nuclear*.

In conclusion, $\mathscr{D}'_\omega(a, b)$ is *complete, barrelled, bornologic, nuclear* and *Montel*.

Let $\varphi \in \mathscr{E}_\omega(a, b)$ and $F \in \mathscr{D}'_\omega(a, b)$; then the functional $\mathscr{D}_\omega(a, b) \ni \psi \to F(\varphi\psi)$ belongs to $\mathscr{D}'_\omega(a, b)$ and we denote it by F. It is easy to verify that $\mathscr{E}_\omega(a, b) \times \mathscr{D}'_\omega(a, b) \ni (\varphi, E) \mapsto \varphi F \in \mathscr{D}'_\omega(a, b)$ is a separately continuous bilinear mapping. Since $\mathscr{E}_\omega(a, b)$ and $\mathscr{D}'_\omega(a, b)$ are barrelled, by [9], Ch. III, § 4, Proposition 6, this mapping is hypocontinuous.

If $F \in L^1_{\mathrm{loc}}(a, b)$, then the functional $\mathscr{D}_\omega(a, b) \ni \varphi \mapsto \int_a^b \varphi(s)F(s)\,ds$ belongs to $\mathscr{D}'_\omega(a, b)$ and we denote it again by F. Hence we have

$$\mathscr{D}_\omega(a, b) \subsetneq \mathscr{E}_\omega(a, b) \subsetneq L^1_{\mathrm{loc}}(a, b) \subsetneq \mathscr{D}'_\omega(a, b)$$

where the inclusions are continuous and with a dense range, as can easily be verified by using the Hahn–Banach Theorem and the fact that $\mathscr{D}_\omega(a, b)$ is Montel.

Let $-\infty \leqslant a \leqslant c < d \leqslant b \leqslant +\infty$; then the inclusion mappings $\mathscr{D}_\omega(c, d) \subsetneq \mathscr{D}_\omega(a, b)$ is continuous, and hence its dual mapping $\mathscr{D}'_\omega(a, b) \to \mathscr{D}'_\omega(c, d)$ is well defined and continuous. We denote the image of $F \in \mathscr{D}'_\omega(a, b)$ in $\mathscr{D}'_\omega(c, d)$ by $F|(c, d)$ and call it the *restriction* of F to (c, d).

Using Corollary 2.6, it is easy to see that the *ω-ultradistributions with the restriction mappings form a sheaf on* $\boldsymbol{R}$. Namely, if $-\infty \leqslant a < b \leqslant +\infty$, $\{(a_\iota, b_\iota)\}_{\iota \in I}$ is a family of open subintervals of (a, b) such that $(a, b) = \bigcup_{\iota \in I} (a_\iota, b_\iota)$ and $F_\iota \in \mathscr{D}'_\omega(a_\iota, b_\iota)$, $\iota \in I$, are such that $F_\iota|(a_\iota, b_\iota) \cap (a_\varkappa, b_\varkappa) = F_\varkappa|(a_\iota, b_\iota) \cap (a_\varkappa, b_\varkappa)$ whenever $(a_\iota, b_\iota) \cap (a_\varkappa, b_\varkappa) \neq \emptyset$, then there exists a unique $F \in \mathscr{D}'_\omega(a, b)$ satisfying $F|(a_\iota, b_\iota) = F_\iota$ for all $\iota \in I$.

Let $-\infty \leqslant a < b \leqslant +\infty$ and $F \in \mathscr{D}'_\omega(a, b)$; then the *support of* F, $\operatorname{supp} F \subset (a, b)$, is defined by:

$s \notin \operatorname{supp} F \Leftrightarrow$ there exists an $\varepsilon > 0$, $(s-\varepsilon, s+\varepsilon) \subset (a, b)$ such that
$$F|(s-\varepsilon, s+\varepsilon) = 0.$$

It is clear that $\operatorname{supp} F$ is a closed subset of (a, b) and that if $\varphi \in \mathscr{D}_\omega(a, b)$, $\operatorname{supp}\varphi \cap \operatorname{supp} F = \emptyset$, then $F(\varphi) = 0$.

PROPOSITION 3.1. *Let $-\infty \leqslant a < b \leqslant +\infty$. Each element of $\mathscr{D}'_\omega(a, b)$ with compact support can be extended to a continuous linear functional on $\mathscr{E}_\omega(a, b)$. Conversely, the restriction of any continuous linear functional on $\mathscr{E}_\omega(a, b)$ to $\mathscr{D}_\omega(a, b)$ is an element of $\mathscr{D}'_\omega(a, b)$ with compact support.*

Proof. Let $F \in \mathscr{D}'_\omega(a, b)$ with compact support. By Theorem 2.5, there exists a $\varphi \in \mathscr{D}_\omega(a, b)$ such that $\varphi(s) = 1$ for s in some neighbourhood of $\operatorname{supp} F$. Then $F = \varphi F$ and the desired extension of F is $\mathscr{E}_\omega(a, b) \ni \psi \mapsto F(\varphi\psi)$.

Conversely, let G be a continuous linear functional on $\mathscr{E}_\omega(a, b)$. Then there exist compact $K \subset (a, b)$, $L > 0$, integer $n \geqslant 1$, and $c > 0$, such that

$$|G(\varphi)| \leqslant c r_{L,n}^{\omega;K}(\varphi), \quad \varphi \in \mathscr{E}_\omega(a, b).$$

Hence the restriction F of G to $\mathscr{D}_\omega(a, b)$ belongs to $\mathscr{D}'_\omega(a, b)$ and $\operatorname{supp} F \subset K$. ∎

Let us denote by $\mathscr{E}'_\omega(a, b)$ the *strong dual space of* $\mathscr{E}_\omega(a, b)$. By Proposition 3.1 the elements of $\mathscr{E}'_\omega(a, b)$ can be identified with ω-ultradistributions with compact support on (a, b). Using Theorem 2.15, it is easy to verify that $\mathscr{E}'(a, b)$ is *complete, barrelled, bornologic, nuclear* and *Montel.*

We have $\mathscr{D}_\omega(a, b) \subsetneq \mathscr{E}'_\omega(a, b)$ where the inclusion is continuous and with a dense range.

We shall denote $\mathscr{D}'_\omega(-\infty, +\infty)$ and $\mathscr{E}'_\omega(-\infty, +\infty)$ briefly by $\mathscr{D}'_\omega$ and $\mathscr{E}'_\omega$, respectively.

Let $F \in \mathscr{E}'_\omega$; then the *Fourier transform* $\hat{F}$ *of* F is defined, as usual, by

$$\hat{F}(t) = \frac{1}{2\pi} F(e^{-it\cdot}), \quad t \in \boldsymbol{R}.$$

Using Lemma 2.18, it is easy to see that F can be extended to an entire function, denoted also by F, such that

$$F(z) = \frac{1}{2\pi} F(e^{-iz\cdot}), \quad z \in \boldsymbol{C}.$$

By Theorem 2.13, an ω-ultradistribution with compact support is uniquely determined by its Fourier transform.

The following technical result is similar to that from [43], Ch. I, Lemma 1.1.6:

LEMMA 3.2. *Let $L > 0$ and let $n \geqslant 1$ be integer. If $f \in L^1(\boldsymbol{R})$ is such that*

$$\int_{-\infty}^{+\infty} f(t)\hat{\varphi}(t)\omega(Lt)^n dt = 0, \quad \varphi \in \mathscr{D}_\omega,$$

then f vanishes almost everywhere.

Proof. Denote

$$\mathscr{B} = \{\hat{\varphi}\cdot|\omega(L\cdot)|^{2n};\ \varphi \in \mathscr{D}_\omega\}.$$

By Theorem 2.3 $\mathscr{B}$ is contained in the involutive algebra $C_0(\boldsymbol{R})$ of all continuous complex functions on $\boldsymbol{R}$ which vanish at ∞.

As $\omega(-iLD)$ is an ω-ultradifferential operator with constant coefficients, for every $\varphi_1, \varphi_2 \in \mathscr{D}_\omega$, we have

$$\hat{\varphi}_1\cdot|\omega(L\cdot)|^{2n}\hat{\varphi}_2\cdot|\omega(L\cdot)|^{2n} = \widehat{[\omega(-iLD)^n\varphi_1] * [\omega(-iLD)^n\varphi_2]}|\omega(L\cdot)|^{2n}$$

and for every $\varphi \in \mathscr{D}_\omega$

$$\overline{\hat{\varphi}|\omega(L\cdot)|^{2n}} = \widehat{\overline{\varphi(-\cdot)}}|\omega(L\cdot)|^{2n}.$$

Hence $\mathscr{B}$ is an involutive subalgebra of $C_0(\boldsymbol{R})$.

Let $t_1, t_2 \in \boldsymbol{R}$ be such that $\psi(t_1) = \psi(t_2)$ for all $\psi \in \mathscr{B}$. Then

$$\int_{-\infty}^{+\infty} \varphi(s)[e^{-it_1 s}|\omega(Lt_1)|^{2n} - e^{-it_2 s}|\omega(Lt_2)|^{2n}]ds = 0, \quad \varphi \in \mathscr{D}_\omega.$$

By using Theorem 2.5, it follows that

$$e^{-it_1 s}|\omega(Lt_1)|^{2n} = e^{-it_2 s}|\omega(Lt_2)|^{2n}, \quad s \in \boldsymbol{R}.$$

In particular, for $s = 0$, we have $|\omega(Lt_1)|^{2n} = |\omega(Lt_2)|^{2n}$, so that

$$e^{-it_1 s} = e^{-it_2 s}, \quad s \in \boldsymbol{R}.$$

Consequently, $t_1 = t_2$.

Applying the Stone–Weierstrass theorem, we can conclude that $\mathscr{B}$ is uniformly dense in $C_0(\boldsymbol{R})$.

By our hypothesis on f, for every $\psi \in \mathscr{B}$ we have

$$\int_{-\infty}^{+\infty} f(t)\psi(t)dt = 0.$$

But then the above equality holds for every $\psi \in C_0(\boldsymbol{R})$, and hence f vanishes almost everywhere. ∎

Next we prove a Paley–Wiener type theorem for ω-ultradistributions with compact support:

THEOREM 3.3. *Let $-\infty < a < b < +\infty$ and let f be an entire function. Then the following statements are equivalent:*

(i) *the restriction of f to $\boldsymbol{R}$ is the Fourier transform of a certain $F \in \mathscr{E}'_\omega$ with $\operatorname{supp} F \subset [a, b]$;*

(ii) *f is of finite exponential type, $h_f(-\pi/2) \leqslant -a$, $h_f(\pi/2) \leqslant b$ and there exist $L > 0$ and integer $n \geqslant 1$ such that*

$$\int_{-\infty}^{+\infty} |f(t)\omega(Lt)^{-n}|dt < +\infty;$$

(iii) *f is of finite exponential type, $h_f(-\pi/2) \leqslant -a$, $h_f(\pi/2) \leqslant b$ and there exist $L > 0$ and integer $n \geqslant 1$ such that*

$$\sup_{t\in\boldsymbol{R}} |f(t)\omega(Lt)^{-n}| < +\infty;$$

(iv) *there exist* $L > 0$ *and integer* $n \geqslant 1$ *such that*

$$c = \sup_{t\in R} |f(t)\omega(Lt)^{-n}| < +\infty$$

and

$$|f(z)| \leqslant \begin{cases} ce^{a\operatorname{Im}z}|\omega(Lz)^n|, & \operatorname{Im}z \leqslant 0, \\ ce^{b\operatorname{Im}z}|\bar{\omega}(Lz)^n|, & \operatorname{Im}z \geqslant 0. \end{cases}$$

Proof. (i) $\Rightarrow$ (iii). By the continuity of F there exist $L > 0$, integer $n \geqslant 1$ and $d > 0$ such that

$$|F(\psi)| \leqslant dp_{2^{-1}L,n}(\psi), \quad \psi \in \mathcal{D}_\omega[a-1, b+1].$$

Let $0 < \varepsilon < 1$ be arbitrary. By Theorem 2.5, there exists a $\varphi \in \mathcal{D}_\omega[a-\varepsilon, b+\varepsilon]$, such that $\varphi(s) = 1$ for s in a neighbourhood of $\operatorname{supp}F$. Then for each $z \in C$ we have

$$f(z) = \hat{F}(z) = \frac{1}{2\pi}F(\varphi e^{-iz\cdot}),$$

so that, using Theorem 2.3, we have

$$|f(z)| \leqslant \frac{d}{2\pi}\sup_{t\in R}|\hat{\varphi}(t+z)\omega(2^{-1}Lt)^n|$$

$$= \frac{d}{2\pi}\sup_{t\in R}|\hat{\varphi}(t+i\operatorname{Im}z)\omega(2^{-1}L(t-\operatorname{Re}z))^n|$$

$$\leqslant \frac{d}{2\pi}\sup_{t\in R}|\hat{\varphi}(t+i\operatorname{Im}z)\omega(Lt)^n||\omega(L\operatorname{Re}z)^n|$$

$$\leqslant \frac{d}{2\pi}p_{L,n}(\varphi)e^{H_{[a-\varepsilon,b+\varepsilon]}(\operatorname{Im}z)}|\omega(L\operatorname{Re}z)^n|.$$

Consequently, f is of finite exponential type and $\sup_{t\in R}|f(t)\omega(Lt)^{-n}| < +\infty$. Since $0 < \varepsilon < 1$ is arbitrary, it follows also that

$$h_f(-\pi/2) \leqslant -a \quad \text{and} \quad h_f(\pi/2) \leqslant b.$$

(iii) $\Rightarrow$ (ii). Clearly,

$$\int_{-\infty}^{+\infty}|f(t)\omega(Lt)^{-n-2}|dt \leqslant \int_{-\infty}^{+\infty}\left(1+\frac{L^2t^2}{t_1^2}\right)^{-1}dt\cdot\sup_{t\in R}|f(t)\omega(Lt)^{-n}| < +\infty.$$

(ii) $\Rightarrow$ (i). We define a linear functional F on $\mathcal{D}_\omega$ by the formula

$$F(\varphi) = 2\pi\int_{-\infty}^{+\infty}f(t)\hat{\varphi}(-t)dt, \quad \varphi \in \mathcal{D}_\omega.$$

Then

$$|F(\varphi)| \leqslant 2\pi\int_{-\infty}^{+\infty}|f(t)\omega(Lt)^{-n}|dt\cdot p_{L,n}(\varphi), \quad \varphi \in \mathcal{D}_\omega,$$

so that $F \in \mathcal{D}_\omega'$.

Let $\varphi \in \mathcal{D}_\omega$, $\operatorname{supp}\varphi \subset [b+\varepsilon, +\infty)$ for a certain $\varepsilon > 0$. By Theorem 2.3, $C \ni z \mapsto f(z)\hat{\varphi}(-z)$ is an entire function of finite exponential type,

$$h_{f(\cdot)\hat{\varphi}(-\cdot)}(\pi/2) \leqslant h_f(\pi/2)+h_{\hat{\varphi}}(-\pi/2) \leqslant b+(-b-\varepsilon) = -\varepsilon$$

and for every $L' > 0$ and integer $n' \geqslant 1$

$$\int_{-\infty}^{+\infty}|f(t)\hat{\varphi}(-t)\omega(L't)^{n'}|dt$$

$$\leqslant \int_{-\infty}^{+\infty}|f(t)\omega(Lt)^{-n}|dt\cdot\sup_{t\in R}|\hat{\varphi}(-t)\omega(Lt)^n\omega(L't)^{n'}| < \infty.$$

By applying again Theorem 2.3, it follows that there exists a $\psi \in \mathcal{D}_\omega$, $\operatorname{supp}\psi \subset (-\infty, -\varepsilon]$, such that

$$f(z)\hat{\varphi}(-z) = \hat{\psi}(z), \quad z \in C.$$

Thus

$$F(\varphi) = 2\pi\int_{-\infty}^{+\infty}\hat{\psi}(t)dt = 2\pi\psi(0) = 0.$$

Consequently, $\operatorname{supp}F \subset (-\infty, b]$. Analogously we can prove that $\operatorname{supp}F \subset [a, +\infty)$, so that $\operatorname{supp}F \subset [a, b]$.

Further, let $\varphi \in \mathcal{D}_\omega$ be arbitrary. By Theorem 2.5, there exists a $\psi \in \mathcal{D}_\omega$ such that $\psi(s) = 1$ for s in a neighbourhood of $[a, b]$ and for $s \in -\operatorname{supp}\varphi$. By the Plancherel formula, for each $r \in R$ we have

$$\int_{-\infty}^{+\infty}\hat{\psi}(t-r)\hat{\varphi}(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\psi(s)e^{irs}\varphi(-s)ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-ir(-s)}\varphi(-s)ds = \hat{\varphi}(r),$$

so that we get

$$\int_{-\infty}^{+\infty}\hat{F}(t)\hat{\varphi}(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\psi e^{-it\cdot})\hat{\varphi}(t)dt$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(r)\hat{\psi}(t-r)\hat{\varphi}(t)drdt$$

$$= \int_{-\infty}^{+\infty}f(r)\hat{\varphi}(r)dr.$$

Using Lemma 3.2, we conclude that $\hat{F} = f$.

Finally, the equivalence (iii) $\Leftrightarrow$ (iv) is a consequence of Corollary 1.2. ■

It is easy to see that the equivalent conditions from Theorem 3.3 are also equivalent to

(v) *there exist $L > 0$ and integer $n \geqslant 1$ such that for each $\varepsilon > 0$ there exists a $c_\varepsilon > 0$ with*

$$|f(z)| \leqslant c_\varepsilon e^{H_{[a-\varepsilon, b+\varepsilon]}(\operatorname{Im} z)} |\omega(L \operatorname{Re} z)^n|, \quad z \in \boldsymbol{C}.$$

Moreover, we formulate the following inversion formula:

COROLLARY 3.4. *For any $F \in \mathscr{E}'_\omega$ we have*

$$F(\varphi) = 2\pi \int_{-\infty}^{+\infty} \hat{F}(t)\hat{\varphi}(-t)\,dt.$$

Proof. By the proof of implication (ii) $\Rightarrow$ (i) from Theorem 3.3, we can define an ω-ultradistribution $G \in \mathscr{E}'_\omega$ by the formula

$$G(\varphi) = 2\pi \int_{-\infty}^{+\infty} \hat{F}(t)\hat{\varphi}(-t)\,dt, \quad \varphi \in \mathscr{D}_\omega,$$

and we have $\hat{G} = \hat{F}$. Hence $F = G$. ∎

Further, let f be an entire function of exponential type 0 such that for a certain $L_0 > 0$, an integer $n_0 \geqslant 1$ and a $c_0 > 0$, we have

$$|f(it)| \leqslant c_0 |\omega(L_0 t)^{n_0}|, \quad t \in \boldsymbol{R}.$$

Then the entire function $z \to f(-z)$ satisfies the same conditions, and so we can consider the ω-ultradifferential operator with constant coefficients $f(-D)$. By Theorem 2.16, it acts continuously on $\mathscr{D}_\omega$ and on $\mathscr{E}_\omega$, and hence we can consider its dual mappings on $\mathscr{D}'_\omega$ and on $\mathscr{E}'_\omega$, respectively. Using the Plancherel formula, it is easy to verify that these dual mappings coincide with $f(D)$ on $\mathscr{D}_\omega$. Consequently, $f(D)$ can be extended to continuous linear mappings on $\mathscr{D}'_\omega$ and on $\mathscr{E}'_\omega$, respectively. We denote these extensions also by $f(D)$.

It is clear that

$$\operatorname{supp} f(D)F \subset \operatorname{supp} F, \quad F \in \mathscr{D}'_\omega.$$

Thus, if $F \in \mathscr{E}'_\omega$, then $f(D)F \in \mathscr{E}'_\omega$ and using Corollary 3.4 and Lemma 3.2, we easily obtain

$$\widehat{f(D)F}(t) = f(it)\hat{F}(t), \quad t \in \boldsymbol{R}.$$

We are now able to characterize ω-ultradistributions with one-point support:

THEOREM 3.5. *Let $s_0 \in \boldsymbol{R}$ and $F \in \mathscr{D}'_\omega$, $\operatorname{supp} F \subset \{s_0\}$. Then the formula*

$$f(z) = e^{s_0 z}\hat{F}(-iz), \quad z \in \boldsymbol{C}, \tag{3.1}$$

defines an entire function f of exponential type 0 such that for a certain $L_0 > 0$, an integer $n_0 \geqslant 1$ and a $c_0 > 0$

$$|f(it)| \leqslant c_0 |\omega(L_0 t)^{n_0}|, \quad t \in \boldsymbol{R},$$

and we have

$$F = 2\pi f(D)\,\delta_{s_0},$$

where δ_{s_0} is the Dirac measure in s_0.

Proof. By Theorem 3.3 there exist an $L_0 > 0$, an integer $n_0 \geqslant 1$ and a $c_0 > 0$ such that

$$|\hat{F}(z)| \leqslant \begin{cases} c_0 e^{s_0 \operatorname{Im} z} |\omega(L_0 z)^{n_0}|, & \operatorname{Im} z \leqslant 0, \\ c_0 e^{s_0 \operatorname{Im} z} |\overline{\omega}(L_0 z)^{n_0}|, & \operatorname{Im} z \geqslant 0. \end{cases}$$

Hence the function f, defined by (3.1), is an entire function of exponential type 0 and

$$|f(it)| \leqslant c_0 |\omega(L_0 t)^{n_0}|, \quad t \in \boldsymbol{R}.$$

Since for every $t \in \boldsymbol{R}$ we have

$$\widehat{f(D)\,\delta_{s_0}}(t) = f(it)\,\hat{\delta}_{s_0}(t) = \frac{1}{2\pi}\hat{F}(t),$$

it follows that

$$F = 2\pi f(D)\,\delta_{s_0}. \ \blacksquare$$

In particular, if $F \in \mathscr{D}'_\omega$ is such that $\operatorname{supp} F \subset \{0\}$, then

$$F = 2\pi\hat{F}(-iD)\,\delta_0.$$

Just as for distributions [17] (for other references see [12], Ch. I, § 4), we have the following

DEFINITION XIII. A one-parameter family $\{F_s\}_{s \in \boldsymbol{R}}$ of ω-ultradistributions is *composable* if for every $\varphi \in \mathscr{D}_\omega$ the function $\boldsymbol{R} \ni s \to F_s(\varphi)$ belongs to $\mathscr{D}_\omega$.

All linear continuous mappings in $\mathscr{L}(\mathscr{D}_\omega)$ can be described in terms of composable one-parameter families of ω-ultradistributions.

PROPOSITION 3.6. *For every composable family $\mathscr{F} = \{F_s\}_{s \in \boldsymbol{R}} \subset \mathscr{D}'_\omega$ the formula*

$$(T_{\mathscr{F}}\varphi)(s) = F_s(\varphi), \quad \varphi \in \mathscr{D}_\omega, \ s \in \boldsymbol{R},$$

defines a continuous linear operator $T_{\mathscr{F}}\colon \mathscr{D}_\omega \to \mathscr{D}_\omega$ and the mapping $\mathscr{F} \to T_{\mathscr{F}}$ is a bijection between all composable one-parameter families of ω-ultradistributions and all continuous linear operators.

Proof. Let $\mathscr{F} = \{F_s\}_{s \in \boldsymbol{R}} \subset \mathscr{D}'_\omega$ be a composable family; then it is obvious that $T_{\mathscr{F}}\colon \mathscr{D}_\omega \to \mathscr{D}_\omega$ is a linear operator.

Let $\{\varphi_n\} \subset \mathscr{D}_\omega$ be such that $\varphi_n \to 0$ and $T_{\mathscr{F}}\varphi_n \to \psi$ in $\mathscr{D}_\omega$. Then for every $s \in \boldsymbol{R}$ we have

$$\psi(s) = \lim_{n\to\infty} (T_{\mathscr{F}}\varphi_n)(s) = \lim_{n\to\infty} F_s(\varphi_n) = 0,$$

so that $\psi = 0$.

Consequently, the graph of $T_{\mathscr{F}}$ is closed. By a theorem of A. Grothendieck (see for example [21], Theorem 6.7.1) we conclude that $T_{\mathscr{F}}$ is continuous.

Moreover, it is clear that the mapping $\mathscr{F} \to T_{\mathscr{F}}$ is injective. On the other hand, if $T\colon \mathscr{D}_\omega \to \mathscr{D}_\omega$ is an arbitrary linear continuous operator, then, considering the adjoint operator $T'\colon \mathscr{D}'_\omega \to \mathscr{D}'_\omega$ and defining

$$T_s = T'\delta_s \in \mathscr{D}'_\omega, \quad s \in \boldsymbol{R},$$

we have

$$F_s(\varphi) = \delta_s(T\varphi) = (T\varphi)(s), \quad \varphi \in \mathscr{D}_\omega,\ s \in \boldsymbol{R},$$

so that $\mathscr{F} = \{F_s\}_{s\in\boldsymbol{R}}$ is a composable family of ω-ultradistributions and $T = T_{\mathscr{F}}$. Hence the surjectivity of $\mathscr{F} \to T_{\mathscr{F}}$ is proved. ∎

Next we characterize ω-ultradifferential operators in terms of composable one-parameter families of ω-ultradistributions:

PROPOSITION 3.7. *Let $\mathscr{F} = \{F_s\}_{s\in\boldsymbol{R}} \subset \mathscr{D}'_\omega$ be a composable family. Then $T_{\mathscr{F}}$ is an ω-ultradifferential operator if and only if*

$$\operatorname{supp} F_s \subset \{s\}, \quad s \subset \boldsymbol{R}.$$

Proof. Clearly, if for every $s \in \boldsymbol{R}$, $\operatorname{supp} F_s \subset \{s\}$, then $T_{\mathscr{F}}$ is an ω-ultradifferential operator.

Conversely, assume that $T_{\mathscr{F}}$ is an ω-ultradifferential operator. It is easy to verify that the adjoint operator $T'_{\mathscr{F}}\colon \mathscr{D}'_\omega \to \mathscr{D}'_\omega$ satisfies the condition

$$\operatorname{supp}(T'_{\mathscr{F}}F) \subset \operatorname{supp} F, \quad F \in \mathscr{D}'_\omega.$$

Since $F_s = T'_{\mathscr{F}} \cdot \delta_s$, for every $s \in \boldsymbol{R}$, it follows that

$$\operatorname{supp} F_s \subset \{s\}, \quad s \in \boldsymbol{R}. \ \blacksquare$$

We remark that a function $g \in \mathscr{G}_\omega$ and the composable family $\{F_s\}_{s\in\boldsymbol{R}} \subset \mathscr{D}'_\omega$ corresponding to the operator $g(\cdot, D)$ are connected via the Fourier transformation:

$$g(s,z) = 2\pi e^{-zs}\hat{F}_s(iz), \quad s \in \boldsymbol{R},\ z \in \boldsymbol{C}.$$

We shall formulate further the result announced at the end of § 2: namely, even when the strong non-quasianalyticity condition for the function ω fails, every ω-ultradifferential operator T can be "locally" developed in a series of the form $T = \sum_{k=1}^{\infty} d_k\omega_k(D)$, where $d_1, d_2, \ldots \in \mathscr{E}_\omega$ and ω_k are particular ω-ultradifferential operators with constant coefficients. This will allow us to approximate T in $\mathscr{L}(\mathscr{E}_\omega)$ by "ω-ultradifferential operators of finite degree".

DEFINITION XIV. For any $L > 0$ and any integer $n \geqslant 1$, we denote by $\mathscr{F}^{L,n}_\omega$ the *vector space of all entire functions f of exponential type* 0 *for which* $\boldsymbol{R} \ni t \mapsto f(t)\omega(Lt)^{-n}$ belongs to $L^2(\boldsymbol{R})$, endowed with the scalar product

$$(f_1|f_2)^{L,n}_\omega = \int_{-\infty}^{+\infty} f_1(t)\overline{f_2(t)}|\omega(Lt)^{-2n}|dt, \quad f_1, f_2 \in \mathscr{F}^{L,n}_\omega.$$

The norm corresponding to the above scalar product will be denoted by $\|\cdot\|^{L,n}_\omega$.

LEMMA 3.8. *For any $L > 0$ and any integer $n \geqslant 1$, $\mathscr{F}^{L,n}_\omega$ is a Hilbert space.*

Proof. Let $\{f_k\}_{k\geqslant 1}$ be a Cauchy sequence in $\mathscr{F}^{L,n}_\omega$. Then the sequence $\{f_k\omega(L\cdot)^{-n}\}_{k\geqslant 1}$ converges in $L^2(\boldsymbol{R})$ to a certain g.

For every $k \geqslant 0$, $f_k\omega(L\cdot)^{-n-1}$ belongs to $L^1(\boldsymbol{R})$, and so by Theorem 3.3 there exists an $F_k \in \mathscr{E}'_\omega$, $\operatorname{supp}(F_k) \subset \{0\}$, such that $f_k = \hat{F}_k$. On the other hand, we can define a linear functional F on $\mathscr{D}_\omega$ by

$$F(\varphi) = 2\pi \int_{-\infty}^{+\infty} g(t)\omega(Lt)^n\hat{\varphi}(-t)dt, \quad \varphi \in \mathscr{D}_\omega.$$

Using Corollary 3.4, for any $k \geqslant 0$ and $\varphi \in \mathscr{D}_\omega$, we get:

$$\begin{aligned}|F(\varphi)-F_k(\varphi)| &= 2\pi\Big|\int_{-\infty}^{+\infty}[g(t)-f_k(t)\omega(Lt)^{-n}]\omega(Lt)^n\hat{\varphi}(-t)dt\Big| \\ &\leqslant 2\pi\|g-f_k\omega(L\cdot)^{-n}\|_{L^2(\boldsymbol{R})}\|\omega(L\cdot)^n\hat{\varphi}\|_{L^2(\boldsymbol{R})} \\ &\leqslant 2\pi\|g-f_k\omega(L\cdot)^{-n}\|_{L^2(\boldsymbol{R})}[p^{\omega}_{L,n}(\varphi)q^{\omega}_{L,n}(\varphi)]^{1/2}.\end{aligned}$$

Hence the sequence $\{F_k\}_{k\geqslant 1}$ is convergent in $\mathscr{D}'_\omega$ and its limit is F. Consequently $F \in \mathscr{D}'_\omega$ and $\operatorname{supp} F \subset \{0\}$.

Let us denote $f = \hat{F}$. By Theorem 3.3, f is an entire function of exponential type 0 and by Corollary 3.4

$$F(\varphi) = 2\pi\int_{-\infty}^{+\infty} f(t)\hat{\varphi}(-t)dt, \quad \varphi \in \mathscr{D}_\omega.$$

By using Lemma 3.2, it follows that $f = g\omega(L\cdot)^n$. Thus $f \in \mathscr{F}^{L,n}_\omega$ and $f_k \to f$ in $\mathscr{F}^{L,n}_\omega$. ∎

In the study of the space $\mathscr{F}^{L,n}_\omega$ we shall need the following Cauchy type integral formula:

LEMMA 3.9. *Let $g\colon \{z \in \boldsymbol{C}; \operatorname{Im} z \leqslant 0\} \to \boldsymbol{C}$ be continuous, analytical and of exponential type* 0 *on* $\{z \in \boldsymbol{C}; \operatorname{Im} z < 0\}$, *such that $\boldsymbol{R} \ni t \mapsto g(t)$ belongs to* $L^2(\boldsymbol{R})$. *Then, for each integer $p \geqslant 0$,*

$$g^{(p)}(z) = -\frac{p!}{2\pi i}\int_{-\infty}^{+\infty}\frac{g(t)}{(t-z)^{p+1}}dt, \quad z \in \boldsymbol{C},\ \operatorname{Im} z < 0.$$

Proof. Let $\varepsilon > 0$ be arbitrary. We define a continuous function g_ε on $\{z \in \boldsymbol{C}; \operatorname{Im} z \leqslant 0\}$ by the formula

$$g_\varepsilon(z) = \frac{1}{\varepsilon}\int_0^\varepsilon g(z+s)ds.$$

It is clear that g_ε is analytical and of exponential type 0 on $\{z \in \boldsymbol{C}; \operatorname{Im} z < 0\}$. Since

$$\sup_{t\in\boldsymbol{R}}|g_\varepsilon(t)| \leqslant \frac{1}{\varepsilon}\Big(\int_0^\varepsilon ds\Big)^{1/2}\Big(\int_0^\varepsilon |g(t+s)|^2ds\Big)^{1/2} \leqslant \frac{1}{\sqrt{\varepsilon}}\Big(\int_{-\infty}^{+\infty}|g(s)|^2ds\Big)^{1/2},$$

by the Phragmén–Lindelöf principle ([38], Ch. I, Th. 22) it follows that

$$|g_\varepsilon(z)| \leqslant \frac{1}{\sqrt{\varepsilon}}\Big(\int_{-\infty}^{+\infty}|g(s)|^2 ds\Big)^{1/2}, \quad z\in \boldsymbol{C},\ \operatorname{Im} z\leqslant 0.$$

We have also

$$\int_{-\infty}^{+\infty}|g_\varepsilon(t)|^2dt = \frac{1}{\varepsilon^2}\int_0^\varepsilon\int_0^\varepsilon\Big(\int_{-\infty}^{+\infty} g(t+s)\overline{g(t+r)}\,dt\Big)ds\,dr \leqslant \int_{-\infty}^{+\infty}|g(t)|^2dt$$

and thus $\boldsymbol{R}\ni t\mapsto g_\varepsilon(t)$ belongs to $L^2(\boldsymbol{R})$.

By using [20], Lemma XI.3.1, it is easy to verify that $\lim\limits_{0<\varepsilon\to 0} g_\varepsilon = g$ in $L^2(\boldsymbol{R})$.

On the other hand, $\lim\limits_{0<\varepsilon\to 0} g^{(p)}(z) = g^{(p)}(z)$ for every integer $p\geqslant 0$ and $z\in\boldsymbol{C}$, $\operatorname{Im} z<0$. Hence it suffices to prove our statement under the additional assumption

$$c = \sup_{\operatorname{Im} z\leqslant 0}|g(z)| < +\infty.$$

Let $p\geqslant 0$ be an integer and $z\in\boldsymbol{C}$, $\operatorname{Im} z<0$.

For any $\delta>0$, denoting

$$G_\delta(w) = \frac{ig(w)}{i-\delta w}, \quad w\in\boldsymbol{C},\ \operatorname{Im} w\leqslant 0,$$

and using the Cauchy integral formula, for each $r>|z|$, we get

$$\Big|G_\delta^{(p)}(z) + \frac{p!}{2\pi i}\int_{-r}^{r}\frac{G_\delta(t)}{(t-z)^{p+1}}dt\Big|$$

$$= \Big|\frac{p!}{2\pi}\int_{-\pi}^{0}\frac{ig(re^{i\theta})}{(i-\delta re^{i\theta})(re^{i\theta}-z)^{p+1}}\Big| re^{i\theta}d\theta \leqslant \frac{cp!}{2\delta(r-|z|)^{p+1}}.$$

Consequently, for every $\delta>0$

$$G_\delta^{(p)}(z) = -\frac{p!}{2\pi i}\int_{-\infty}^{+\infty}\frac{G_\delta(t)}{(t-z)^{p+1}}dt.$$

Letting $\delta\to 0$, we obtain our statement. ∎

Let $L>0$ and let $n\geqslant 1$ be an integer; for any integer $k\geqslant 1$ we write

$$r_k^{L,n} = \frac{tp}{L}, \quad n(p-1)<k\leqslant np.$$

Then

$$\omega(Lz)^n = \prod_{k=1}^{\infty}\Big(1+\frac{iz}{r_k^{L,n}}\Big), \quad z\in\boldsymbol{C}.$$

Following V. Gurariĭ ([28]), we can define in $\mathscr{F}_\omega^{L,n}$ the sequence $\{\omega_k^{L,n}\}_{k\geqslant 1}$ by the formula

$$\omega_k^{L,n}(z) = \sqrt{\frac{r_k^{L,n}}{\pi}}\,\frac{(z+ir_1^{L,n})\ldots(z+ir_{k-1}^{L,n})}{(z-ir_1^{L,n})\ldots(z-ir_{k-1}^{L,n})(z-ir_k^{L,n})}\cdot\omega^n(Lz).$$

The next lemma was established in [28], § 2, in a more general context:

LEMMA 3.10. *For any $L>0$ and any integer $n\geqslant 1$, the sequence $\{\omega_k^{L,n}\}_{k\geqslant 1}$ is an orthonormal basis in $\mathscr{F}_\omega^{L,n}$.*

Proof. A simple computation shows that $\{\omega_k^{L,n}\}_{k\geqslant 1}$ is an orthonormal sequence in $\mathscr{F}_\omega^{L,n}$. We need only to prove that if $f\in\mathscr{F}_\omega^{L,n}$ is such that

$$(f|\omega_k^{L,n})_\omega^{L,n} = 0, \quad k\geqslant 1,$$

then f vanishes identically.

For simplicity we shall denote $r_k^{L,n}$ simply by r_k. Using Lemma 3.9, we have

$$0 = (f|\omega_1^{L,n})_\omega^{L,n} = \sqrt{\frac{r_1}{\pi}}\int_{-\infty}^{+\infty}\frac{f(t)}{\omega(Lt)^n}\cdot\frac{1}{t+ir_1}dt = -2\pi i\sqrt{\frac{r_1}{\pi}}\cdot\frac{f(-ir_1)}{\omega(-iLr_1)^n},$$

so that $f(-ir_1)=0$. Consequently we can define $f_1\in\mathscr{F}_\omega^{L,n}$ by the formula

$$f_1(z) = f(z)\frac{z-ir_1}{z+ir_1}, \quad z\in\boldsymbol{C},\ z\neq -ir_1.$$

Using again Lemma 3.9 as above to compute $(f_1|\omega_2^{L,n})_\omega^{L,n}$, we find that $f(-ir_2)=0$. Hence we can similarly define $f_2\in\mathscr{F}_\omega^{L,n}$ by

$$f_2(z) = f_1(z)\frac{z-ir_2}{z+ir_2} = f(z)\frac{(z-ir_1)(z-ir_2)}{(z+ir_1)(z+ir_2)}, \quad z\in\boldsymbol{C},\ z\neq -ir_1, -ir_2.$$

By induction we conclude that for every integer $k\geqslant 1$ we can define $f_k\in\mathscr{F}_\omega^{L,n}$ by

$$f_k(z) = f(z)\prod_{p=1}^{k}\frac{z-ir_p}{z+ir_p}, \quad z\in\boldsymbol{C},\ z\neq -ir_1,\ldots,-ir_k.$$

Then the formula

$$g(z) = \frac{f(z)}{\omega(Lz)^n}, \quad z\in\boldsymbol{C},\ z\neq -ir_1, -ir_2, \ldots,$$

defines an entire function g. By [38], Ch. I, Corollary to Theorem 12, g is of exponential type 0. Further, as $\boldsymbol{R}\ni t\mapsto g(t)$ belongs to $L^2(\boldsymbol{R})$, the formula

$$h(z) = \int_0^z g(\zeta)^2 d\zeta, \quad z\in\boldsymbol{C},$$

defines an entire function h of exponential type 0, which is bounded on the real axis. Using the Phragmén–Lindelöf principle ([38], Ch. I, Th. 22), we infer that h is constant and this successively implies $g\equiv 0$ and $f\equiv 0$. ∎

We shall now give a weighted approximation result (compare with [28], § 3):

LEMMA 3.11. *Let $L > 0$, let $n \geqslant 1$ be an integer, $f \in \mathscr{F}_\omega^{L,n}$ and*

$$d_k = (f|\omega_k^{L,n})_\omega^{L,n}, \qquad k \geqslant 1.$$

Then for any integer $m \geqslant 1$ and any $t \in \boldsymbol{R}$, we have

$$\Big|f(t) - \sum_{k=1}^{m} d_k \omega_k^{L,n}(t)\Big| \leqslant \frac{1}{\sqrt{\pi}} \Big(\sum_{k>m} |d_k|^2\Big)^{1/2} \Big(\sum_{k>1} \frac{1}{r_k^{L,n}}\Big) |\omega(Lt)^n|.$$

In particular

$$\lim_{m\to+\infty} \sup_{t\in R} \frac{\Big|f(t) - \sum\limits_{k=1}^{m} d_k \omega_k^{L,n}(t)\Big|}{|\omega(Lt)^n|} = 0.$$

Proof. For every integer $k \geqslant 1$ and $t \in \boldsymbol{R}$ we have

$$|\omega_k^{L,n}(t)|^2 = \frac{r_k^{L,n}}{\pi} \cdot \frac{1}{t^2 + (r_k^{L,n})^2} |\omega(Lt)^{2n}| \leqslant \frac{1}{\pi r_k^{L,n}} |\omega(Lt)^{2n}|.$$

Consequently, for every integer $m \geqslant 1$ and $t \in \boldsymbol{R}$, we have

$$\sum_{k>m} |d_k \omega_k^{L,n}(t)| \leqslant \Big(\sum_{k>m} |d_k|^2\Big)^{1/2} \Big(\sum_{k>m} |\omega_k^{L,n}(t)|^2\Big)^{1/2} \tag{3.2}$$

$$\leqslant \frac{1}{\sqrt{\pi}} \Big(\sum_{k>m} |d_k|^2\Big)^{1/2} \Big(\sum_{k>m} \frac{1}{r_k^{L,n}}\Big)^{1/2} |\omega(Lt)^n|.$$

In particular, the series $\sum\limits_{k=1}^{\infty} d_k \omega_k^{L,n}$ converges uniformly on each compact subset of $\boldsymbol{R}$. By Lemma 3.10, its limit is f and our statement results from (3.2). ∎

Let $g \in \mathscr{G}_\omega$, $L > 0$ and let $n \geqslant 1$ be an integer. For any integer $k \geqslant 1$, we can choose $S_k \subset \{1, \ldots, k\}$ such that

$$r_{p_1}^{L,n} \neq r_{p_2}^{L,n} \text{ for } p_1, p_2 \in S_k, p_1 \neq p_2 \quad \text{and} \quad \{r_p^{L,n}; p \in S_k\} = \{r_p^{L,n}; 1 \leqslant p \leqslant k\}.$$

Then there exist uniquely determined constants $\alpha_{p,q}^k \in \boldsymbol{C}$, $k \geqslant 1$, $p \in S_k$, $1 \leqslant q \leqslant k$, such that

$$\frac{(t - ir_1^{L,n}) \ldots (t - ir_{k-1}^{L,n})}{(t + ir_1^{L,n}) \ldots (t + ir_{k-1}^{L,n})(t + ir_k^{L,n})} = \sum_{p\in S_k} \sum_{q=1}^{k} \frac{\alpha_{p,q}^k}{(t + ir_p^{L,n})^q}, \qquad k \geqslant 1, \ t \in \boldsymbol{R}.$$

DEFINITION XV. The functions $d_k^{g,L,n}\colon \boldsymbol{R} \to \boldsymbol{C}$, $k \geqslant 1$, defined by

$$d_k^{g,L,n}(s) = -2i\sqrt{\pi r_k^{L,n}} \sum_{p\in S_k} \sum_{q=1}^{k} \frac{\alpha_{p,q}^k}{(q-1)!} \big(g(s, i\,\cdot)\omega(L\,\cdot)^{-n}\big)^{(q-1)}(-ir_p^{L,n})$$

are called the (L, n)-*weighted coefficients of the ω-ultradifferential operator* $g(\cdot\,, D)$.

By Corollary 2.21, we have

$$d_k^{g,L,n} \in \mathscr{E}_\omega, \qquad k \geqslant 1.$$

The above "artificial" definition becomes more natural if we observe that if $s \in \boldsymbol{R}$ is such that $g(s, i\,\cdot) \in \mathscr{F}_\omega^{L,n}$, then, by Lemma 3.9,

$$d_k^{g,L,n}(s) = \sqrt{\frac{r_k^{L,n}}{\pi}} \sum_{p\in S_k} \sum_{q=1}^{k} \alpha_{p,q}^k \int_{-\infty}^{+\infty} \frac{g(s, it)}{\omega(Lt)^n} \cdot \frac{1}{(t + ir_p^{L,n})^q}\, dt$$

$$= \int_{-\infty}^{+\infty} \frac{g(s, it)}{\omega(Lt)^n} \sqrt{\frac{r_k^{L,n}}{\pi}} \frac{(t - ir_1^{L,n}) \ldots (t - ir_{k-1}^{L,n})}{(t + ir_1^{L,n}) \ldots (t + ir_{k-1}^{L,n})(t + ir_k^{L,n})}\, dt.$$

Hence

$$d_k^{g,L,n} = \big(g(s, i\,\cdot)|\omega_k^{L,n}\big)_\omega^{L,n}, \qquad k \geqslant 1 \text{ integer}. \tag{3.3}$$

The equality (3.3) will play an important role in what follows. However, the "artificial" Definition XV of the functions $d_k^{g,L,n}$ has the advantage of allowing us to define them even for $g(s, i\,\cdot) \notin \mathscr{F}_\omega^{L,n}$ and to verify easily that $d_k^{g,L,n} \in \mathscr{E}_\omega$.

It is clear that the (L, n)-weighted coefficients of an ω-ultradifferential operator with constant coefficients are also constant.

LEMMA 3.12. *Let $g \in \mathscr{G}_\omega$, $-\infty < a < b < +\infty$, $L > 0$ and let $n \geqslant 1$ be an integer. If $L_0 > 0$, an integer $n_0 \geqslant 1$ and $c_0 > 0$ are such that*

$$r_{L,n}^{\omega,[a,b]}\big(g(\cdot\,, it)\big) \leqslant c_0 |\omega(L_0 t)^{n_0}|, \qquad t \in \boldsymbol{R}, \tag{3.4}$$

then for every integer $m \geqslant 1$ and $t \in \boldsymbol{R}$ we have

$$r_{L,n}^{\omega,a,b}\Big(g(\cdot\,, it) - \sum_{k=1}^{m} d_k^{g,L_0,n_0+1} \omega_k^{L_0,n_0+1}(t)\Big)$$

$$\leqslant c_0 \sqrt{\frac{t_1}{L_0}} \Big(\sum_{k>m} \frac{1}{r_k^{L_0,n_0+1}}\Big)^{1/2} |\omega(L_0 t)^{n_0+1}|.$$

Proof. Let $p \geqslant 0$ be a fixed integer. By Corollary 2.21, $\dfrac{\partial^p g}{\partial s^p} \in \mathscr{G}_\omega$. It is clear that the (L_0, n_0+1)-weighted coefficients of $L^p a_p^{\omega,n} \dfrac{\partial^p g}{\partial s^p}$ are

$$L^p a_p^{\omega,n} (d_k^{g,L_0,n_0+1})^{(p)}, \qquad k \geqslant 1.$$

Let $s \in [a, b]$ be also fixed. By (3.4), we have

$$\Big|L^p a_p^{\omega,n} \frac{\partial^p g}{\partial s^p}(s, it)\Big| \leqslant c_0 |\omega(L_0 t)^{n_0}|, \qquad t \in \boldsymbol{R},$$

so that for $k \geqslant 1$ we have

$$L^p a_p^{\omega,n} (d_k^{g,L_0,n_0+1})^{(p)}(s) = \Big(L^p a_p^{\omega,n} \frac{\partial^p g}{\partial s^p}(s, i\,\cdot)\Big|\omega_k^{L_0,n_0+1}\Big)_\omega^{L_0,n_0+1}.$$

Using Lemma 3.11, we get for any integer $m \geqslant 1$ and $t \in \boldsymbol{R}$

$$\left|L^p a_p^{\omega,n}\left(\frac{\partial^p g}{\partial s^p}(s, it) - \sum_{k=1}^{m} (d_k^{g,L_0,n_0+1})^{(p)}(s)\,\omega_k^{L_0,n_0+1}(t)\right)\right|$$

$$\leqslant \frac{1}{\sqrt{\pi}}\left\|L^p a_p^{\omega,n}\frac{\partial^p g}{\partial s^p}(s, i\,\cdot)\right\|_k^{L_0,n_0+1}\left(\sum_{k>m}\frac{1}{r_k^{L_0,n_0+1}}\right)^{1/2}|\omega(L_0 t)^{n_0+1}|$$

$$\leqslant \frac{c_0}{\sqrt{\pi}}\left(\int_{-\infty}^{+\infty}|\omega(L_0 t)^{-2}|dt\right)^{1/2}\left(\sum_{k>m}\frac{1}{r_k^{L_0,n_0+1}}\right)^{1/2}|\omega(L_0 t)^{n_0+1}|$$

$$\leqslant \frac{c_0}{\sqrt{\pi}}\left(\int_{-\infty}^{+\infty}\left(1+\frac{L_0^2 t^2}{t_1^2}\right)^{-1}dt\right)^{1/2}\left(\sum_{k>m}\frac{1}{r_k^{L_0,n_0+1}}\right)^{1/2}|\omega(L_0 t)^{n_0+1}|$$

$$= c_0\sqrt{\frac{t_1}{L_0}}\left(\sum_{k>m}\frac{1}{r_k^{L_0,n_0+1}}\right)^{1/2}|\omega(L_0 t)^{n_0+1}|.$$

Since $p \geqslant 0$ and $s \in [a, b]$ are arbitrary, the statement follows.

We can now give our first approximation result:

PROPOSITION 3.13. *Let $g \in \mathscr{G}_\omega$, $-\infty < a < b < +\infty$, $\varepsilon > 0$, $L > 0$ and let $n \geqslant 1$ be an integer. If $L_0 > 0$, an integer $n_0 \geqslant 1$ and $c_0 > 0$ are such that*

$$r_{4L,n+2}^{\omega,[a-\varepsilon,b+\varepsilon]}(g(\cdot, it)) \leqslant c_0|\omega(L_0 t)^{n_0}|, \quad t \in \boldsymbol{R},$$

then there exists a $c > 0$ depending only on $b-a$, ε, L, n, L_0, n_0, such that for every integer $m \geqslant 1$ and $\varphi \in \mathscr{E}_\omega$ we have

$$r_{L,n}^{\omega,[a,b]}\left(g(\cdot, D)\varphi - \sum_{k=1}^{m} d_k^{g,L_0,n_0+1}\,\omega_k^{L_0,n_0+1}(-iD)\varphi\right)$$

$$\leqslant c_0 c\left(\sum_{k>m}\frac{1}{r_k^{L_0,n_0+1}}\right)^{1/2} r^{\omega,[a-\varepsilon,b+\varepsilon]}_{4\max\{L,L_0\},n+n_0+3}(\varphi).$$

Proof. By Theorem 2.5, there are $\varphi, \theta \in \mathscr{D}_\omega$ such that $\psi(s) = 1$ for $s \in [-\varepsilon/4, b-a+\varepsilon/4]$, $\operatorname{supp}\psi \subset [-\varepsilon/2, b-a+\varepsilon/2]$, $\theta(s) = 1$ for $s \in [-3\varepsilon/4, b-a+3\varepsilon/4]$, $\operatorname{supp}\theta \subset [-\varepsilon, b-a+\varepsilon]$.

For every $s_0 \in \boldsymbol{R}$, we can define $\psi_{s_0}, \theta_{s_0} \in \mathscr{D}_\omega$ by the formula

$$\psi_{s_0}(s) = \psi(s-s_0), \quad \theta_{s_0}(s) = \theta(s-s_0), \quad s \in \boldsymbol{R}.$$

Then $r_{L',n'}^{\omega}(\psi_{s_0})$ and $r_{L',n'}^{\omega}(\theta_{s_0})$ depend only on ψ, L', n' and on θ, L', n', respectively.

Let $m \geqslant 1$ be an integer and $\varphi \in \mathscr{E}_\omega$. Defining $g_m \in \mathscr{G}_\omega$ by

$$g_m(s, z) = g(s, z) - \sum_{k=1}^{m} d_k^{g,L_0,n_0+1}(s)\,\omega_k^{L_0,n_0+1}(-iz), \quad s \in \boldsymbol{R},\ z \in \boldsymbol{C},$$

we have

$$g_m(\cdot, D) = g(\cdot, D) - \sum_{k=1}^{m} d_k^{g,L_0,n_0+1}\omega_k^{L_0,n_0+1}(-iD)g.$$

By Proposition 2.7

$$r_{L,n}^{\omega,[a,b]}(g_m(\cdot, D)\varphi) \leqslant r_{L,n}^{\omega}(g_m(\cdot, D)(\varphi\psi_a)) \leqslant q_{L,n}^{\omega}(g_m(\cdot, D)(\varphi\psi_a)).$$

A computation similar to that at the beginning of the proof of Theorem 2.21 shows us that

$$\widehat{g_m(\cdot, D)(\varphi\psi_a)}(r) = \int_{-\infty}^{+\infty}\widehat{\theta_a g_m(\cdot, it)}(r-t)\widehat{(\varphi\psi_a)}(t)\,dt, \quad r \in \boldsymbol{R}.$$

Hence

$$r_{L,n}^{\omega,[a,b]}(g_m(\cdot, D)\varphi)$$

$$\leqslant \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|\widehat{\theta_a g_m(\cdot, it)}(r-t)|\,|\widehat{\varphi\psi_a}(t)|\,|\omega(Lr)^n|\,dt\,dr$$

$$\leqslant \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\left|\widehat{\theta_a g_m(\cdot, it)}(r-t)\,\omega(\sqrt{2}L(r-t))^n\right|dr\right)\left|\widehat{\varphi\psi_a}(t)\,\omega(\sqrt{2}Lt)^n\right|dt$$

$$= \int_{-\infty}^{+\infty} q_{\sqrt{2}L,n}^{\omega}(\theta_a g_m(\cdot, it))\left|\varphi\psi_a(t)\,\omega(\sqrt{2}Lt)^n\right|dt.$$

By Propositions 2.1, 2.7 and 2.10, for every $t \in \boldsymbol{R}$

$$q_{\sqrt{2}L,n}^{\omega}(\theta_a g_m(\cdot, it)) \leqslant \frac{\pi t_1}{L}p_{\sqrt{2}L,n+2}^{\omega}(\theta_a g_m(\cdot, it))$$

$$\leqslant \frac{t_1}{L}(b-a+2\varepsilon)r_{2L,n+2}^{\omega}(\theta_a g_m(\cdot, it))$$

$$\leqslant \frac{t_1}{L}(b-a+2\varepsilon)r_{4L,n+2}(\theta_a)r_{4L,n+2}^{\omega,[a-\varepsilon,b+\varepsilon]}(g_m(\cdot, it)).$$

On the other hand, by Lemma 3.12, for every $t \in \boldsymbol{R}$, we have

$$r_{4L,n+2}^{\omega,[a-\varepsilon,b+\varepsilon]}(g_m(\cdot, it)) \leqslant c_0\sqrt{\frac{t_1}{L_0}}\left(\sum_{k>m}\frac{1}{r_k^{L_0,n_0+1}}\right)^{1/2}|\omega(L_0 t)^{n_0+1}|.$$

Hence, if we write

$$c' = \frac{t_1}{L}(b-a+2\varepsilon)r_{4L,n+2}^{\omega}(\theta)\sqrt{\frac{t_1}{L_0}},$$

we get

$$q_{\sqrt{2}L,n}^{\omega}(\theta_a g_m(\cdot, it)) \leqslant c_0 c'\left(\sum_{k>m}\frac{1}{r_k^{L_0,n_0+1}}\right)^{1/2}|\omega(L_0 t)^{n_0+1}|, \quad t \in \boldsymbol{R}.$$

Consequently

$$r_{L,n}^{\omega,[a,b]}\big(g_m(\cdot\,,D)\varphi\big) \leqslant c_0 c' \Big(\sum_{k>m} \frac{1}{r_k^{L_0,n_0+1}}\Big)^{1/2} q^{\omega}_{\max\{\sqrt{2}L,L_0\},n+n_0+1}(\varphi\theta_a).$$

Again by Propositions 2.1, 2.7 and 2.10

$$q^{\omega}_{\max\{\sqrt{2}L,L_0\},n+n_0+1}(\varphi\theta_a) \leqslant \frac{t_1}{L}(b-a+\varepsilon) r^{\omega}_{\max\{4L,2\sqrt{2}L_0\},n+n_0+3}(\psi_a) r^{\omega,[a-\varepsilon,b+\varepsilon]}_{\max\{4L,2\sqrt{2}L_0\},n+n_0+3}(\varphi),$$

so that writing

$$c = c'\frac{t_1}{L}(b-a+\varepsilon) r^{\omega}_{\max\{4L,2\sqrt{2}L_0\},n+n_0+3}(\psi),$$

we conclude that

$$r_{L,n}^{\omega,[a,b]}\big(g_m(\cdot\,,D)\varphi\big) \leqslant c_0 c \Big(\sum_{k>m} \frac{1}{r_k^{L_0,n_0+1}}\Big)^{1/2} r^{\omega,[a-\varepsilon,b+\varepsilon]}_{4\max\{L,L_0\},n+n_0+3}(\varphi). \blacksquare$$

Let $\mathscr{L}(\mathscr{E}_\omega)$ be the space of all linear continuous operators on $\mathscr{E}_\omega$, endowed with the topology of the uniform convergence on the bounded subsets of $\mathscr{E}_\omega$. Then, by the above proposition, the space of all ω-ultradifferential operators is the closure of

$$\Big\{\sum_{k=1}^{m} c_k \omega_k^{L,n}(-iD);\ L>0,\ n,m\geqslant 1 \text{ integers},\ c_1,\ldots,c_m\in\mathscr{E}_\omega\Big\}$$

in $\mathscr{L}(\mathscr{E}_\omega)$. Moreover, we obtain the following

COROLLARY 3.14. *Let f be an entire function of exponential type* 0 *such that for a certain $L_0>0$, an integer $n_0\geqslant 1$ and a $c_0>0$*

$$|f(it)| \leqslant c_0|\omega(L_0 t)^{n_0}|, \quad t\in\boldsymbol{R}. \tag{3.5}$$

Then for any $-\infty<a<b<+\infty$, $\varepsilon>0$, $L>0$ and integer $n\geqslant 1$, there exist $c>0$ depending only on $b-a$, ε, L, n, L_0, n_0, such that for every integer $m\geqslant 1$ and $\varphi\in\mathscr{E}_\omega$

$$r_{L,n}^{\omega,[a,b]}\Big(f(D)\varphi - \sum_{k=1}^{m} d_k^{f,L_0,n_0+1}\omega_k^{L_0,n_0+1}(-iD)\varphi\Big) \leqslant c_0 c\Big(\sum_{k>m}\frac{1}{r_k^{L_0,n_0+1}}\Big)^{1/2} r^{\omega,[a-\varepsilon,b-\varepsilon]}_{4\max\{L,L_0\},n+n_0+3}(\varphi).$$

In particular, if f satisfies (3.5), *then*

$$f(D) = \sum_{k=1}^{\infty} d_k^{f,L_0,n_0+1}\omega_k^{L_0,n_0+1}(-iD),$$

where the series on the right side converges in $\mathscr{L}(\mathscr{E}_\omega)$.

Let $L>0$ and let $n\geqslant 1$ be an integer. For any integers $l\geqslant k\geqslant 1$ we define the polynomial $\omega_{k,l}^{L,n}$ by the formula

$$\omega_{k,l}^{L,n}(z) = \sqrt{\frac{r_k^{L,n}}{\pi}}\cdot\frac{(z+ir_1^{L,n})\ldots(z+ir_{k-1}^{L,n})}{(z-ir_1^{L,n})\ldots(z-ir_{k-1}^{L,n})(z-ir_k^{L,n})}\prod_{j=1}^{l}\Big(1+\frac{iz}{r_j^{L,n}}\Big).$$

LEMMA 3.15. *Let $L>0$ and let $n\geqslant 1$ be an integer. Then for any integers $l\geqslant k\geqslant 1$ and any $t\in\boldsymbol{R}$, we have*

$$|\omega_k^{L,n}(t)-\omega_{k,l}^{L,n}(t)| \leqslant \sqrt{\frac{1}{\pi}\max\Big\{2,\frac{1}{r_k^{L,n}}\Big\}}\Big(\sum_{p=1}^{\infty}\frac{1}{r_p^{L,n}}\Big)^{1/3}|\omega(Lt)^n|. \tag{3.6}$$

Proof. Let $l\geqslant k\geqslant 1$ be integers and $t\in\boldsymbol{R}$. For convenience we denote $r_p^{L,n}$ simply by r_p. It is clear that we have

$$\left|\frac{\omega_k^{L,n}(t)-\omega_{k,l}^{L,n}(t)}{\omega(Lt)^n}\right| = \Big(\frac{r_k}{\pi(t^2+r_k^2)}\Big)^{1/2}\left|1-\frac{1}{\prod_{p>l}\Big(1+\frac{it}{r_p}\Big)}\right|.$$

If $|t|\geqslant\big(\sum_{p>l}1/r_p\big)^{-2/3}$, then

$$\left|\frac{\omega_k^{L,n}(t)-\omega_{k,l}^{L,n}(t)}{\omega(Lt)^n}\right| \leqslant 2\Big(\frac{1}{2\pi|t|}\Big)^{1/2} = \sqrt{\frac{2}{\pi}}\Big(\sum_{p>l}\frac{1}{r_p}\Big)^{1/3}.$$

If $|t|\leqslant\big(\sum_{p>l}1/r_p\big)^{-2/3}$, then

$$\left|1-\frac{1}{\prod_{p>l}\Big(1+\frac{it}{r_p}\Big)}\right| \leqslant \sum_{p>l}\left|\frac{1}{\prod_{l<q<p}\Big(1+\frac{it}{r_q}\Big)}-\frac{1}{\prod_{l<q\leqslant p}\Big(1+\frac{it}{r_q}\Big)}\right|$$

$$\leqslant |t|\sum_{p>l}\frac{1}{r_p} \leqslant \Big(\sum_{p>l}\frac{1}{r_p}\Big)^{1/3},$$

so that finally we get

$$\left|\frac{\omega_k^{L,n}(t)-\omega_{k,l}^{L,n}(t)}{\omega(Lt)^n}\right| \leqslant \sqrt{\frac{1}{\pi r_k}}\Big(\sum_{p>l}\frac{1}{r_p}\Big)^{1/3}.$$

Consequently (3.6) holds. ■

We are now able to give our second approximation result, namely to approximate every ω-ultradifferential operator with "ω-ultradifferential operators of finite degree".

PROPOSITION 3.16. *Let $g\in\mathscr{G}_\omega$, $-\infty<a<b<+\infty$, $\varepsilon>0$, $L>0$ and let $n\geqslant 1$ be an integer. If $L_0>0$, an integer $n_0\geqslant 1$ and $c_0>1$ are such that*

$$r^{\omega,[a-\varepsilon,b+\varepsilon]}_{4L,n+2}\big(g(\cdot\,,it)\big) \leqslant c_0|\omega(L_0 t)^{n_0}|, \quad t\in\boldsymbol{R},$$

then there exists a $d>0$ *depending only on* $b-a$, ε, L, n, L_0, n_0, *such that for every integers* $l \geqslant m \geqslant 1$ *and* $\varphi \in \mathscr{E}_\omega$

$$r_{L,n}^{\omega,[a,b]}\Big(g(\cdot\,,D)\varphi-\sum_{k=1}^{m} d_k^{g,L_0,n_0+1}\omega_{k,l}^{L_0,n_0+1}(-iD)\varphi\Big)$$
$$\leqslant c_0 d\left[\Big(\sum_{k>m}\frac{1}{r_k^{L_0,n_0+1}}\Big)^{1/2}+m\Big(\sum_{k>l}\frac{1}{r_k^{L_0,n_0+1}}\Big)^{1/3}\right] r_{8\max\{L,L_0\},n+n_0+3}^{\omega,[a-\varepsilon,b+\varepsilon]}(\varphi).$$

Proof. By Theorem 2.5 there exists a $\psi \in \mathscr{D}_\omega[-\varepsilon, b-a+\varepsilon]$ such that $\psi(s)=1$ for s in some neighbourhood of $[0, b-a+\varepsilon]$. If we denote $\psi_{s_0}(t)=\psi(t-s_0)$, $s_0 \in \boldsymbol{R}$, $r_{L',n'}(\psi_{s_0})$ depends only on ψ, L', n'.

Let $l \geqslant k \geqslant 1$ be integers and $\varphi \in \mathscr{E}_\omega$. By Propositions 2.10, 2.7 and 2.1, we have

$$r_{L,n}^{\omega,[a,b]}\big(d_k^{g,L_0,n_0+1}\omega_k^{L_0,n_0+1}(-iD)\varphi-d_k^{g,L_0,n_0+1}\omega_{k,l}^{L_0,n_0+1}(-iD)\varphi\big)$$
$$= r_{L,n}^{\omega,[a,b]}\big(d_k^{g,L_0,n_0+1}\omega_k^{L_0,n_0+1}(-iD)(\varphi\psi_a)-d_k^{g,L_0,n_0+1}\omega_{k,l}^{L_0,n_0+1}(-iD)(\varphi\psi_a)\big)$$
$$\leqslant r_{L,n}^{\omega,[a,b]}(d_k^{g,L_0,n_0+1})\, r_{2L,n}^{\omega,[a,b]}\big(\omega_k^{L_0,n_0+1}(-iD)(\varphi\psi_a)-\omega_{k,l}^{L_0,n_0+1}(-iD)(\varphi\psi_a)\big)$$
$$\leqslant \frac{\pi t_1}{2L}\, r_{2L,n}^{\omega,[a,b]}(d_k^{g,L_0,n_0+1})\, p_{2L,n+2}^{\omega}\big([\omega_k^{L_0,n_0+1}-\omega_{k,l}^{L_0,n_0+1}](-iD)(\varphi\psi_a)\big).$$

For any integer $p \geqslant 0$ and $s \in [a,b]$

$$\left|(2L)^p a_p^{\omega,n}\frac{\partial^p g}{\partial s^p}(s,it)\right| \leqslant r_{2L,n}^{\omega,[a,b]}(g(\cdot\,,it)) \leqslant c_0|\omega(L_0 t)|^{n_0}, \qquad t \in \boldsymbol{R},$$

and so

$$|(2L)^p a_p^{\omega,n}(d_k^{g,L_0,n_0+1})^{(p)}(s)|$$
$$= \left\|\Big((2L)^p a_p^{\omega,n}\frac{\partial^p g}{\partial s^p}(s,i\,\cdot)|\omega_k^{L_0,n_0+1}|\Big)^{L_0,n_0+1}\right\|_\omega$$
$$= \left\|(2L)^p a_p^{\omega,n}\frac{\partial^p g}{\partial s^p}(s,i\,\cdot)\right\|_\omega^{L_0,n_0+1} \leqslant c_0\Big(\int_{-\infty}^{+\infty}|\omega(L_0 t)^{-2}|\,dt\Big)^{1/2}$$
$$\leqslant c_0\left(\int_{-\infty}^{+\infty}\Big(1+\frac{L_0^2t^2}{t_1^2}\Big)^{-2}dt\right)^{1/2} = c_0\sqrt{\frac{\pi t_1}{L_0}}.$$

Hence

$$r_{2L,n}^{\omega,[a,b]}(d_k^{g,L_0,n_0+1}) \leqslant c_0\sqrt{\frac{\pi t_1}{L_0}}.$$

On the other hand, by Lemma 3.15 and by Propositions 2.7 and 2.10,

$$p_{2L,n+2}^{\omega}\big([\omega_k^{L_0,n_0+1}-\omega_{k,l}^{L_0,n_0+1}](-iD)(\varphi\psi_a)\big)$$
$$\leqslant \sqrt{\frac{1}{\pi}\max\Big\{2,\frac{1}{r_k^{L_0,n_0+1}}\Big\}}\,\Big(\sum_{p>l}\frac{1}{r_p^{L,n}}\Big)^{1/3} p_{\max\{2L,L_0\},n+n_0+3}^{\omega}(\varphi\psi_a)$$

$$\leqslant \frac{1}{\pi}(b-a+2\varepsilon)\sqrt{\frac{1}{\pi}\max\Big\{2,\frac{1}{r_k^{L_0,n_0+1}}\Big\}}\,\Big(\sum_{p>l}\frac{1}{r_p^{L,n}}\Big)^{1/3} r_{8\max\{L,L_0\},n+n_0+3}^{\omega}(\psi_a)\times$$
$$\times r_{8\max\{L,L_0\},n+n_0+3}^{\omega,[a-\varepsilon,b+\varepsilon]}(\varphi).$$

Denoting

$$c'=\frac{\pi t_1}{2L}\sqrt{\frac{\pi t_1}{L_0}}\cdot\frac{1}{\pi}(b-a+2\varepsilon)\sqrt{\frac{1}{\pi}\max\Big\{2,\sup_{q\geqslant 1}\frac{1}{r_q^{L_0,n_0+1}}\Big\}}\, r_{8\max\{L,L_0\},n+n_0+3}^{\omega}(\psi),$$

we conclude that

$$r_{L,n}^{\omega,[a,b]}\big(d_k^{g,L_0,n_0+1}\omega_k^{L_0,n_0+1}(-iD)\varphi-d_k^{g,L_0,n_0+1}\omega_{k,l}^{L_0,n_0+1}(-iD)\varphi\big) \tag{3.7}$$
$$\leqslant c_0c'\Big(\sum_{p>l}\frac{1}{r_p^{L_0,n_0+1}}\Big)^{1/3} r_{8\max\{L,L_0\},n+n_0+3}^{\omega,[a-\varepsilon,b+\varepsilon]}(\varphi).$$

Finally, let $c>0$ be as in the statement of Proposition 3.13. Then, by using Proposition 3.13 and the inequality (3.7), we obtain our statement for $d=\max\{c,c'\}$. ∎

From the above proposition we immediately get

THEOREM 3.17. *The linear subspace of* $\mathscr{L}(\mathscr{E}_\omega)$, *formed by all* ω-*ultradifferential operators, is the closure of*

$$\Big\{\sum_{k=0}^{m} c_k D^k;\ m \geqslant 0 \text{ integer},\ c_0,\ldots,c_m \in \mathscr{E}_\omega\Big\}.$$

In particular, the linear subspace of $\mathscr{L}(\mathscr{E}_\omega)$, *formed by all* ω-*ultradifferential operators with constant coefficients, is the closure of*

$$\Big\{\sum_{k=0}^{m} c_k D^k;\ m \geqslant 0 \text{ integer},\ c_0,\ldots,c_m \in \boldsymbol{C}\Big\}.$$

Using Theorems 3.5 and 3.17, we immediately obtain

COROLLARY 3.18. *The linear subspace of* $\mathscr{E}'_\omega$ *formed by all* ω-*ultradistributions with the support in* $\{0\}$ *is the closure of*

$$\Big\{\sum_{k=0}^{m} c_k \delta_0^{(k)};\ m \geqslant 0 \text{ integer},\ c_0,\ldots,c_m \in \boldsymbol{C}\Big\}.$$

We note that, by using the Hahn–Banach theorem, the above statement results also from Corollary 2.17.

We end our considerations on ω-ultradifferential operators with the remark that such an operator T can be extended to a continuous linear mapping $\mathscr{D}'_\omega \to \mathscr{D}'_\omega$ if and only if its adjoint T' invariates $\mathscr{D}_\omega$. As we have seen, this holds for example for ω-ultradifferential operators with constant coefficients, but there exist also larger classes of "extendable" ω-ultradifferential operators. We do not insist here on this problem.

Further, we shall give a local structure theorem for ω-ultradistributions, analogous to [34], Theorem 10.3.

THEOREM 3.19. *Let $\mathscr{B}$ be a bounded subset of $\mathscr{D}'_\omega$ and $-\infty < a < b < +\infty$. Then there exist $L > 0$, integer $n \geqslant 1$ and $c > 0$ such that for each $F \in \mathscr{B}$ there exists a continuous complex function G on $\boldsymbol{R}$ such that $\sup_{s \in \boldsymbol{R}} |G(s)| \leqslant c$ and*

$$F|_{(a,b)} = \omega(-iD)^n G|_{(a,b)}.$$

Proof. Since $\mathscr{D}_\omega$ is barrelled and $\mathscr{B}$ is equicontinuous, there exist $L > 0$, integer $n \geqslant 1$ and $c > 0$ such that

$$|F(\varphi)| \leqslant c \sup_{t \in \boldsymbol{R}} |\hat{\varphi}(t)\omega(-Lt)^n|, \quad F \in \mathscr{B},\ \varphi \in \mathscr{D}_\omega[a, b].$$

Let $F \in \mathscr{B}$; by the Hahn–Banach theorem and by the Riesz–Kakutani representation theorem, there exists a bounded regular measure μ on $\boldsymbol{R}$, $\|\mu\| \leqslant c$, such that

$$F(\varphi) = \int_{-\infty}^{+\infty} \hat{\varphi}(t)\,\omega(-Lt)^n d\mu(t), \quad \varphi \in \mathscr{D}_\omega[a, b].$$

Denoting by G the Fourier transform of μ, for each $\varphi \in \mathscr{D}_\omega[a, b]$, we have

$$F(\varphi) = \int_{-\infty}^{+\infty} \widehat{\big(\omega(iLD)^n\varphi\big)}(t)\,d\mu(t)$$

$$= \int_{-\infty}^{+\infty} \big(\omega(iLD)^n\varphi\big)(s)\,G(s)\,ds = \big(\omega(-iLD)^n G\big)(\varphi).$$

Since the inclusion $\mathscr{D}_\omega(a, b) \subset \mathscr{D}_\omega[a, b]$ is continuous, we conclude that

$$F|_{(a,b)} = \omega(-iLD)^n G|_{(a,b)}. \quad \blacksquare$$

We end this section by remarking that sometimes it is convenient to change ω into a "better" one, without changing $\mathscr{D}_\omega$. For this purpose we give the following

THEOREM 3.20. *Let $t_1, t_2, \ldots > 0$, $t_1 < +\infty$, $\sum_{k=1}^{\infty} 1/t_k < +\infty$ and $r_1, r_2, \ldots > 0$, $r_1 < +\infty$, $\sum_{k=1}^{\infty} 1/r_k < +\infty$. Then the following statements are equivalent:*

(i) $\mathscr{D}_{\omega_{\{t_k\}}} \subset \mathscr{D}_{\omega_{\{r_k\}}}$;

(ii) $\mathscr{D}_{\omega_{\{t_k\}}} \subset \mathscr{D}_{\omega_{\{r_k\}}}$, *where the inclusion is continuous and has a dense range*;

(iii) $\mathscr{E}_{\omega_{\{t_k\}}} \subset \mathscr{E}_{\omega_{\{r_k\}}}$;

(iv) $\mathscr{E}_{\omega_{\{t_k\}}} \subset \mathscr{E}_{\omega_{\{r_k\}}}$, *where the inclusion is continuous and has a dense range*;

(v) *there exist an $L_0 > 0$, an integer $n_0 \geqslant 1$ and a $c_0 > 0$ with*

$$|\omega_{\{r_k\}}(t)| \leqslant c_0 |\omega_{\{t_k\}}(L_0 t)^{n_0}|, \quad t \in \boldsymbol{R};$$

(vi) *there exist an $L_1 > 0$, an integer $n_1 \geqslant 1$ and a $c_1 > 0$ with*

$$a_k^{\omega\{r_k\},1} \leqslant c_1 L_1^k a_k^{\omega\{t_k\},n_1}, \quad k \geqslant 0;$$

(vii) *there exist an $L_2 > 0$, an integer $n_2 \geqslant 1$ and a $c_2 > 0$ with*

$$a_k^{\omega\{r_k\},n} \leqslant c_1 L_2^k a_k^{\omega\{t_k\},nn_2}, \quad n \geqslant 1,\ k \geqslant 0.$$

Proof. (i) $\Rightarrow$ (iii). Let $\varphi \in \mathscr{E}_{\omega_{\{t_k\}}}$ be arbitrary, $\boldsymbol{K} \subset \boldsymbol{R}$ compact, $L > 0$ and $n \geqslant 1$ integer. By Theorem 3.5 there exists a $\psi \in \mathscr{D}_{\omega_{\{t_k\}}}$ such that $\psi(s) = 1$ for s in some neighbourhood of $\boldsymbol{K}$.

Then $\varphi\psi \in \mathscr{D}_{\omega_{\{t_k\}}} \subset \mathscr{D}_{\omega_{\{r_k\}}} \subset \mathscr{E}_{\omega_{\{r_k\}}}$, so that

$$r_{L,n}^{\omega_{\{r_k\}},K}(\varphi) = r_{L,n}^{\omega_{\{r_k\}},K}(\varphi\psi) < +\infty.$$

Since, $\boldsymbol{K}, L, n$ are arbitrary, it follows that $\varphi \in \mathscr{E}_{\omega_{\{r_k\}}}$.

Since $\mathscr{E}_{\omega_{\{t_k\}}}$ and $\mathscr{E}_{\omega_{\{r_k\}}}$ are Fréchet spaces, by the usual closed graph theorem the inclusion $\mathscr{E}_{\omega_{\{t_k\}}} \subset \mathscr{E}_{\omega_{\{r_k\}}}$ is continuous. By Theorem 2.13 it has also a dense range.

(iv) $\Rightarrow$ (ii). By Proposition 2.7, for any $-\infty < a < b < +\infty$ we have $\mathscr{D}_{\omega_{\{t_k\}}}[a, b] \subset \mathscr{D}_{\omega_{\{r_k\}}}[a, b]$, where the inclusion is continuous. Hence $\mathscr{D}_{\omega_{\{t_k\}}} \subset \mathscr{D}_{\omega_{\{r_k\}}}$ and the inclusion is continuous.

On the other hand, if $\varphi \in \mathscr{D}_{\omega_{\{r_k\}}}$ then there exists a net $\{\varphi_\iota\} \subset \mathscr{E}_{\omega_{\{t_k\}}}$ which converges in $\mathscr{E}_{\omega_{\{r_k\}}}$ to φ; so, choosing $\psi \in \mathscr{D}_{\omega_{\{t_k\}}}$ such that $\psi(s) = 1$ for s in some neighbourhood of $\operatorname{supp}\varphi$, we have $\{\psi\varphi_\iota\} \subset \mathscr{D}_{\omega_{\{t_k\}}}$ and $\psi\varphi_\iota \to \varphi$ in $\mathscr{D}_{\omega_{\{r_k\}}}$. Hence $\mathscr{D}_{\omega_{\{t_k\}}}$ is dense in $\mathscr{D}_{\omega_{\{r_k\}}}$.

Obviously, (ii) $\Rightarrow$ (i), and so we conclude that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

(iv) $\Rightarrow$ (v). The restriction of $\omega_{\{r_k\}}(-iD)\delta$ to $\mathscr{E}_{\omega_{\{t_k\}}}$ belongs to $\mathscr{E}'_{\omega_{\{t_k\}}}$ and so by Theorem 3.3 there exist an $L_0 > 0$, an integer $n_0 \geqslant 1$ and a $c_0 > 0$ such that

$$|(\omega_{\{r_k\}}(-iD)\,\delta_0)(e^{-it\cdot})| \leqslant c_0 |\omega_{\{t_k\}}(L_0 t)^{n_0}|, \quad t \in \boldsymbol{R}.$$

Since for every $t \in \boldsymbol{R}$

$$(\omega_{\{r_k\}}(-iD)\,\delta_0)(e^{-it\cdot}) = (\omega_{\{r_k\}}(iD)\,e^{-it\cdot})(0) = \omega_{\{r_k\}}(t),$$

the assertion (v) follows.

(v) $\Rightarrow$ (vii). For all integers $n \geqslant 1$, $k \geqslant 1$, by a computation similar to that from the proof of the implication (i) $\Rightarrow$ (ii) in Theorem 2.24, we get

$$a_k^{\omega\{r_k\},n} \leqslant \left(\frac{2L_0 a_k^{\omega\{t_k\},nn_0}}{a_{k-1}^{\omega\{t_k\},nn_0}}\right)^k \left|\omega_{\{r_k\}}\left(\frac{a_{k-1}^{\omega\{t_k\},nn_0}}{2L_0 a_k^{\omega\{t_k\},nn_0}}\right)^n\right|$$

$$\leqslant c_0^n \left(\frac{2L_0 a_k^{\omega\{t_k\},nn_0}}{a_{k-1}^{\omega\{t_k\},nn_0}}\right)^k \left|\omega_{\{t_k\}}\left(\frac{a_k^{\omega\{t_k\},nn_0}}{2a_k^{\omega\{t_k\},nn_0}}\right)^{nn_0}\right|$$

$$\leqslant 2c_0^n\left(\frac{2L_0 a_k^{\omega\{t_k\},nn_0}}{a_{k-1}^{\omega\{t_k\},nn_0}}\right)^k \sup_{p\geqslant 0} a_p^{\omega\{t_k\},nn_0}\left(\frac{a_{k-1}^{\omega\{t_k\},nn_0}}{a_k^{\omega\{t_k\},nn_0}}\right)^p$$

$$= \sqrt{2}\, c_0^n \left(\sqrt{2}\, L_0\right)^k a_k^{\omega\{t_k\},nn_0}.$$

Hence (vii) is satisfied with $L_2 = \sqrt{2}L_0$, $n_2 = n_0$ and $c_2 = \sqrt{2}c_0$.

(vii) $\Rightarrow$ (vi). For every $t \in \boldsymbol{R}$ we have

$$|\omega_{\{r_k\}}(t)| \leqslant \sqrt{2}\sup_{p\geqslant 0} a_p^{\omega\{r_k\},1}\left|\sqrt{2}\, t\right|^p$$
$$\leqslant \sqrt{2}\, c_1 \sup_{p\geqslant 0} a_p^{\omega\{t_k\},n_1}\left|\sqrt{2}\, L_1 t\right|^p \leqslant \sqrt{2}\, c_1\left|\omega_{\{t_k\}}\left(\sqrt{2}\, L_1 t\right)^{n_1}\right|,$$

so that (v) results for $L_0 = 2L_1$, $n_0 = n_1$ and $c_0 = \sqrt{2}c_1$.

(v) $\Rightarrow$ (i). This implication follows directly from Definition 2.1. ∎

We remark that if the equivalent statements of Theorem 3.19 are fulfilled, then $\mathscr{D}'_{\omega\{r_k\}} \subset \mathscr{D}'_{\omega\{t_k\}}$ and $\mathscr{E}'_{\omega\{r_k\}} \subset \mathscr{E}'_{\omega\{t_k\}}$, where the inclusions are continuous and have dense ranges.

4. ω-ultradistribution semi-groups and the abstract Cauchy problem

In the whole of this section $0 < t_1 \leqslant t_2 \leqslant \ldots, t_1 < +\infty$, $\sum_{k=1}^{\infty} 1/t_k < +\infty$, will be fixed and $\omega_{\{t_k\}}$ will be denoted simply by ω.

Let X be a fixed Banach space.

For $-\infty < a < b < +\infty$ we denote by $\mathscr{L}\left(\mathscr{D}_\omega(a;b);X\right)$ the vector space of all continuous linear mappings $\mathscr{D}_\omega(a;b) \mapsto X$ endowed with the topology of the uniform convergence on the bounded subsets of $\mathscr{D}_\omega(a,b)$.

Analogously, one can define the locally convex linear space $\mathscr{L}\left(\mathscr{E}_\omega(a,b);X\right)$.

Most of the results of the preceeding section can be extended to the X-valued ω-ultradistribution defined above.

So, if $\varphi \in \mathscr{E}_\omega(a,b)$ and $F \in \mathscr{L}\left(\mathscr{D}_\omega(a,b);X\right)$, then the mapping $\mathscr{D}_\omega(a,b) \ni \psi \mapsto F(\varphi\psi)$ belongs to $\mathscr{L}\left(\mathscr{D}_\omega(a,b);X\right)$ and we denote it by φF.

If $F\colon (a,b) \to X$ is a continuous mapping, then $\mathscr{D}_\omega(a,b) \ni \varphi \mapsto \int_a^b \varphi(s)F(s)\,ds$ belongs to $\mathscr{L}\left(D_\omega(a,b);X\right)$ and we denote it also by F.

If $F \in \mathscr{L}\left(\mathscr{D}_\omega(a,b);X\right)$, then there exists a smallest closed subset S of (a,b) such that $\varphi \in \mathscr{D}_\omega(a,b)$, $S \cap \operatorname{supp}\varphi = \emptyset \Rightarrow F(\varphi) = 0$, called the *support* of F and denoted by $\operatorname{supp} F$.

We remark further that there exists a correspondence between $\mathscr{L}\left(\mathscr{E}_\omega(a,b);X\right)$ and the elements of $\mathscr{L}\left(\mathscr{D}_\omega(a,b);X\right)$ with a compact support, analogous to that established in Proposition 3.1 for scalar-valued ω-ultradistributions.

For $\varphi \in \mathscr{D}_\omega$ we put

$$\tilde{\varphi}(z) = 2\pi\hat{\varphi}(iz) = \int_{-\infty}^{+\infty} e^{zt}\varphi(t)\,dt, \quad z \in \boldsymbol{C}.$$

Then, by Theorem 2.3, for $-\infty < a < b < +\infty$, $\varphi \in \mathscr{D}_\omega[a,b]$ and for every $L > 0$ and integer $n \geqslant 1$, we have

$$|\tilde{\varphi}(z)| \leqslant \begin{cases} 2\pi p^\omega_{L,n}(\varphi)\,|\omega(iLz)|^{-n}e^{a\operatorname{Re}z}, & \operatorname{Re}z \leqslant 0,\\ 2\pi p^\omega_{L,n}(\varphi)\,|\omega(-iLz)|^{-n}e^{b\operatorname{Re}z}, & \operatorname{Re}z \geqslant 0.\end{cases} \tag{4.1}$$

If $F \in \mathscr{L}(\mathscr{E}_\omega; X)$, $\operatorname{supp}F \subset [a,b]$, then, writing

$$\tilde{F}(z) = 2\pi\hat{F}(iz) = F(e^{z\cdot}), \quad z \in \boldsymbol{C},$$

and using similar arguments to those used in the proof of Theorem 3.3, we infer that there exist $L, c > 0$ and integer $n \geqslant 1$ such that

$$\|\tilde{F}(z)\| \leqslant \begin{cases} c|\omega(iLz)|^n e^{a\operatorname{Re}z}, & \operatorname{Re}z \leqslant 0,\\ c|\omega(-iLz)|^n e^{b\operatorname{Re}z}, & \operatorname{Re}z \geqslant 0.\end{cases} \tag{4.2}$$

In the whole of this section we shall write

$$\mathscr{D}^0_\omega = \{\varphi \in \mathscr{D}_\omega;\ \operatorname{supp}\varphi \subset (0,+\infty)\}.$$

Definition XVI. We say that a closed linear operator A in X *satisfies the condition* C_ω, or that $A \in (C_\omega)$, if there are $\alpha, \beta, c_0 > 0$ and an integer $n_0 \geqslant 1$ such that the resolvent $R(z;A) = (z-A)^{-1} \in \mathscr{L}(X)$ exists for z in the domain

$$\Lambda_{\omega,\alpha,\beta} = \left\{z \in \boldsymbol{C};\ \operatorname{Re}z \geqslant \alpha\ln\left|\omega(|\operatorname{Im}z|)\right| + \beta\right\} \tag{4.3}$$

and satisfies

$$\|R(z;A)\| \leqslant c_0|\omega(-iz)|^{n_0}, \quad z \in \Lambda_{\omega,\alpha,\beta}. \tag{4.4}$$

We denote by $\Gamma_{\omega,\alpha,\beta}$ the boundary of $\Lambda_{\omega,\alpha,\beta}$ oriented from the lower to the upper half plane.

Lemma 4.1. *If $A \in (C_\omega)$ and α, β, c_0, n_0 are as in Definition XVI, then the integral in the formula*

$$\mathscr{E}(\varphi) = \frac{1}{2\pi i}\int_{\Gamma_{\omega,\alpha,\beta}} \tilde{\varphi}(z)R(z;A)\,dz, \quad \varphi \in \mathscr{D}_\omega, \tag{4.5}$$

converges in the norm-topology of $\mathscr{L}(X)$ and defines an ω-ultradistribution $\mathscr{E} \in \mathscr{L}\left(\mathscr{D}_\omega; \mathscr{L}(X)\right)$ with the properties

(i) $\operatorname{supp}\mathscr{E} \subset [0,+\infty)$;

(ii) $\mathscr{E}(\varphi)X \subset D(A)$ *for all* $\varphi \in \mathscr{D}_\omega$;

(iii) $\mathscr{E}' - A\mathscr{E} = \delta_0 I_X$, $\mathscr{E}' - \mathscr{E}A = \delta_0 I_{D(A)}$.

Proof. Let $-\infty < a < b < +\infty$. By (4.1) and by the evident inequality

$$\left|\omega(|z|)\right| \leqslant |\omega(-iz)|, \quad \operatorname{Re}z \geqslant 0,$$

we have for every $\varphi \in \mathscr{D}_\omega[a,b]$, $L > 0$, integer $n \geqslant 1$ and $z \in \Gamma_{\omega,\alpha,\beta}$

$$\|\tilde{\varphi}(z)R(z;A)\| \leqslant 2\pi c_0 p^\omega_{L,n}(\varphi)|\omega(-iLz)|^{-n}e^{b(\alpha\ln|\omega(|\operatorname{Im}z|)|+\beta)}|\omega(-iz)|^{n_0}$$
$$\leqslant 2\pi c_0 e^{b\beta}p^\omega_{L,n}(\varphi)\,|\omega(-iLz)|^{-n}|\omega(-iz)|^{b\alpha+n_0}.$$

Hence, taking $L = 1$ and $n \geqslant b\alpha+n_0+2$, we get for all $\varphi \in \mathscr{D}_\omega[a,b]$ and $z \in \Gamma_{\omega,\alpha,\beta}$

$$\|\tilde{\varphi}(z)R(z;A)\| \leqslant 2\pi c_0 e^{b\beta}p^\omega_{1,n}(\varphi)\,|\omega(-iz)|^{-2}.$$

Consequently, the integral in (4.5) converges in the norm-topology of $\mathscr{L}(X)$ and

$$\|\mathscr{E}(\varphi)\| \leqslant \Big(\int\limits_{\Gamma_{\omega,\alpha,\beta}} c_0 e^{b\beta} |\omega(-iz)|^{-2} d|z| \Big) p^{\omega}_{1,n}(\varphi), \qquad \varphi \in \mathscr{D}_\omega[a, b].$$

We conclude that $\mathscr{E}$ is a well-defined ω-ultradistribution from $\mathscr{L}(\mathscr{D}_\omega; \mathscr{L}(X))$.

Let $\varphi \in \mathscr{D}_\omega$, $\operatorname{supp}\varphi \subset (-\infty, 0)$. Then by (4.1) the analytic function $\Lambda_{\omega,\alpha,\beta} \ni z \mapsto \tilde{\varphi}(z) R(z; A)$ is $O(|z|^{-2})$ at ∞, and so by the Cauchy integral theorem $\mathscr{E}(\varphi) = 0$. Consequently, (i) is satisfied.

Using the closedness of A, it is easy to see that for every $\varphi \in \mathscr{D}_\omega$ we have $\mathscr{E}(\varphi)X \subset D(A)$ and

$$\mathscr{E}(\varphi)A \subset A\mathscr{E}(\varphi). \tag{4.6}$$

Hence also (ii) holds.

Finally, for every $\varphi \in \mathscr{D}_\omega$

$$\begin{aligned}(\mathscr{E}' - A\mathscr{E})(\varphi) &= \mathscr{E}(-\varphi') - A\mathscr{E}(\varphi) \\ &= \frac{1}{2\pi i} \int\limits_{\Gamma_{\omega,\alpha,\beta}} \tilde{\varphi}(z)(z-A) R(z; A)\,dz \\ &= \Big(\frac{1}{2\pi i} \int\limits_{\Gamma_{\omega,\alpha,\beta}} \tilde{\varphi}(z)\,dz \Big) I_X = \delta_0(\varphi) I_X;\end{aligned}$$

thus $\mathscr{E}' - A\mathscr{E} = \delta_0 I_X$. Using this fact and (4.6), we obtain also

$$\mathscr{E}' - \mathscr{E}A = \delta_0 I_{D(A)}. \ \blacksquare$$

We shall now prove a converse of the above lemma:

LEMMA 4.2. *Let A be a closed linear operator in X. If there exists an $\mathscr{E} \in \mathscr{L}(\mathscr{D}_\omega; \mathscr{L}(X))$ satisfying the conditions* (i), (ii), (iii) *from Lemma* 4.1, *then $A \in (C_\omega)$.*

Proof. By Theorem 2.5 there exists a $\psi \in \mathscr{D}_\omega$ such that $\psi(s) = 1$ for $s \in [-1, 1]$. For every $z \in \boldsymbol{C}$ we define $\psi_z \in \mathscr{D}_\omega$ by

$$\psi_z(s) = e^{-zs}\psi(s), \qquad s \in \boldsymbol{R}.$$

By the first equality in (iii), we have

$$\mathscr{E}'(\psi_z) - A\mathscr{E}(\psi_z) = \psi_z(0) I_X, \qquad z \in \boldsymbol{C}.$$

Since $\mathscr{E}'(\psi_z) = \mathscr{E}(-\psi_z') = z\mathscr{E}(\psi_z) - \mathscr{E}(e^{-z\cdot}\psi')$ and $\psi_z(0) = 1$, we get

$$(z-A)\mathscr{E}(\psi_z) = I_X + \mathscr{E}(e^{-z\cdot}\psi'), \qquad z \in \boldsymbol{C}.$$

Using the second equality in (iii), we deduce similarly that

$$\mathscr{E}(\psi_z)(z-A) = I_{D(A)} + \mathscr{E}(e^{-z\cdot}\psi'), \qquad z \in \boldsymbol{C}.$$

Hence if

$$\|\mathscr{E}(e^{-z\cdot}\psi')\| \leqslant \tfrac{1}{2}, \tag{4.7}$$

then $R(z; A) \in \mathscr{L}(X)$ exists and

$$R(z; A) = \mathscr{E}(\psi_z)[I_X + \mathscr{E}(e^{-z\cdot}\psi')]^{-1},$$

so that

$$\|R(z; A)\| \leqslant \|\mathscr{E}(\psi_z)\| \sum_{k=0}^{\infty} \|\mathscr{E}(e^{-z\cdot}\psi')\|^k \leqslant 2\|\mathscr{E}(\psi_z)\|. \tag{4.8}$$

Further, we remark that

$$\mathscr{E}(e^{-z\cdot}\psi') = (\psi'\mathscr{E})(e^{-z\cdot}) = \widetilde{(\psi'\mathscr{E})}(-z), \qquad z \in \boldsymbol{C},$$

where $\operatorname{supp}(\psi'\mathscr{E})$ is compact and contained in $[1, +\infty)$. By (4.2), there exist an L', a $c' > 0$ and an integer $n' \geqslant 1$ such that

$$\|\mathscr{E}(e^{-z\cdot}\psi')\| \leqslant c' |\omega(-iL'z)|^{n'} e^{-\operatorname{Re} z}, \qquad \operatorname{Re} z \geqslant 0.$$

Since

$$|\omega(-iz)| \leqslant \omega(-i\operatorname{Re} z)\, |\omega(|\operatorname{Im} z|)|, \qquad \operatorname{Re} z \geqslant 0,$$

for every $\varepsilon > 0$ there exists a $c_\varepsilon > 0$ such that for $\operatorname{Re} z \geqslant 0$

$$|\omega(-iL'z)|^{n'} \leqslant c_\varepsilon e^{\varepsilon \operatorname{Re} z} |\omega(L'|\operatorname{Im} z|)|^{n'} \leqslant c_\varepsilon e^{\varepsilon \operatorname{Re} z} |\omega(|\operatorname{Im} z|)|^{(L')^2 n'}.$$

Taking $\varepsilon = \frac{1}{2}$, we get

$$\|\mathscr{E}(e^{-z\cdot}\psi')\| \leqslant c' c_{1/2} e^{-\frac{1}{2}\operatorname{Re} z} |\omega(|\operatorname{Im} z|)|^{(L')^2 n'}, \qquad \operatorname{Re} z \geqslant 0.$$

Hence (4.7) holds whenever $\operatorname{Re} z \geqslant 0$ and

$$c' c_{1/2} e^{-\frac{1}{2}\operatorname{Re} z} |\omega(|\operatorname{Im} z|)|^{(L')^2 n'} \leqslant \tfrac{1}{2},$$

that is,

$$\operatorname{Re} z \geqslant 2(L')^2 n' \ln|\omega(|\operatorname{Im} z|)| + 2\ln(2c' c_{1/2}).$$

Consequently, for

$$\alpha = 2(L')^2 n', \qquad \beta = \max\{0, 2\ln(2c' c_{1/2})\},$$

$R(z; A) \in \mathscr{L}(X)$ exists for $z \in \Lambda_{\omega,\alpha,\beta}$.

Finally

$$\mathscr{E}(\psi_z) = (\psi\mathscr{E})(e^{-z\cdot}) = \widetilde{(\psi\mathscr{E})}(-z), \qquad z \in \boldsymbol{C},$$

where $\operatorname{supp}(\psi\mathscr{E})$ is compact and contained in $[0, +\infty)$. Again, by (4.2), there exist an L'', a $c'' > 0$ and an integer $n'' \geqslant 1$ such that

$$\|\mathscr{E}(\psi_z)\| \leqslant c'' |\omega(-iL''z)|^{n''} \leqslant c'' |\omega(-iz)|^{(L'')^2 n''}, \qquad \operatorname{Re} z \geqslant 0.$$

Hence by (4.8), taking

$$c_0 = 2c'', \qquad n_0 \geqslant (L'')^2 n'',$$

we get

$$\|R(z; A)\| \leqslant c_0 |\omega(-iz)|^{n_0}, \qquad z \in \Lambda_{\omega,\alpha,\beta}. \ \blacksquare$$

With regard to the class (C_ω), we recall the following definition:

DEFINITION XVII. Let A be a linear operator in X and $x_0 \in X$. By a *solution of the abstract Cauchy problem*, denoted briefly by (ACP), for (A, x_0) we understand a continuous function u: $[0, +\infty) \to X$ which is strongly differentiable on $(0, +\infty)$ and satisfies

$$u(0) = x_0,$$

$$u(t) \in D(A), \quad u'(t) = Au(t), \quad t > 0.$$

If $A \in (C_\omega)$ then we have the following uniqueness result for the (ACP):

LEMMA 4.3. *Let $A \in (C_\omega)$ and $x_0 \in X$. Then the* (ACP) *has at most one solution for* (A, x_0).

Proof. Since $R(t; A) \in \mathscr{L}(X)$ exists for $t \in [t_0, +\infty)$, where t_0 is some real number, and

$$\overline{\lim_{t\to+\infty}} \frac{\ln \|R(t; A)\|}{t} \leqslant 0,$$

our statement is an immediate consequence of [42], Theorem 1. ■

Using the above lemmas, we can infer

THEOREM 4.4. *Let A be a closed linear operator in X. Then $A \in (C_\omega)$ if and only if there exists an $\mathscr{E} \in \mathscr{L}(\mathscr{D}_\omega; \mathscr{L}(X))$ satisfying the properties:*

(i) $\operatorname{supp} \mathscr{E} \subset [0, +\infty)$;

(ii) $\mathscr{E}(\varphi)X \subset D(A)$ *for all* $\varphi \in \mathscr{D}_\omega$;

(iii) $\mathscr{E}' - A\mathscr{E} = \delta_0 I_X$, $\mathscr{E}' - \mathscr{E}A = \delta_0 I_{D(A)}$.

Moreover, if $A \in (C_\omega)$ then the above three conditions define a unique $\mathscr{E} \in \mathscr{L}(\mathscr{D}_\omega; \mathscr{L}(X))$, given by the formula

$$\mathscr{E}(\varphi) = \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} \tilde{\varphi}(z) R(z; A)\, dz, \quad \varphi \in \mathscr{D}_\omega,$$

where $\alpha, \beta > 0$ are as in Definition XVI *and the integral on the right side converges in the norm-topology of $\mathscr{L}(X)$.*

Proof. Let $\mathscr{E}, \mathscr{F} \in \mathscr{L}(\mathscr{D}_\omega; \mathscr{L}(X))$ be such that (i), (ii), (iii) hold for each of them. We write $\mathscr{G} = \mathscr{E} - \mathscr{F}$. Then

$$\operatorname{supp} \mathscr{G} \subset [0, +\infty),$$

$$\mathscr{G}(\varphi)X \subset D(A) \quad \text{for all } \varphi \in \mathscr{D}_\omega,$$

$$\mathscr{G}' - A\mathscr{G} = 0.$$

Let $\varphi \in \mathscr{D}_\omega$ and $x \in X$ be arbitrary. We take a real number $t_0 > 0$ such that $\operatorname{supp}\varphi \subset (-\infty, t_0)$ and we define u: $[0, +\infty) \to X$ by

$$u(t) = \mathscr{G}(\tau_{t-t_0}\varphi)x, \quad t \geqslant 0,$$

where, as usual, $\tau_r \varphi \in \mathscr{D}_\omega$, $r \in \boldsymbol{R}$, is defined by $(\tau_r\varphi)(s) = \varphi(s-r)$, $s \in \boldsymbol{R}$. Then u is continuous and strongly differentiable on $(0, +\infty)$, and for every $t > 0$ we have

$$u'(t) = \mathscr{G}\big(-(\tau_{t-t_0}\varphi)'\big)x = \mathscr{G}'(\tau_{t-t_0}\varphi)x = Au(t).$$

Since $\operatorname{supp}(\tau_{-t_0}\varphi) = -t_0 + \operatorname{supp}\varphi \subset (-\infty, 0)$, we have also

$$u(0) = \mathscr{G}(\tau_{-t_0}\varphi)x = 0.$$

Consequently, u is a solution of the (ACP) for $(A, 0)$ and thus by Lemma 4.3 $u \equiv 0$. In particular,

$$\mathscr{G}(\varphi)x = u(t_0) = 0.$$

We conclude that $\mathscr{G} = 0$, that is $\mathscr{E} = \mathscr{F}$.

Using the uniqueness result proved above and Lemmas 4.1 and 4.2, we obtain the theorem. ■

We remark that a closed linear operator A in X satisfies the condition C_ω if and only if there are $\alpha, \beta > 0$ such that $R(z; A) \in \mathscr{L}(X)$ exists for $z \in \Lambda_{\omega,\alpha,\beta}$ and for every $\varepsilon > 0$ there exist a $c_\varepsilon > 0$ and an integer $n_\varepsilon \geqslant 1$ with

$$\|R(z; A)\| \leqslant c_\varepsilon |\omega(-iz)|^{n_\varepsilon} e^{\varepsilon \operatorname{Re} z}, \quad z \in \Lambda_{\omega,\alpha,\beta}.$$

Indeed, if A satisfies the above condition, then a similar reasoning to that used in the proof of Lemma 4.1 shows us that there exists an $\mathscr{E} \in \mathscr{L}(\mathscr{D}_\omega; \mathscr{L}(X))$ satisfying the conditions (i), (ii), (iii) from Theorem 4.4.

DEFINITION XVIII. Let $A \in (C_\omega)$; then the ω-ultradistribution $\mathscr{E} \in \mathscr{L}(\mathscr{D}_\omega; \mathscr{L}(X))$ defined by the conditions (i), (ii), (iii) from Theorem 4.4 is called the *ω-ultradistribution semi-group generated by A.*

We recall that distribution semi-groups were defined by J. L. Lions in [40] and their generators were characterized by C. Foiaş in a particular case ([23]) and by J. Chazarain in the general case ([11]). In [11] J. Chazarain considered in the frame of the existing ultradistribution theories also ultradistribution semi-groups and he also characterized their generators. Theorem 4.4 extends Chazarain's results.

The "semi-group property" of ω-ultradistribution semi-groups consists in the following one:

PROPOSITION 4.5. *If $\mathscr{E}$ is an ω-ultradistribution semi-group, then*

$$\mathscr{E}(\varphi * \psi) = \mathscr{E}(\varphi)\mathscr{E}(\psi), \quad \varphi, \psi \in \mathscr{D}_\omega^0.$$

Proof. Let $A \in (C_\omega)$ be the generator of $\mathscr{E}$. By (4.5), the proof is a routine exercise in the functional calculus theory; since we shall use similar arguments several times, we give the proof in detail.

Let $\varphi, \psi \in \mathscr{D}_\omega^0$. Using the formulas

$$\mathscr{E}(\varphi) = \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} \tilde{\varphi}(z) R(z; A)\, dz,$$

$$\mathscr{E}(\psi) = \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta+1}} \tilde{\psi}(\lambda) R(\lambda; A)\, d\lambda$$

and the resolvent equation, we infer that

$$\mathscr{E}(\varphi)\mathscr{E}(\psi) = \frac{1}{2\pi i}\int_{\Gamma_{\omega,\alpha,\beta}} \tilde{\varphi}(z)R(z;A)\left[\frac{1}{2\pi i}\int_{\Gamma_{\omega,\alpha,\beta+1}} \frac{\tilde{\psi}(\lambda)}{\lambda-z}d\lambda\right]dz+$$

$$+\frac{1}{2\pi i}\int_{\Gamma_{\omega,\alpha,\beta+1}} \tilde{\psi}(\lambda)R(\lambda;A)\left[\frac{1}{2\pi i}\int_{\Gamma_{\omega,\alpha,\beta}} \frac{\tilde{\varphi}(z)}{z-\lambda}dz\right]d\lambda.$$

Since $\operatorname{supp}\varphi$, $\operatorname{supp}\psi \subset (0,+\infty)$, by (4.1) and by the Cauchy integral formula we get

$$\frac{1}{2\pi i}\int_{\Gamma_{\omega,\alpha,\beta+1}} \frac{\tilde{\psi}(\lambda)}{\lambda-z}d\lambda = \tilde{\psi}(z), \quad z\in\Gamma_{\omega,\alpha,\beta},$$

$$\frac{1}{2\pi i}\int_{\Gamma_{\omega,\alpha,\beta}} \frac{\tilde{\varphi}(z)}{z-\lambda}dz = 0, \quad \lambda\in\Gamma_{\omega,\alpha,\beta+1},$$

so that finally we obtain

$$\mathscr{E}(\varphi)\mathscr{E}(\psi) = \frac{1}{2\pi i}\int_{\Gamma_{\omega,\alpha,\beta}} \tilde{\varphi}(z)\tilde{\psi}(z)R(z;A)dz = \mathscr{E}(\varphi*\psi). \ \blacksquare$$

For every $\mathscr{E}\in\mathscr{L}(\mathscr{D}_\omega;\mathscr{L}(X))$ we put

$$\mathscr{N}_{\mathscr{E}} = \{x\in X;\ \mathscr{E}(\varphi)x = 0 \text{ for all } \varphi\in\mathscr{D}_\omega^0\},$$

$$\mathscr{R}_{\mathscr{E}} = \Big\{\sum_{j=1}^{n}\mathscr{E}(\varphi_j)x_j;\ n\geqslant 1 \text{ integer},\ \varphi_j\in\mathscr{D}_\omega^0,\ x_j\in X\Big\}.$$

DEFINITION XIX. We say that an ω-ultradistribution semi-group is *regular* if $\mathscr{N}_{\mathscr{E}} = \{0\}$ and $\overline{\mathscr{R}_{\mathscr{E}}} = X$. We denote by (C_ω^0) the class of all operators from (C_ω) which generate regular ω-ultradistribution semi-groups.

We note that if $A\in(C_\omega^0)$ then A is densely defined, because $D(A)\supset\mathscr{R}_{\mathscr{E}}$. However, in general, there exist densely defined operators in (C_ω) which do not belong to (C_ω^0), as the following example, due to D. Voiculescu, shows.

Let us assume that $t_1\leqslant t_2\leqslant\ldots<+\infty$ and let H be an infinite-dimensional separable Hilbert space with an orthonormal basis $\{e_n\}_{-\infty<n<+\infty}$. We define the weighted shift $T\in\mathscr{L}(H)$ by

$$Te_n = \frac{1}{t_{n+1}}e_{n+1}, \quad n\geqslant 0,$$

$$Te_n = \frac{1}{t_{-n+1}}e_{n+1}, \quad n<0.$$

It is easy to verify that T is injective and has a dense range, so we can consider the densely defined closed linear operator $A = T^{-1}$. Since

$$\|T^k\| = \frac{1}{t_1\ldots t_k}, \quad k\geqslant 1,$$

$R(z;A)\in\mathscr{L}(H)$ exists for every $z\in\boldsymbol{C}$ and it is given by the formula

$$R(z;A) = -\sum_{k=0}^{\infty}z^kT^{k+1}.$$

Hence

$$\|R(z;A)\|\leqslant\sum_{k=0}^{\infty}\frac{|z|^k}{t_1\ldots t_{k+1}}\leqslant\Big(\sum_{k=1}^{\infty}\frac{1}{t_k}\Big)|\omega(|z|)|, \quad z\in\boldsymbol{C}.$$

In particular

$$\|R(z;A)\|\leqslant\Big(\sum_{k=1}^{\infty}\frac{1}{t_k}\Big)|\omega(-iz)|, \quad \operatorname{Re}z\geqslant 0,$$

so that $A\in(C_\omega)$. On the other hand, denoting by $\mathscr{E}$ the ω-ultradistribution semi-group generated by A and using (4.5), (4.1) and the Cauchy integral theorem, we can easily see that $\operatorname{supp}\mathscr{E} = \{0\}$, so that

$$\mathscr{N}_{\mathscr{E}} = H, \quad \mathscr{R}_{\mathscr{E}} = \{0\}.$$

Consequently, $A\notin(C_\omega^0)$.

The following result shows that the class (C_ω^0) is large enough:

PROPOSITION 4.6. *Let A be a densely defined closed linear operator in X such that for some α,β, a $c_0>0$ and an integer $n_0\geqslant 1$, $R(z;A)\in\mathscr{L}(X)$ exists whenever $z\in\Lambda_{\omega,\alpha,\beta}$ and satisfies*

$$\|R(z;A)\|\leqslant c_0(1+|z|)^{n_0}, \quad z\in\Lambda_{\omega,\alpha,\beta}.$$

Then $A\in(C_\omega^0)$.

Proof. It is obvious that $A\in(C_\omega)$.

Let $x\in D(A^{n_0+2})$ be arbitrary. Then

$$R(z;A)x = \sum_{j=0}^{n_0+1}z^{-j-1}A^jx+z^{-n_0-2}R(z;A)A^{n_0+2}x, \quad z\in\Lambda_{\omega,\alpha,\beta}. \tag{4.9}$$

Further, let $\{\varphi_k\}_{k\geqslant 1}\subset\mathscr{D}_\omega^0$ be a sequence such that $\varphi_k\to\delta_0$ in $\mathscr{E}'_\omega$ (for example, choosing some $\varphi_1\in\mathscr{D}_\omega^0$, $\varphi_1\geqslant 0$, $\int_{-\infty}^{+\infty}\varphi_1(s)ds = 1$, we can take $\varphi_k(s) = k\varphi_1(ks)$, $s\in\boldsymbol{R}$). Then

$$\tilde{\varphi}_k\mapsto 1 \quad\text{and}\quad (\tilde{\varphi}_k)^{(j)}\mapsto 0,\ j\geqslant 1$$

uniformly on the bounded subsets of $\boldsymbol{C}$.

By (4.1), for each integer $k\geqslant 1$, $\tilde{\varphi}_k$ is analytical and $O(|z|^{-2})$ at ∞ in $\boldsymbol{C}\setminus\Lambda_{\omega,\alpha,\beta}$; hence by the Cauchy integral formula

$$\frac{1}{2\pi i}\int_{\Gamma_{\omega,\alpha,\beta}}\tilde{\varphi}_k(z)z^{-j-1}A^jx\,dz = (\tilde{\varphi}_k)^{(j)}(0)A^jx, \quad 0\leqslant j\leqslant n_0+1.$$

Using (4.9), we get for all $k \geqslant 1$

$$\mathscr{E}(\varphi_k)x = \sum_{j=0}^{n_0+1} (\tilde{\varphi}_k)^{(j)}(0)A^j x + \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} \tilde{\varphi}_k(z) z^{-n_0-2} R(z;A) A^{n_0+2} x\, dz.$$

Letting $k \to +\infty$, we deduce that

$$\lim_{k\to\infty} \mathscr{E}(\varphi_k)x = x + \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} z^{-n_0-2} R(z;A) A^{n_0+2} x\, dz.$$

Since $z \mapsto z^{-n_0-2}R(z;A)A^{n_0+2}x$ is analytical and $O(|z|^{-2})$ at ∞ in $\Lambda_{\omega,\alpha,\beta}$, its integral along $\Gamma_{\omega,\alpha,\beta}$ vanishes. So

$$\lim_{k\to+\infty} \mathscr{E}(\varphi_k)x = x.$$

We conclude that $\mathscr{R}_{\mathscr{E}}$ is dense in $D(A^{n_0+2})$. Since A is densely defined and its resolvent set is not empty, $D(A^{n_0+2})$ is dense in X, and so $\overline{\mathscr{R}_{\mathscr{E}}} = X$.

Similar arguments show that $\mathscr{R}_{\mathscr{E}^*}$ is X-dense in X^*, where $\mathscr{E}^*$ is the ω-ultra-distribution semi-group generated by A^*. Since $\mathscr{N}_{\mathscr{E}}$ is the annihilator of $\mathscr{R}_{\mathscr{E}^*}$ in X, it follows that $\mathscr{N}_{\mathscr{E}} = \{0\}$. ∎

For operators in (C_ω^0) the abstract Cauchy problem has a unique solution for a dense set of initial conditions. Namely, if $A \in (C_\omega^0)$ and $\mathscr{E}$ is the ω-ultradistribution semi-group generated by A, then, by condition (iii) from Theorem 4.4, for every $x \in X$ and $\varphi \in \mathscr{D}_\omega^0$ the function $[0, +\infty) \ni t \mapsto \mathscr{E}(\tau_t\varphi)x$ is a solution of the (ACP) for $(A, \mathscr{E}(\varphi)x)$. Hence, if $x_0 \in \mathscr{R}_{\mathscr{E}}$, then the (ACP) has a solution for (A, x_0) and by Lemma 4.3 this solution is unique.

We shall give later a more precise result, restricting ourselves now to the following extension of a result of R. Beals from [3]:

COROLLARY 4.7. *Let A be a densely defined closed linear operator in X such that for an increasing function f: $[0, +\infty) \to [1, +\infty)$ with $\int_1^{+\infty} t^{-2}\ln f(t)\,dt < +\infty$, $c_0 > 0$ and an integer $n_0 \geqslant 1$, $R(z;A) \in \mathscr{L}(X)$ exists for z in the region*

$$\Lambda_f = \{z \in \boldsymbol{C};\ \operatorname{Re} z \geqslant \ln f(|\operatorname{Im} z|)\}$$

and satisfies the estimation

$$\|R(z;A)\| \leqslant c_0(1+|z|)^{n_0}, \quad z \in \Lambda_f.$$

Then there exists a dense linear subspace $\mathscr{R}$ in X such that the (ACP) *has a unique solution for (A, x_0) whenever $x_0 \in \mathscr{R}$.*

Proof. By Theorem 1.6 there exist $0 < t_1 \leqslant t_2 \leqslant \ldots < +\infty$, $\sum_{k=1}^{\infty} 1/t_k < +\infty$, and $c > 1$ such that, writing $\omega = \omega_{\{t_k\}}$, we have

$$f(t) \leqslant c|\omega(t)|, \quad t \in [0, +\infty).$$

Hence $R(z;A) \in \mathscr{L}(X)$ exists for $z \in \Lambda_{\omega,1,\ln c}$ and

$$\|R(z;A)\| \leqslant c_0(1+|z|)^{n_0}, \quad z \in \Lambda_{\omega,1,\ln c}.$$

By Proposition 4.6, it follows that $A \in (C_\omega^0)$.

Finally, by the remarks preceding the corollary, denoting by $\mathscr{E}$ the ω-ultra-distribution semi-group generated by A, we can take $\mathscr{R} = \mathscr{R}_{\mathscr{E}}$. ∎

We remark that in [3] R. Beals proved the above statement for $f = e^{\psi}$, where ψ: $[0, +\infty) \to [0, +\infty)$ is continuous and concave and satisfies

$$\lim_{t\to+\infty} \psi(t) = +\infty, \quad \lim_{t\to+\infty} t^{-1}\psi(t) = 0, \quad \int_1^{+\infty} t^{-2}\psi(t)\,dt < +\infty.$$

If ψ is as above, then it is increasing. Indeed, for $0 \leqslant s < t < r$ we have

$$(r-t)\psi(s) + (t-s)\psi(r) \leqslant (r-s)\psi(t),$$

so that, dividing by r and letting $r \to +\infty$, we get

$$\psi(s) \leqslant \psi(t).$$

Hence Corollary 4.7 extends Beals' result.

The following technical lemma is essential for our further considerations:

LEMMA 4.8. *Let $0 < r_1 \leqslant r_2 \leqslant \ldots, r_1 < +\infty$, $\sum_{k=1} 1/r_k < +\infty$. Then for any $\alpha > 0$, $0 < \beta < r_1$ and $0 < \varepsilon < 1$, there exists a $c > 0$ such that*

$$|\omega_{\{r_k\}}(iz)| \geqslant c|\omega_{\{r_k\}}(|z|)|^{\varepsilon}$$

for $\operatorname{Re} z \leqslant \alpha \ln|\omega_{\{r_k\}}(|\operatorname{Im} z|)| + \beta$.

Proof. For convenience, we denote $\omega_{\{r_k\}}$ simply by τ. Let us write $\Omega = \{z \in \boldsymbol{C};\ \operatorname{Re} z < \alpha \ln|\tau(|\operatorname{Im} z|)| + \beta\}$. As Ω is simply connected and $z \mapsto \tau(iz)$ does not vanish on Ω, there exists an analytic function f on Ω such that

$$e^{f(z)} = \tau(iz), \quad z \in \Omega.$$

Then

$$e^{\operatorname{Re} f(z)} = |\tau(iz)|, \quad z \in \Omega \tag{4.10}$$

and

$$f'(z) = i\frac{\tau'(iz)}{\tau(iz)} = \sum_{p=1}^{\infty} \frac{1}{z - r_p}, \quad z \in \Omega. \tag{4.11}$$

We put $\Omega_1 = \{z \in \Omega;\ \operatorname{Re} z < 0\}$ and $\Omega_2 = \{z \in \Omega;\ \operatorname{Re} z \geqslant 0\}$.

It is easy to see that for every $z \in \Omega_1$

$$|\tau(iz)| \geqslant |\tau(|z|)| \geqslant |\tau(|z|)|^{\varepsilon}.$$

Next we shall prove that

$$\lim_{\substack{z\in\Omega_2\\ |\operatorname{Im} z|\to+\infty}} |f'(z)| = 0. \tag{4.12}$$

Indeed, for any integer $p_0 \geqslant 1$ and $z \in \Omega_2$, we have

$$|f'(z)| \leqslant \sum_{p=1}^{p_0} \frac{1}{|z-r_p|} + \sum_{p>p_0} \frac{1}{|z-r_p|}$$

$$\leqslant \frac{p_0}{|\mathrm{Im}\, z|} + \sum_{p>p_0} \frac{1}{\sqrt{(|z|^2+r_p^2-2r_p \mathrm{Re}\, z)}}$$

$$\leqslant \frac{p_0}{|\mathrm{Im}\, z|} + \left(1+\frac{\alpha \ln\left|\tau(|\mathrm{Im}\, z|)\right|+\beta}{|\mathrm{Im}\, z|}\right) \sum_{p>p_0} \frac{1}{r_p},$$

so that

$$\varlimsup_{\substack{z\in\Omega_2 \\ |\mathrm{Im}\, z| \to +\infty}} |f'(z)| \leqslant \sum_{p>p_0} \frac{1}{r_p}.$$

Since $p_0 \geqslant 1$ is arbitrary, we obtain (4.12).

Let $\delta > 0$ be arbitrary; by (4.12) there is a constant $d_1 > 0$ such that $|f'(z)| \leqslant \delta$ whenever $z \in \Omega_2$ and $|\mathrm{Im}\, z| \geqslant d_1$. Hence for $z \in \Omega_2$, $|\mathrm{Im}\, z| \geqslant d_1$, we have

$$\mathrm{Re} f(i \mathrm{Im}\, z) - \mathrm{Re} f(z) \leqslant |f(i\mathrm{Im}\, z) - f(z)|$$

$$= \Big| \int_{[i\mathrm{Im}\, z,\, z]} f'(\lambda)\, d\lambda \Big| \leqslant \delta \mathrm{Re}\, z \leqslant \alpha\delta \ln\left|\tau(|\mathrm{Im}\, z|)\right| + \beta\delta.$$

By (4.10) it follows that

$$\frac{\left|\tau(|\mathrm{Im}\, z|)\right|}{|\tau(iz)|} \leqslant e^{\beta\delta} |\tau(|\mathrm{Im}\, z|)|^{\alpha\delta}, \quad z\in\Omega_2,\ |\mathrm{Im}\, z| \geqslant d_1;$$

hence

$$|\tau(iz)| \leqslant e^{-\beta\delta} \left|\tau(|\mathrm{Im}\, z|)\right|^{1-\alpha\delta}, \quad z \in \Omega_2,\ |\mathrm{Im}\, z| \geqslant d_1. \tag{4.13}$$

On the other hand, for any $\gamma > 1$, there is a constant $d_2 > 0$ such that

$$\gamma \geqslant \left(1+\frac{\alpha \ln|\tau(t)|+\beta}{|t|}\right)^2, \quad t \in \boldsymbol{R},\ |t| \geqslant d_2.$$

Thus for $z \in \Omega_2$, $|\mathrm{Im}\, z| \geqslant d_2$ and integer $k \geqslant 1$, we have

$$\left(1+\left(\frac{\mathrm{Im}\, z}{t_k}\right)^2\right)^{\gamma} \geqslant 1+\gamma\left(\frac{\mathrm{Im}\, z}{t_k}\right)^2$$

$$\geqslant 1+\left(1+\frac{\mathrm{Re}\, z}{|\mathrm{Im}\, z|}\right)^2 \left(\frac{\mathrm{Im}\, z}{t_k}\right)^2 \geqslant 1+\left(\frac{|z|}{t_k}\right)^2.$$

Consequently, we obtain

$$\left|\tau(|\mathrm{Im}\, z|)\right| \geqslant \left|\tau(|z|)\right|^{1/\gamma}, \quad z \in \Omega_2,\ |\mathrm{Im}\, z| \geqslant d_2. \tag{4.14}$$

Using (4.13) and (4.14), we conclude that, for every $0 < \delta < \alpha^{-1}$ and $\gamma > 1$, there exists a $d > 0$ such that

$$|\tau(iz)| \geqslant e^{-\beta\delta} \left|\tau(|z|)\right|^{(1-\alpha\delta)/\gamma}, \quad z \in \Omega_2,\ |\mathrm{Im}\, z| \geqslant d.$$

Choosing $0 < \delta < (1-\varepsilon)/\alpha$ and $\gamma = (1-\alpha\delta)/\varepsilon > 1$, we infer that for $z \in \Omega_2$ with $|\mathrm{Im}\, z|$ large enough

$$|\tau(iz)| \geqslant e^{-\beta(1-\varepsilon)/\alpha} |\tau(|z|)|^{\varepsilon}.$$

Now our statement is immediate. ∎

We shall next define "abstract ω-spaces", similarly to the abstract Gevrey spaces of R. Beals from [4] and to the abstract Beurling spaces of I. Ciorănescu from [14]. For this purpose, we prove a lemma, inspired by the techniques used in [14], § 3.

For every $\gamma > 0$ we define the entire function ω_γ by the formula

$$\omega_\gamma(z) = \omega(z) \prod_{t_k<\gamma} \frac{1+(iz/\gamma)}{1+(iz/t_k)}.$$

LEMMA 4.9. *Let $A \in (C_\omega^0)$, α, β, $c_0 > 0$ and let $n_0 \geqslant 1$ be an integer as in Definition* XVI, *let $\mathscr{E}$ be the ω-ultradistribution semi-group generated by A and let $\gamma > \beta$. Then for each integer $n \geqslant 2n_0+6$ we conclude that*

(i) *the integral in the formula*

$$B_{\gamma,n} = \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} \omega_\gamma^{-n}(iz) R(z; A)\, dz$$

converges in the norm-topology of $\mathscr{L}(X)$ and $B_{\gamma,n}$ does not depend on the choice of α, β, c_0 and n_0;

(ii) *for every $\varphi \in \mathscr{D}_\omega^0$*

$$B_{\gamma,n}\mathscr{E}\big(\omega_\gamma^n(-iD)\varphi\big) = \mathscr{E}\big(\omega_\gamma^n(-iD)\varphi\big)B_{\gamma,n} = \mathscr{E}(\varphi);$$

(iii) *$B_{\gamma,n}$ is injective and $B_{\gamma,n}X \supset \mathscr{R}_{\mathscr{E}}$, so $B_{\gamma,n}^{-1}$ is a well-defined closed linear operator with $D(B_{\gamma,n}^{-1}) \supset \mathscr{R}_{\mathscr{E}}$;*

(iv) $\overline{B_{\gamma,n}^{-1}|\mathscr{R}_{\mathscr{E}}} = B_{\gamma,n}^{-1}$.

Moreover, for all integers $n, m \geqslant 2n_0+6$

(v) $B_{\gamma,n}B_{\gamma,m} = B_{\gamma,n+m}$.

Finally, if $m \geqslant 1$ and $n \geqslant 2n_0+m+6$ are integers, then

(vi) *$B_{\gamma,n}X \subset D(A^m)$ and*

$$A^m B_{\gamma,n} = \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} z^m \omega_\gamma^{-n}(iz) R(z; A)\, dz,$$

where the integral on the right side converges in the norm-topology of $\mathscr{L}(X)$.

Proof. Let us first establish an estimation for $\|R(z; A)\|$ on $\Gamma_{\omega,\alpha,\beta}$. Since

$$|\omega(-iz)| \leqslant \omega(-i\mathrm{Re}\, z)\, |\omega(|z|)|, \quad \mathrm{Re}\, z \geqslant 0,$$

for every $\varepsilon > 0$ there exists a $c_\varepsilon > 0$ such that

$$|\omega(-iz)|^{n_0} \leqslant c_\varepsilon e^{\varepsilon \mathrm{Re}\, z} |\omega(|z|)|^{n_0}, \quad \mathrm{Re}\, z \geqslant 0.$$

Hence, for $z \in \Gamma_{\omega,\alpha,\beta}$

$$\|R(z; A)\| \leqslant c_0 c_\varepsilon e^{\varepsilon \mathrm{Re}\, z} |\omega(|z|)|^{n_0} \leqslant c_0 c_\varepsilon e^{\beta\varepsilon} |\omega(|z|)|^{n_0+\alpha\varepsilon}.$$

In particular,

(4.15) $$\|R(z;A)\| \leqslant c_0 c_{1/\alpha} e^{\beta/\alpha} |\omega(|z|)|^{n_0+1}, \quad z \in \Gamma_{\omega,\alpha,\beta}.$$

On the other hand, choosing an integer $m \geqslant 1$ with

$$m \geqslant \max_{t_k<\gamma} \frac{\gamma^2}{t_k^2},$$

we have

$$|\omega(t)| \leqslant |\omega_\gamma(t)|^m, \quad t \in \boldsymbol{R},$$

so that

$$\boldsymbol{C} \setminus \Lambda_{\omega,\alpha,\beta} \subset \{z \in \boldsymbol{C};\ \operatorname{Re} z < \alpha m \ln|\omega_\gamma(|\operatorname{Im} z|)| + \beta\}.$$

Applying Lemma 4.8 with $\varepsilon = \frac{1}{2}$, we deduce that there exists a $c > 0$ such that

$$|\omega_\gamma(iz)| \geqslant c|\omega_\gamma(|z|)|^{1/2}, \quad z \in \boldsymbol{C} \setminus \Lambda_{\omega,\alpha,\beta}.$$

Hence, writing

$$d = \inf_{t \in R} \prod_{t_k<\gamma} \left|\frac{1+(it/\gamma)}{1+(it/t_k)}\right| > 0,$$

we get

(4.16) $$|\omega_\gamma(iz)| \geqslant cd|\omega(|z|)|^{1/2}, \quad z \in \boldsymbol{C} \setminus \Lambda_{\omega,\alpha,\beta}.$$

Using (4.15) and (4.16), we get for each integer $n \geqslant 2n_0+6$

$$\|\omega_\gamma^{-n}(iz)R(z;A)\| \leqslant \frac{c_0 c_{1/\alpha} e^{\beta/\alpha}}{c^n d^n} |\omega(|z|)|^{-2}, \quad z \in \Gamma_{\omega,\alpha,\beta};$$

hence the integral in the formula from (i) converges in the norm topology of $\mathscr{L}(X)$.

Taking into account that $\widetilde{(\omega_\gamma^n(-iD)\varphi)}(z) = \omega_\gamma^n(iz)\tilde{\varphi}(z)$, $z \in \boldsymbol{C}$, and that $z \mapsto \omega_\gamma^{-n}(iz)$ is an analytic function on a neighbourhood of $\overline{\boldsymbol{C} \setminus \Lambda_{\omega,\alpha,\beta}}$ and $O(|z|^{-n/2})$ at ∞ on $\boldsymbol{C} \setminus \Lambda_{\omega,\alpha,\beta}$, and using similar arguments to those used in the proof of Proposition 4.5, we can easily verify (ii) and (v).

(iii) is a direct consequence of (ii). Indeed, if $x \in X$ is such that $B_{\gamma,n}x = 0$, then by (ii)

$$\mathscr{E}(\varphi)x = \mathscr{E}(\omega_\gamma^n(-iD)\varphi)B_{\gamma,n}x = 0, \quad \varphi \in \mathscr{D}_\omega^0;$$

so $x = 0$. On the other hand, again by (ii)

$$B_{\gamma,n}X \supset \bigcup_{\varphi \in \mathscr{D}_\omega^0} B_{\gamma,n}\mathscr{E}(\omega_\gamma^n(-iD)\varphi)X = \bigcup_{\varphi \in \mathscr{D}_\omega^0} \mathscr{E}(\varphi)X;$$

hence $B_{\gamma,n}X \supset \mathscr{R}_{\mathscr{E}}$.

In order to prove (iv), let $n \geqslant 2n_0+6$ be integer and $\varphi \in \mathscr{D}_\omega^0$. Since $B_{\gamma,n}$ and $\mathscr{E}(\varphi)$ commute, we have successively

$$B_{\gamma,n}^{-1}\mathscr{E}(\varphi)B_{\gamma,n} = \mathscr{E}(\varphi),$$

$$\overline{(B_{\gamma,n}^{-1}|\mathscr{R}_{\mathscr{E}})}\,(\mathscr{E}(\varphi)B_{\gamma,n}) = \mathscr{E}(\varphi).$$

Using again the permutability of $B_{\gamma,n}$ and $\mathscr{E}(\varphi)$, we get

$$B_{\gamma,n}\mathscr{E}(\varphi)X \subset D\overline{(B_{\gamma,n}^{-1}|\mathscr{R}_{\mathscr{E}})} \quad \text{and} \quad \overline{(B_{\gamma,n}^{-1}|\mathscr{R}_{\mathscr{E}})}\,(B_{\gamma,n}\mathscr{E}(\varphi)) = \mathscr{E}(\varphi).$$

Since $\varphi \in \mathscr{D}_\omega^0$ is arbitrary, we conclude that $B_{\gamma,n}\mathscr{R}_{\mathscr{E}} \subset D\overline{(B_{\gamma,n}^{-1}|\mathscr{R}_{\mathscr{E}})}$ and

$$\overline{(B_{\gamma,n}^{-1}|\mathscr{R}_{\mathscr{E}})}B_{\gamma,n}x = x, \quad x \in \mathscr{R}_{\mathscr{E}}.$$

Using the density of $\mathscr{R}_{\mathscr{E}}$, the continuity of $B_{\gamma,n}$ and the closedness of $\overline{B_{\gamma,n}^{-1}|\mathscr{R}_{\mathscr{E}}}$, we infer that $B_{\gamma,n}X \subset D(\overline{B_{\gamma,n}^{-1}|\mathscr{R}_{\mathscr{E}}})$, that is $D(B_{\gamma,n}^{-1}) \subset D(\overline{B_{\gamma,n}^{-1}|\mathscr{R}_{\mathscr{E}}})$. Consequently, $B_{\gamma,n}^{-1} = \overline{B_{\gamma,n}^{-1}|\mathscr{R}_{\mathscr{E}}}$, and so (iv) is proved.

By (ii) we have

$$B_{\gamma,n}^{-1}\mathscr{E}(\varphi)x = \mathscr{E}(\omega_\gamma^n(-iD)\varphi)x, \quad \varphi \in \mathscr{D}_\omega^0,\ x \in X,$$

and by (iv)

$$\overline{B_{\gamma,n}^{-1}|\mathscr{R}_{\mathscr{E}}} = B_{\gamma,n}^{-1}.$$

Thus $B_{\gamma,n}^{-1}$, and hence also $B_{\gamma,n}$, does not depend on the choice of α, β, c_0 and n_0. So (i) is completely proved.

Finally, let $m \geqslant 1$ and $n \geqslant 2n_0+m+6$ be integers. By (4.15) and (4.16), it follows that the integral in the formula from (vi) converges in the norm-topology of $\mathscr{L}(X)$. We put

$$T = \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} z^m \omega_\gamma^{-n}(iz)R(z;A)dz \in \mathscr{L}(X).$$

Let $x \in D(A^m)$. Using the formula

$$A^m R(z;A)x = z^m R(z;A)x - \sum_{j=0}^{m-1} z^{m-j-1}A^j x, \quad x \in \Gamma_{\omega,\alpha,\beta},$$

and the closedness of A^m, we can easily verify that $B_{\gamma,n}x \in D(A^m)$ and

$$A^m B_{\gamma,n}x = Tx.$$

By the density of $D(A^m)$, the continuity of $B_{\gamma,n}$ and the closedness of A^m it follows that $B_{\gamma,n}X \subset D(A^m)$ and

$$A^m B_{\gamma,n} = T.$$

Hence (vi) is proved. ∎

We shall study next the dependence of $B_{\gamma,n}$ on γ:

LEMMA 4.10. *Let $A \in (C_\omega^0)$, $\alpha, \beta, c_0 > 0$ and let $n_0 \geqslant 1$ be an integer as in Definition XVI, and let $\gamma_1, \gamma_2 > \beta$. Then for each integer $n \geqslant 2n_0+6$ the integral in the formula*

$$C_{\gamma_1,\gamma_2,n} = \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} \omega_{\gamma_1}^n(z)\omega_{\gamma_2}^{-2n}(z)R(z;A)dz$$

converges in the norm-topology of $\mathscr{L}(X)$ and we have

(4.17) $$B_{\gamma_2,2n} = B_{\gamma_1,n}C_{\gamma_1,\gamma_2,n} = C_{\gamma_1,\gamma_2,n}B_{\gamma_1,n}.$$

Proof. By the estimations (4.15) and (4.16), the integral in the above formula converges in the norm-topology of $\mathscr{L}(X)$. Using similar arguments to those used in the proof of Proposition 4.5, we get the equalities (4.17). ∎

The above two lemmas permit us to infer

THEOREM 4.11. *Let $A \in (C_\omega^0)$, $\alpha, \beta, c_0 > 0$ and let $n_0 \geqslant 1$ be an integer as in Definition XVI, let $\mathscr{E}$ be the ω-ultradistribution semi-group generated by A, $\gamma > \beta$ and let $p_0 \geqslant 2n_0+6$ be an integer. Then*

$$\bigcap_{n \geqslant p_0} B_{\gamma,n} X,$$

endowed with the norms $\|\cdot\|_{\gamma,n}$, $n \geqslant p_0$, defined by

$$\|x\|_{\gamma,n} = \|B_{\gamma,n}^{-1} x\|, \quad x \in \bigcap_{n \geqslant p_0} B_{\gamma,n} X,$$

is a Fréchet space $X_{\omega,A}$ which does not depend on the choice of $\alpha, \beta, c_0, n_0, \gamma, p_0$ and satisfies:

(i) *$X_{\omega,A} \subset D(A)$, $AX_{\omega,A} \subset X_{\omega,A}$ and $A|X_{\omega,A} \in \mathscr{L}(X_{\omega,A})$;*
(ii) *$\mathscr{R}_{\mathscr{E}}$ is a dense linear subspace of $X_{\omega,A}$;*
(iii) *the inclusion $X_{\omega,A} \subset X$ is continuous and has a dense range.*

Proof. By Lemma 4.9 (v) and by Lemma 4.10, for all $\gamma_1, \gamma_2 > \beta$ and $n_1, n_2 \geqslant 2n_0+6$, $n_2 \geqslant 3n_1$, we have:

$$B_{\gamma_2,n_2} X \subset B_{\gamma_2,2n_1} X \subset B_{\gamma_1,n_1} X;$$

so $\bigcap_{n \geqslant p_0} B_{\gamma,n} X$ does not depend on γ and p_0. On the other hand, again by Lemma 4.9(v) and Lemma 4.10, for all $x \in \bigcap_{n \geqslant p_0} B_{\gamma,n} X$ and $\gamma_1, \gamma_2, n_1, n_2$ as above, we have

$$\|x\|_{\gamma_1,n_1} = \|B_{\gamma_1,n_1}^{-1} x\| = \|C_{\gamma_1,\gamma_2,n_1} B_{\gamma_2,n_2-2n_1} B_{\gamma_2,n_2}^{-1} x\| \tag{4.18}$$
$$\leqslant \|C_{\gamma_1,\gamma_2,n_1}\| \, \|B_{\gamma_2,n_2-2n_1}\| \, \|x\|_{\gamma_2,n_2};$$

so the topology defined on $\bigcap_{n \geqslant p_0} B_{\gamma,n} X$ by the norms $\|\cdot\|_{\gamma,n}$, $n \geqslant p_0$, does not depend on γ and p_0 either. Taking into account Lemma 4.9 (i), we conclude that $X_{\omega,A}$ does not depend on the choice of $\alpha, \beta, c_0, n_0, \gamma, p_0$. Clearly, it is a Fréchet space.

In order to prove (i), we remark that by Lemma 4.9 (vi), $X_{\omega,A} \subset D(A)$. By the same lemma, for $y \in D(A)$, $\gamma > \beta$ and integer $n \geqslant 2n_0+7$, we get

$$B_{\gamma,n} A y = \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} \omega_\gamma^{-n}(iz) R(z; A) A y \, dz$$

$$= \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} z \omega_\gamma^{-n}(iz) R(z; A) y \, dz - \left[\frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} \omega_\gamma^{-n}(iz) dz\right] y = AB_{\gamma,n} y$$

and $AB_{\gamma,n} \in \mathscr{L}(X)$. Hence, if $x \in X_{\omega,A}$ then for all $\gamma > \beta$ and integer $n \geqslant 2n_0+7$, we successively have

$$B_{\gamma,n}^{-1} x \in X_{\omega,A} \subset D(A),$$

$$Ax = AB_{\gamma,n}(B_{\gamma,n}^{-1} x) = B_{\gamma,n} A (B_{\gamma,n}^{-1} x) \in B_{\gamma,n} X,$$

so that $Ax \in X_{\omega,A}$. Consequently, $AX_{\omega,A} \subset X_{\omega,A}$. Finally, if $x \in X_{\omega,A}$, $\gamma > \beta$ and $n \geqslant 2n_0+7$ is integer, then

$$\|Ax\|_{\gamma,n} = \|B_{\gamma,n}^{-1} AB_{\gamma,n} B_{\gamma,n} B_{\gamma,2n}^{-1} x\| \leqslant \|AB_{\gamma,n}\| \, \|x\|_{\gamma,2n},$$

and thus $A|X_{\omega,A} \in \mathscr{L}(X_{\omega,A})$.

By Lemma 4.9 (iii), $\mathscr{R}_{\mathscr{E}}$ is a linear subspace of $X_{\omega,A}$, and using Lemma 4.9 (iv) and (4.18), we can easily verify that $\mathscr{R}_{\mathscr{E}}$ is dense in $X_{\omega,A}$. So we also obtain (ii).

(iii) is an immediate consequence of the inequality

$$\|x\| \leqslant \|B_{\gamma,n}\| \, \|x\|_{\gamma,n}, \quad x \in X_{\omega,A}, \; n \geqslant 2n_0+6,$$

and of the density of $\mathscr{R}_{\mathscr{E}}$ in X. ∎

We now define another class of "abstract ω-spaces", using the coefficients $a_k^{\omega,n}$ as in the definition of the space $\mathscr{E}_\omega$.

DEFINITION XX. Let $A \in (C_\omega^0)$. We define the *Fréchet space* $Y_{\omega,A}$ by endowing the linear subspace of X

$$\Big\{x \in \bigcap_{k=0}^{\infty} D(A^k);\ \sup_{k \geqslant 0} L^k a_k^{\omega,n} \|A^k x\| < +\infty \text{ for every } L > 0 \text{ and integer } n \geqslant 1\Big\}$$

with the semi-norms

$$x \mapsto \sup_{k \geqslant 0} L^k a_k^{\omega,n} \|A^k x\|, \quad L > 0, \; n \geqslant 1 \text{ integer}.$$

LEMMA 4.12. *For each $A \in (C_\omega^0)$ we have*

$$X_{\omega,A} \subset Y_{\omega,A},$$

where the inclusion is continuous.

Proof. Let $L > 0$ and integer $n \geqslant 1$ be arbitrary. We choose α, β, c_0, n_0 as in Definition XVII, $\gamma > \beta$ and integer $m \geqslant 2L^2 n + 2n_0 + 6$. By (4.15) there exists a $c_1 > 0$ such that

$$\|R(z; A)\| \leqslant c_1 |\omega(|z|)|^{n_0+1}, \quad z \in \Gamma_{\omega,\alpha,\beta}. \tag{4.19}$$

On the other hand, by (4.16) there exists a $c_2 > 0$ with

$$|\omega_\gamma(iz)| \geqslant c_2 |\omega(|z|)|^{1/2}, \quad z \in C \setminus \Lambda_{\omega,\alpha,\beta}; \tag{4.20}$$

so

$$|(L^k a_k^{\omega,n} z^k) \omega_\gamma^{-m}(iz)| \geqslant c_2^{-m} |\omega(|z|)|^{-n_0-3}, \quad k \geqslant 0, \; z \in C \setminus \Lambda_{\omega,\alpha,\beta}. \tag{4.21}$$

Let $x \in X_{\omega,A}$ and integer $k \geqslant 0$ be arbitrary. By Theorem 4.11(i) $A|X_{\omega,A} \in \mathscr{L}(X_{\omega,A})$ and it obviously commutes with $B_{\gamma,m}|X_{\omega,A} \in \mathscr{L}(X_{\omega,A})$; hence

$$\begin{aligned} L^k a_k^{\omega,n} A^k x &= L^k a_k^{\omega,n} B_{\gamma,m} A^k B_{\gamma,m}^{-1} x \\ &= \frac{1}{2\pi i} \int\limits_{\Gamma_{\omega,\alpha,\beta}} L^k a_k^{\omega,n} \omega_\gamma^{-m}(iz) R(z;A) A^k B_{\gamma,m}^{-1} x\, dz \\ &= \frac{1}{2\pi i} \int\limits_{\Gamma_{\omega,\alpha,\beta}} (L^k a_k^{\omega,n} z^k) \omega_\gamma^{-m}(iz) R(z;A) B_{\gamma,m}^{-1} x - \\ &\qquad - (L^k a_k^{\omega,n} z^k) \omega_\gamma^{-m}(iz) \sum_{j=0}^{k-1} z^{-j-1} A^j B_{\gamma,m}^{-1} x\, dz. \end{aligned}$$

Since $z \mapsto (L^k a_k^{\omega,n} z^k)\omega_\gamma^{-m}(iz) \sum_{j=0}^{k-1} z^{-j-1} A^j B_{\gamma,m}^{-1} x$ is analytical on a neighbourhood of $\overline{C \setminus \Lambda_{\omega,\alpha,\beta}}$ and by (4.20) it is $O(|z|^{-2})$ at ∞ on $\overline{C \setminus \Lambda_{\omega,\alpha,\beta}}$, using the Cauchy integral theorem we deduce

$$L^k a_k^{\omega,n} A^k x = \frac{1}{2\pi i} \int\limits_{\Gamma_{\omega,\alpha,\beta}} (L^k a_k^{\omega,n} z^k)\, \omega_\gamma^{-m}(iz) R(z;A) B_{\gamma,m}^{-1} x\, dz.$$

Now using (4.19) and (4.20), we can easily see that

$$L^k a_k^{\omega,n} \|A^k x\| \leqslant \frac{c_1 c_2^{-m}}{2\pi} \Big(\int\limits_{\Gamma_{\omega,\alpha,\beta}} |\omega(|z|)|^{-2} dz \Big) \|x\|_{\gamma,m}.$$

We conclude that, writing

$$c = \frac{c_1 c_2^{-m}}{2} \int\limits_{\Gamma_{\omega,\alpha,\beta}} |\omega(|z|)|^{-2} d|z|,$$

we have

$$\sup_{k \geqslant 0} L^k a_k^{\omega,n} \|A^k x\| \leqslant c\|x\|_{\gamma,m}, \quad x \in X_{\omega,A},$$

which proves our statement. ■

Assuming that ω satisfies the strong non-quasianalyticity condition, we can prove also the converse inclusion, extending [4], Lemma 4, and [14], Theorem 3.8:

THEOREM 4.13. *Let us assume that ω satisfies the strong non-quasianalyticity condition and let $A \in (C_\omega^0)$. Then the Fréchet spaces $X_{\omega,A}$ and $Y_{\omega,A}$ coincide.*

Proof. By the above lemma we need only to prove that $Y_{\omega,A} \subset X_{\omega,A}$ where the inclusion is continuous. We note that, by statement (iii) from Theorem 2.25, we have

$$Y_{\omega,A} = \{x \in \bigcap_{k=0}^{\infty} D(A^k);\ \sup_{k \geqslant 0} L^k c_k^{\omega,n} \|A^k x\| < +\infty \text{ for each } L > 0,\ n \geqslant 1\}$$

and the topology of $Y_{\omega,A}$ is given by the semi-norms

$$x \mapsto \sup_{k \geqslant 0} L^k c_k^{\omega,n} \|A^k x\|, \quad L > 0,\ n \geqslant 1 \text{ integer.}$$

Moreover, by statement (i) from Theorem 2.25, there exist a $c' > 0$ and an integer $n' > 1$ such that

$$\omega(-it) \leqslant c'|\omega(t)|^{n'}, \quad t \geqslant 0. \tag{4.22}$$

Let α, β, c_0, n_0 be as in Definition XVI, let $\gamma > \beta$ and let $n \geqslant 2n_0+6$ be integer. Writing

$$c_\gamma = \sup_{t \in R} \prod_{t_k < \gamma} \left| \frac{1+(it/\gamma)}{1+(it/t_k)} \right| < +\infty,$$

we have

$$|\omega_\gamma(t)| \leqslant c_\gamma |\omega(t)|, \quad t \in \boldsymbol{R}.$$

By Theorem 1.3, for all integers $k \geqslant 0$ we have

$$|(\omega_\gamma^n)^{(k)}(t)| \leqslant c_\gamma^n |(\omega^n)^{(k)}(t)|, \quad t \in \boldsymbol{R}.$$

In particular,

$$\frac{1}{k!} |(\omega^n)^{(k)}(0)| \leqslant c^n \frac{1}{k!} |(\omega^n)^{(k)}(0)|, \quad k \geqslant 0,$$

that is

$$c_k^{\omega_\gamma,n} \leqslant c^n c_k^{\omega,n}, \quad k \geqslant 0. \tag{4.23}$$

Then, using (4.22), we get

$$\sum_{k=0}^{\infty} c_k^{\omega_\gamma,n} t^k \leqslant (cc')^n |\omega(t)|^{nn'}, \quad t \geqslant 0$$

and this implies by (4.20)

$$\sum_{k=0}^{\infty} c_k^{\omega_\gamma,n} |z|^k |\omega_\gamma^{-n-2nn'}(iz)| \leqslant (cc')^n |\omega(|z|)|^{-n_0-3}, \quad z \in C \setminus \Lambda_{\omega,\alpha,\beta}. \tag{4.24}$$

Now let $x \in Y_{\omega,A}$ be arbitrary. Since by (4.23)

$$c_k^{\omega_\gamma,n} \|A^k x\| \leqslant 2^{-k} c^n \sup_{p \geqslant 0} 2^p c_p^{\omega,n} \|A^p x\|, \quad k \geqslant 0,$$

we can define the elements $y_1, y_2, \ldots, y_\infty \in X$ by

$$y_m = \sum_{k=0}^{\infty} (-1)^k c_k^{\omega_\gamma,n} A^k x, \quad m = 1, 2, \ldots, \infty;$$

then $y_m \to y_\infty$ in the norm topology of X. Moreover,

$$\|y_m\| \leqslant 2c^n \sup_{p \geqslant 0} 2^p c_p^{\omega,n} \|A^p x\|, \quad m = 1, 2, \ldots, \infty. \tag{4.25}$$

For every integer $1 \leqslant m < +\infty$ we have

$$B_{\gamma,n+2nn'}y_m = \frac{1}{2\pi i}\int_{\Gamma_{\omega,\alpha,\beta}}\left[\sum_{k=0}^{m}(-1)^k c_k^{\omega_\gamma,n}\omega_\gamma^{-n-nn'}(iz)R(z;A)A^k x\right]dz$$

$$= \frac{1}{2\pi i}\int_{\Gamma_{\omega,\alpha,\beta}}\left[\left(\sum_{k=0}^{m}(-1)^k c_k^{\omega_\gamma,n}z^k\right)\omega_\gamma^{-n-nn'}(iz)R(z;A)x - \right.$$

$$\left. - \sum_{k=0}^{m}(-1)^k c_k^{\omega_\gamma,n}z^k\omega_\gamma^{-n-nn'}(iz)\sum_{j=0}^{k-1}z^{-j-1}A^j x\right]dz.$$

Using (4.24) and the Cauchy integral theorem, we can easily verify that for every integer $1 \leqslant m < +\infty$

$$B_{\gamma,n+2nn'}y_m = \frac{1}{2\pi i}\int_{\Gamma_{\omega,\alpha,\beta}}\left(\sum_{k=0}^{m}(-1)^k c_k^{\omega_\gamma,n}z^k\right)\omega_\gamma^{-n-2nn'}(iz)R(z;A)x\,dz.$$

Since $y_m \to y_\infty$, by (4.19), (4.24) and the Lebesgue dominated convergence theorem, we deduce

$$B_{\gamma,n+2nn'}y_\infty = \frac{1}{2\pi i}\int_{\Gamma_{\omega,\alpha,\beta}}\omega_\gamma^n(iz)\omega_\gamma^{-n-2nn'}(iz)R(z;A)x\,dz = B_{\gamma,2nn'}x.$$

By Lemma 4.9 (v) and (iii), it follows that

$$B_{\gamma,n}y_\infty = x;$$

so, using (4.25), we conclude that

(4.26) $\quad x \in B_{\gamma,n}X \quad$ and $\quad \|B_{\gamma,n}^{-1}x\| = \|y_\infty\| \leqslant 2c^n \sup_{p\geqslant 0} 2^p c_p^{\omega,n}\|A^p x\|.$

Since $x \in Y_{\omega,A}$ and the integer $n \geqslant 2n_0+6$ are arbitrary, (4.26) proves that $Y_{\omega,A} \subset X_{\omega,A}$ and that this inclusion is continuous. ∎

Definition XXI. Let $-\infty \leqslant a < b \leqslant +\infty$ and let E be a locally convex topological vector space. Then we denote by $\mathscr{E}_\omega((a,b);E)$ the vector space of all infinitely strongly differentiable functions $\varphi\colon (a,b) \to E$ such that for any compact $K \subset (a,b)$, $L > 0$, integer $n \geqslant 1$ and continuous seminorm p on E

$$r_{L,n}^{\omega,K,p}(\varphi) = \sup_{k\geqslant 0}\left[L^k a_k^{\omega,n}\sup_{s\in K}p\big(\varphi^{(k)}(s)\big)\right] < +\infty.$$

We endow $\mathscr{E}_\omega((a,b);E)$ with the locally convex topology defined by the seminorms $r_{L,n}^{\omega,K,p}$, compact $K \subset (a,b)$, $L > 0$, integer $n \geqslant 1$, p continuous seminorm on E.

We next recall the following terminology from [36]:

Definition XXII. Let E be a locally convex topological vector space.

We say that a family $\{U_t\}_{t\geqslant 0} \subset \mathscr{L}(E)$ is a *semi-group of class* (C_0) if

$$U_0 = I, \quad U_{t+s} = U_t U_s, \; t,s \geqslant 0,$$

and for every $x \in E$, the mapping $[0,+\infty) \ni t \mapsto U_t x \in E$ is continuous.

A semi-group $\{U_t\}_{t\geqslant 0} \subset \mathscr{L}(E)$ is called *locally equi-continuous* if for every compact $K \subset [0,+\infty)$, the family $\{U_t\}_{t\in K}$ is equi-continuous.

We can now give our main result concerning the abstract Cauchy problem:

Theorem 4.14. *Let $A \in (C_\omega^0)$. Then*

(i) *for each $x \in X_{\omega,A}$ the* (ACP) *has a unique solution*

$$[0,+\infty) \ni t \mapsto U_t^{\omega,A}x \in X$$

for (A,x);

(ii) *the linear operators $U_t^{\omega,A}$, $t \geqslant 0$, belong to $\mathscr{L}(X_{\omega,A})$ and commute with $A|X_{\omega,A} \in \mathscr{L}(X_{\omega,A})$;*

(iii) *$\{U_t^{\omega,A}\}_{t\geqslant 0}$ is a locally equi-continuous semi-group of class (C_0) of operators from $\mathscr{L}(X_{\omega,A})$ and its infinitesimal generator is $A|X_{\omega,A} \in \mathscr{L}(X_{\omega,A})$;*

(iv) *for each $x \in X_{\omega,A}$ the function*

$$(0,+\infty) \ni t \mapsto U_t^{\omega,A}x \in X_{\omega,A}$$

belongs to $\mathscr{E}_\omega((0,+\infty);X_{\omega,A})$ and it depends continuously on x;

(v) *denoting by $\mathscr{E}$ the ω-ultradistribution semi-group generated by A, we have*

$$\mathscr{E}(\varphi)x = \int_0^{+\infty}\varphi(t)U_t^{\omega,A}x\,dt, \quad \varphi \in \mathscr{D}_\omega^0,\; x \in X_{\omega,A}.$$

Proof. Let α, β, c_0, n_0 be as in Definition XVI and $\gamma > \beta$. Let $t \geqslant 0$ and let $n \geqslant 2\alpha t + 2n_0 + 6$ be integer. By (4.19) and (4.20) we have for all $z \in \Gamma_{\omega,\alpha,\beta}$

$$\|e^{tz}\omega_\gamma^{-n}(iz)R(z;A)\| \leqslant c_1 c_2^{-n}e^{\beta t}|\omega(|z|)|^{\alpha t - n/2 + n_0 + 1} \leqslant c_1 c_2^{-n}|\omega(|z|)|^{-2}e^{\beta t};$$

so the integral in the formula

$$E_{t,n} = \frac{1}{2\pi i}\int_{\Gamma_{\omega,\alpha,\beta}}e^{tz}\omega_\gamma^{-n}(iz)R(z;A)dz$$

converges in the norm topology of $\mathscr{L}(X)$ and

(4.27) $$E_{t,n} \leqslant \frac{c_1 c_2^{-n}}{2\pi}\left(\int_{\Gamma_{\omega,\alpha,\beta}}|\omega(|z|)|^{-2}dz\right)e^{\beta t}.$$

It is clear that $E_{t,n}$ commutes with all operators $B_{\gamma,m}$, $m \geqslant 2n_0+6$, so that

(4.28) $$E_{t,n}X_{\omega,A} \subset X_{\omega,A}, \quad E_{t,n}|X_{\omega,A} \in \mathscr{L}(X_{\omega,A}).$$

On the other hand, using Lemma 4.9 (ii), (4.5) and a reasoning similar to that used in the proof of Proposition 4.5, we get for every $\varphi \in \mathscr{D}^0_\omega$

$$E_{t,n}B^{-1}_{\gamma,n}\mathscr{E}(\varphi) = E_{t,n}\mathscr{E}(\omega^n_\gamma(-iD)\varphi)$$
$$= \frac{1}{2\pi i}\int\limits_{\Gamma_{\omega,\alpha,\beta}} e^{tz}\omega_\gamma^{-n}(iz)R(z;A)\,dz\cdot\frac{1}{2\pi i}\int\limits_{\Gamma_{\omega,\alpha,\beta}}\omega^n_\gamma(iz)\tilde{\varphi}(z)R(z;A)dz$$
$$= \frac{1}{2\pi i}\int\limits_{\Gamma_{\omega,\alpha,\beta}} e^{tz}\tilde{\varphi}(z)R(z;A)\,dz = \mathscr{E}(\varphi(\cdot-t)),$$

that is, with the usual notation $\tau_t\varphi = \varphi(\cdot-t)$,

$$E_{t,n}B^{-1}_{\gamma,n}\mathscr{E}(\varphi) = \mathscr{E}(\tau_t\varphi),\quad \varphi\in\mathscr{D}^0_\omega. \tag{4.29}$$

By (4.28), (4.29) and Theorem 4.11 (ii), $U_t^{\omega,A} = E_{t,n}B^{-1}_{\gamma,n}|X_{\omega,A}$ belongs to $\mathscr{L}(X_{\omega,A})$ and is completely defined by

$$U_t^{\omega,A}\mathscr{E}(\varphi)x = \mathscr{E}(\tau_t\varphi)x,\quad \varphi\in\mathscr{D}^0_\omega,\ x\in X, \tag{4.30}$$

so that it does not depend on n. Moreover, by Theorem 4.4 (iii)

$$\mathscr{E}(-\varphi') = A\mathscr{E}(\varphi),\quad \varphi\in\mathscr{D}^0_\omega;$$

thus (4.30) implies for every $\varphi\in\mathscr{D}^0_\omega$ and $x\in X$

$$AU_t^{\omega,A}\mathscr{E}(\varphi)x = A\mathscr{E}(\tau_t\varphi)x = \mathscr{E}(-\tau_t\varphi')x = U_t^{\omega,A}\mathscr{E}(-\varphi')x = U_t^{\omega,A}A\mathscr{E}(\varphi)x.$$

By Theorem 4.11 (ii) it follows that $U_t^{\omega,A}$ commutes with $A|X_{\omega,A}\in\mathscr{L}(X_{\omega,A})$.

Again by (4.30) and Theorem 4.11 (ii), the semi-group property of the family $\{U_t^{\omega,A}\}_{t\geqslant 0}$ follows immediately.

An easy computation shows us that, choosing for example $n\geqslant 2\alpha+2n_0+6$,

$$\lim_{0\leqslant t\to 0}\|E_{t,n}-B_{\gamma,n}\| = 0.$$

Hence, for every $x\in X_{\omega,A}$ and integer $m\geqslant 2n_0+6$ we have

$$\overline{\lim_{0\leqslant t\to 0}}\|U_t^{\omega,A}x-x\|_{\gamma,m} = \overline{\lim_{0\leqslant t\to 0}}\|B^{-1}_{\gamma,m}(E_{t,n}-B_{\gamma,n})B^{-1}_{\gamma,n}x\|$$
$$\leqslant \overline{\lim_{0\leqslant t\to 0}}\|E_{t,n}-B_{\gamma,n}\|\,\|B^{-1}_{\gamma,m}B^{-1}_{\gamma,n}x\| = 0;$$

so the semi-group $\{U_t^{\omega,A}\}_{t\geqslant 0}$ is of class (C_0).

Next let $K\subset[0,+\infty)$ be compact. We choose $t_0>0$ such that $K\subset[0,t_0]$ and $n\geqslant 2\alpha t_0+2n_0+6$. Then, using (4.27) and Lemma 4.9 (v), for every $t\in[0,t_0]$, integer $m\geqslant 2n_0+6$ and $x\in X_{\omega,A}$, we get

$$\|U_t^{\omega,A}x\|_{\gamma,m} = \|B^{-1}_{\gamma,m}E_{t,n}B^{-1}_{\gamma,n}x\|$$
$$\leqslant \frac{c_1c_2^{-n}}{2\pi}\Bigg(\int\limits_{\Gamma_{\omega,\alpha,\beta}}|\omega(|z|)|^{-2}d|z|\Bigg)e^{\beta t_0}\|x\|_{\gamma,n+m},$$

so that the semi-group $\{U_t^{\omega,A}\}_{t\geqslant 0}$ is locally equi-continuous.

Further, let $t\geqslant 0$ and choose $t_0>t$ and an integer $n\geqslant 2\alpha t_0+2n_0+8$. Using the Cauchy integral theorem and the closedness of A, we can easily verify that

$$E_{t,n}X\subset D(A)\quad\text{and}\quad AE_{t,n} = \frac{1}{2\pi i}\int\limits_{\Gamma_{\omega,\alpha,\beta}} ze^{tz}\omega_\gamma^{-n}(iz)R(z;A)dz.$$

Since, for every $0<s<t_0$, $s\neq t$, we have

$$\left|\frac{1}{t-s}(e^{tz}-e^{sz})-ze^{tz}\right|\leqslant |t-s|\,|z^2e^{t_0z}|,\qquad \operatorname{Re}z\geqslant 0,$$

writing

$$c_3 = \frac{1}{2\pi}\int\limits_{\Gamma_{\omega,\alpha,\beta}}|z^2e^{t_0z}\omega_\gamma^{-n}(iz)|\,\|R(z;A)\|\,|d|z|| < +\infty,$$

we infer that

$$\left\|\frac{1}{t-s}(E_{t,n}-E_{s,n})-AE_{t,n}\right\|\leqslant c_3|s-t|.$$

Consequently, for each $x\in X_{\omega,A}$ and integer $m\geqslant 2n_0+6$, we have

$$\left\|\frac{1}{t-s}(U_t^{\omega,A}x-U_s^{\omega,A}x)-AU_t^{\omega,A}x\right\|_{\gamma,m}\leqslant c_3|s-t|\,\|x\|_{\gamma,n+m},\qquad 0<s<t_0,\ s\neq t;$$

hence

$$\lim_{0<s\to t}\left\|\frac{1}{t-s}(U_t^{\omega,A}x-U_s^{\omega,A}x)-AU_t^{\omega,A}x\right\|_{\gamma,m} = 0. \tag{4.31}$$

By the facts proved above concerning the operators $\{U_t^{\omega,A}\}_{t\geqslant 0}$ and by Lemma 4.3, we conclude that (i), (ii), (iii) of the theorem hold.

In order to prove (iv), let $x\in X_{\omega,A}$. By (4.31) the function

$$(0,+\infty)\ni t\mapsto U_t^{\omega,A}x\in X_{\omega,A}$$

is strongly differentiable and

$$\frac{d}{dt}U_t^{\omega,A}x = AU_t^{\omega,A}x = U_t^{\omega,A}Ax.$$

By induction, we infer that the above function is infinitely strongly differentiable and

$$\frac{d^k}{dt}U_t^{\omega,A}x = A^kU_t^{\omega,A}x = U_t^{\omega,A}A^kx,\qquad k\geqslant 0. \tag{4.32}$$

Further, let $K\subset(0,+\infty)$ be compact, let $L>0$ and let $n\geqslant 1$, $m\geqslant 2n_0+6$ be integers. By the local equi-continuity of $\{U_t^{\omega,A}\}_{t\geqslant 0}$ proved above, there exist an integer $m'\geqslant 2n_0+6$ and a $c_4>0$ such that

$$\|U_t^{\omega,A}y\|_{\gamma,m}\leqslant c_4\|y\|_{\gamma,m'},\qquad t\in K,\ y\in X_{\omega,A},$$

and by Lemma 4.12 there exist an integer $m''\geqslant 2n_0+6$ and a $c_5>0$ with

$$\sup_{k\geqslant 0}L^ka_k^{\omega,n}\|A^ky\|\leqslant c_5\|y\|_{\gamma,m''},\qquad y\in X_{\omega,A}.$$

Hence, using (4.32), we deduce

$$\begin{aligned} r_{L,n}^{\omega,K\|\cdot\|_{\gamma,m}}(U_t^{\omega,A}x) &= \sup_{k\geqslant 0}\,[L^k a_k^{\omega,n}\sup_{t\in K}\|U_t^{\omega,A}A^k x\|_{\gamma,m}] \\ &\leqslant c_4 \sup_{k\geqslant 0} L^k a_k^{\omega,n}\|A^k x\|_{\gamma,m'} \\ &= c_4 \sup_{k\geqslant 0} L^k a_k^{\omega,n}\|A^k B_{\gamma,m'}^{-1}x\| \\ &\leqslant c_4 c_5 \|B_{\gamma,m'}^{-1}x\|_{\gamma,m''} = c_4 c_5\|x\|_{\gamma,m'+m''}, \end{aligned}$$

which proves (iv).

Finally, let $\varphi\in\mathscr{D}_\omega^0$ and $x\in X_{\omega,A}$. Let us take $t_0>0$ such that $\operatorname{supp}\varphi\subset[0,t_0]$ and $n\geqslant 2\alpha t_0+2n_0+6$ is an integer. Using arguments similar to those used in the proof of Proposition 4.5, we get

$$\begin{aligned} \mathscr{E}(\varphi)B_{\gamma,n} &= \frac{1}{2\pi i}\int_{\Gamma_{\omega,\alpha,\beta}}\tilde{\varphi}(z)R(z;A)\,dz\cdot\frac{1}{2\pi i}\int_{\Gamma_{\omega,\alpha,\beta}}\omega_\gamma^{-n}(iz)R(z;A)\,dz \\ &= \frac{1}{2\pi i}\int_{\Gamma_{\omega,\alpha,\beta}}\tilde{\varphi}(z)\omega_\gamma^{-n}(iz)R(z;A)\,dz \\ &= \int_0^{t_0}\varphi(t)\left[\frac{1}{2\pi i}\int_{\Gamma_{\omega,\alpha,\beta}}e^{tz}\omega_\gamma^{-n}(iz)R(z;A)\,dz\right]dt \\ &= \int_0^{t_0}\varphi(t)E_{t,n}\,dt, \end{aligned}$$

so that

$$\mathscr{E}(\varphi)x = \mathscr{E}(\varphi)B_{\gamma,n}B_{\gamma,n}^{-1}x = \int_0^{t_0}\varphi(t)E_{t,n}B_{\gamma,n}^{-1}x\,dt = \int_0^{+\infty}\varphi(t)U_t^{\omega,A}x\,dt.$$

Hence (v) is also proved. ∎

Assuming that ω satisfies the strong non quasi-analyticity condition, by Theorem 4.13 we can replace the space $X_{\omega,A}$ in Theorem 4.14 by the space $Y_{\omega,A}$; we thus obtain an extension of some results of T. Ushijima from [64] (see also [26], Theorem 3.1), [4], Theorem 1 and [14], Theorem 4.1.

We finally remark that using Theorem 1.6, Proposition 4.6 and Theorem 4.14, we can easily obtain the following extension of Corollary 4.7:

COROLLARY 4.15. *Let A be a densely defined closed linear operator in X such that for some increasing function $f\colon[0,+\infty)\to[1,+\infty)$ with $\int_1^{+\infty}t^{-2}\ln f(t)\,dt<+\infty$, $c_0>0$ and an integer $n_0\geqslant 1$, $R(z;A)\in\mathscr{L}(X)$ exists for z in the region*

$$\Lambda_f = \{z\in\boldsymbol{C};\ \operatorname{Re}z\geqslant\ln f(|\operatorname{Im}z|)\}$$

and satisfies the estimation

$$\|R(z;A)\|\leqslant c_0(1+|z|)^{n_0},\quad z\in\Lambda_f.$$

Then there exists a Fréchet space $\mathscr{X}$, continuously and densely imbedded in X, and a semi-group $\{U_t\}_{t\geqslant 0}$ of class (C_0) of continuous linear operator on $\mathscr{X}$ such that

$$\mathscr{X}\subset D(A),\quad A\mathscr{X}\subset\mathscr{X},\quad A|\mathscr{X}\subset\mathscr{L}(\mathscr{X})$$

and $\{U_t\}_{t\geqslant 0}$ is generated by $A|\mathscr{X}$. Moreover, for each $x\in\mathscr{X}$

$$[0,+\infty)\ni t\mapsto U_t x\in\mathscr{X}\subset X$$

is the unique solution of the (ACP) *for* (A,x).

5. ω-ultradistributions as hyperfunctions

The purpose of this section is to represent the X-valued ω-ultradistributions as boundary values of X-valued analytic functions on $\boldsymbol{C}\setminus\boldsymbol{R}$, where X is an arbitrary Banach space.

In the whole of this section $t_1,t_2,\ldots>0$, $t_1<+\infty$, $\sum_{k=1}^{\infty}1/t_k<+\infty$ will be fixed and $\omega_{\{t_k\}}$ will be denoted simply by ω. We fix also a Banach space X.

For each integer $n\geqslant 1$ we shall denote by ω_n^* the *Borel transform of* $|\omega(\cdot)^n|$, that is

$$\omega_n^*(t) = t\int_0^{+\infty}|\omega(ts)^n|e^{-s}\,ds = \int_0^{+\infty}|\omega(s)^n|e^{-s/t}\,ds,\quad t>0.$$

It is easy to verify that

$$\sup_{k\geqslant 0}k!\,a_k^{\omega,n}t^{k+1}\leqslant\omega_n^*(t)\leqslant 3\sup_{k\geqslant 0}k!\,a_k^{\omega,n}(3t)^{k+1},\quad t>0. \tag{5.1}$$

We first give the following representation of ω-ultradistributions with compact support:

PROPOSITION 5.1. *Let $F\in\mathscr{L}(\mathscr{E}_\omega;X)$, $\operatorname{supp}F\subset(a,b)$, $-\infty<a<b<+\infty$ and define the Cauchy transform of F by*

$$\Phi(z) = \frac{1}{2\pi i}F\left(\frac{1}{\cdot - z}\right),\quad z\in\boldsymbol{C}\setminus\operatorname{supp}F.$$

Then the X-valued function Φ is analytical on $\boldsymbol{C}\setminus\operatorname{supp}F$ and there exist $L>0$, integer $n\geqslant 1$ and $c>0$ such that

$$\|\Phi(z)\|\leqslant c\,\omega_n^*\left(\frac{L}{\operatorname{dist}(z,[a,b])}\right),\quad z\in\boldsymbol{C}\setminus[a,b].$$

Moreover, the following convergence holds in $\mathscr{L}(\mathscr{D}_\omega;X)$:

$$\lim_{0<\varepsilon\to 0}\big(\Phi(\cdot+i\varepsilon)-\Phi(\cdot-i\varepsilon)\big) = F.$$

Proof. It is easy to verify that Φ is analytical. By the continuity of F there exist $L>0$, integer $n\geqslant 1$ and $d>0$ such that

$$\|F(\varphi)\|\leqslant d r_{L,n}^{\omega;[a,b]}(\varphi),\qquad \varphi\in\mathscr{E}_\omega.$$

Hence, for each $z\in C\diagdown[a,b]$,

$$\|\Phi(z)\|\leqslant\frac{d}{2\pi}r_{L,n}^{\omega;[a,b]}\left(\frac{1}{\cdot-z}\right)$$

$$=\frac{d}{2\pi}\sup_{k\geqslant 0}\left(L^k a_k^{\omega,n}\sup_{s\in[a,b]}\frac{k!}{|s-z|^{k+1}}\right)$$

$$\leqslant\frac{d}{2\pi L}\,\omega_n^*\left(\frac{L}{\operatorname{dist}(z,[a,b])}\right).$$

Now let $\varphi\in\mathscr{D}_\omega$ and $\varepsilon>0$. Then for each integer $k\geqslant 0$ and $s\in R$ we have

$$\left|\left[\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\varphi(t)\left(\frac{1}{s-(t+i\varepsilon)}-\frac{1}{s-(t-i\varepsilon)}\right)dt\right]^{(k)}-\varphi^{(k)}(s)\right|$$

$$=\left|\left[\frac{1}{\pi}\int_{-\infty}^{+\infty}\varphi(t)\frac{\varepsilon}{(s-t)^2+\varepsilon^2}\,dt\right]^{(k)}-\varphi^{(k)}(s)\right|$$

$$=\left|\left[\frac{1}{\pi}\int_{-\infty}^{+\infty}\varphi(s+t)\frac{\varepsilon}{t^2+\varepsilon^2}\,dt\right]^{(k)}-\varphi^{(k)}(s)\right|$$

$$=\left|\frac{1}{\pi}\int_{-\infty}^{+\infty}[\varphi^{(k)}(s+t)-\varphi^{(k)}(s)]\frac{\varepsilon}{t^2+\varepsilon^2}\,dt\right|$$

$$\leqslant\frac{1}{\pi}\int_{-\sqrt{\varepsilon}}^{+\sqrt{\varepsilon}}|\varphi^{(k)}(s+t)-\varphi^{(k)}(s)|\frac{\varepsilon}{t^2+\varepsilon^2}\,dt+$$

$$+\frac{1}{\pi}\int_{|t|>\sqrt{\varepsilon}}|\varphi^{(k)}(s+t)-\varphi^{(k)}(s)|\frac{\varepsilon}{t^2+\varepsilon^2}\,dt$$

$$\leqslant\frac{\sqrt{\varepsilon}}{\pi}\sup_{t\in R}|\varphi^{(k+1)}(t)|+\frac{4}{\pi}\sup_{t\in R}|\varphi^{(k)}(t)|\left(\frac{\pi}{2}-\arctan\frac{1}{\sqrt{\varepsilon}}\right);$$

thus, using Proposition 2.7, we get

$$\left\|\int_{-\infty}^{+\infty}\varphi(t)\big(\Phi(t+i\varepsilon)-\Phi(t-i\varepsilon)\big)dt-F(\varphi)\right\|$$

$$=\left\|F\left[\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\varphi(t)\left(\frac{1}{\cdot-(t+i\varepsilon)}-\frac{1}{\cdot-(t-i\varepsilon)}\right)dt-\varphi\right]\right\|$$

$$\leqslant d r_{L,n}^{\omega;[a,b]}\left(\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\varphi(t)\left(\frac{1}{\cdot-(t+i\varepsilon)}-\frac{1}{\cdot-(t-i\varepsilon)}\right)dt-\varphi\right)$$

$$\leqslant d\left(\frac{\sqrt{\varepsilon}}{\pi}r_{L,n}^{\omega}(\varphi')+\frac{4}{\pi}\left(\frac{\pi}{2}-\arctan\frac{1}{\sqrt{\varepsilon}}\right)r_{L,n}^{\omega}(\varphi)\right)$$

$$\leqslant d\left(\frac{\sqrt{\varepsilon}}{\pi}q_{L,n}^{\omega}(\varphi')+\frac{4}{\pi}\left(\frac{\pi}{2}-\arctan\frac{1}{\sqrt{\varepsilon}}\right)q_{L,n}^{\omega}(\varphi)\right)$$

$$\leqslant d\left(\frac{\sqrt{\varepsilon}}{\pi}\frac{t_1}{L}q_{L,n+1}^{\omega}(\varphi)+\frac{4}{\pi}\left(\frac{\pi}{2}-\arctan\frac{1}{\sqrt{\varepsilon}}\right)q_{L,n+1}^{\omega}(\varphi)\right)$$

$$=d\left(\frac{\sqrt{\varepsilon}t_1}{L}+4\left(\frac{\pi}{2}-\arctan\frac{1}{\sqrt{\varepsilon}}\right)\right)q_{L,n+1}^{\omega}(\varphi).$$

Consequently $\lim\limits_{\varepsilon\to+\infty}\big(\Phi(\cdot+i\varepsilon)-\Phi(\cdot-i\varepsilon)\big)=F$ in $\mathscr{L}(\mathscr{D}_\omega;X)$. ∎

We remark that as $\lim\limits_{t\to+\infty}\omega_n^*(L/t)=0$ for each $L>0$ and integer $n\geqslant 1$, the Cauchy transform of any $F\in\mathscr{L}(\mathscr{E}_\omega;X)$ vanishes at ∞.

Definition XXIII. We denote by $\mathscr{H}_\omega(X)$ the vector space of all X-valued analytic functions Φ, defined on $C\diagdown R$, with the property that for every compact $K\subset R$ there exist δ, L, $c>0$ and integer $n\geqslant 1$ (depending on Φ and K) such that

$$\|\Phi(z)\|\leqslant c\,\omega_n^*\left(\frac{L}{|\operatorname{Im}z|}\right),\qquad z\in C,\ \operatorname{Re}z\in K,\ 0\neq|\operatorname{Im}z|<\delta.$$

General ω-ultradistributions can also be represented as boundary values of functions from $\mathscr{H}_\omega(X)$, but not in a canonical way:

Theorem 5.2. *Let $F\in\mathscr{L}(\mathscr{D}_\omega;X)$. Then there exists an X-valued analytic function Φ on $C\diagdown\operatorname{supp}F$ such that $\Phi|_{C\diagdown R}\in\mathscr{H}_\omega(X)$ and the following convergence holds in $\mathscr{L}(\mathscr{D}_\omega;X)$:*

$$\lim_{0<\varepsilon\to 0}\big(\Phi(\cdot+i\varepsilon)-\Phi(\cdot-i\varepsilon)\big)=F.$$

Proof. By Theorem 2.5, for each integer $-\infty<m<+\infty$ we can take a function $\varphi_m\in\mathscr{D}_\omega[m-\frac{2}{3},m+\frac{2}{3}]$ such that $\sum\limits_m\varphi_m=1$. Then putting $F_m=\varphi_m F$, we obtain $F_m\in\mathscr{E}'_\omega$ and

$$\operatorname{supp}F_m\subset\operatorname{supp}\varphi_m\cap\operatorname{supp}F\subset[m-\tfrac{2}{3},m+\tfrac{2}{3}]\cap\operatorname{supp}F.$$

By Proposition 5.1 for each integer m, there exists an X-valued analytic function Φ_m on $C\diagdown\operatorname{supp}F_m\supset(C\diagdown[m-\frac{2}{3},m+\frac{2}{3}])\cup(C\diagdown\operatorname{supp}F)$, such that $\Phi_m\in\mathscr{H}_\omega(X)$ and

$$\lim_{0<\varepsilon\to 0}\big(\Phi_m(\cdot+i\varepsilon)-\Phi_m(\cdot-i\varepsilon)\big)=F_m\quad\text{ in }\mathscr{L}(\mathscr{D}_\omega;X).$$

Let m be an integer with $|m|\geqslant 2$; then Φ_m is analytical on $\{z\in C;\ |z|<|m|-\frac{2}{3}\}$ and we have the expansion

$$\Phi_m(z)=\sum_{k=0}^{\infty}c_k^m z^k,\qquad z\in C,\ |z|\leqslant|m|-\tfrac{2}{3}.$$

Let us choose the integer k_m such that

$$\Big\|\Phi_m(z)-\sum_{k=0}^{\infty} c_k^m z^k\Big\| \leqslant 2^{-|m|}, \qquad |z| \leqslant |m|-1$$

and denote by P_m the polynomial defined by $P_m(z) = \sum_{k=0}^{k_m} c_k^m z^k$. We can now define an X-valued analytic function Φ on $C\setminus \operatorname{supp} F$ by

$$\Phi = \sum_{|m|\leqslant 1} \Phi_m + \sum_{|m|\geqslant 2} (\Phi_m - P_m).$$

If K is compact and $\delta > 0$, then if we take an integer $m_0 \geqslant 1 + \sup_{\substack{\operatorname{Re} z\in K\\ |\operatorname{Im} z|\leqslant\delta}} |z|$, on the set

$$\{z\in C;\ \operatorname{Re} z\in K,\ |\operatorname{Im} z|\leqslant \delta\}\setminus \operatorname{supp} F,$$

we have the following expansion

$$\Phi = \sum_{|m|<m_0} \Phi_m - \sum_{2\leqslant |m|<m_0} P_m + \sum_{|m|\geqslant m_0} (\Phi_m - P_m),$$

where the last two terms are analytical on $\{z\in C;\ |z| < m_0 - \frac{2}{3}\}$.

Hence

$$\lim_{0<\varepsilon\to 0} \big(\Phi(\cdot + i\varepsilon) - \Phi(\cdot - i\varepsilon)\big) = F \quad \text{in } \mathscr{L}(\mathscr{D}_\omega; X). \ ■$$

We remark that in proving the above result we adapted the idea of H. G. Tillmann from [60], [61] (see also [10], Ch. 5).

Next we shall prove that, under some additional assumptions concerning ω, any function from $\mathscr{H}_\omega(X)$ has boundary value in $\mathscr{L}(\mathscr{D}_\omega; X)$.

DEFINITION XXIV. Let $n\geqslant 1$ be an integer; for each integer $k\geqslant 0$ we define

$$b_k^{\omega,n} = \inf_{t>0} \frac{\sup_{m\geqslant 0} m!\, a_m^{\omega,n} t^n}{k!\, t^k}.$$

It is clear that

$$a_0^{\omega,n} = b_0^{\omega,n} = 1$$

and

$$a_k^{\omega,n} \leqslant b_k^{\omega,n} < +\infty, \qquad k\geqslant 0;$$

hence

$$\sup_{k\geqslant 0} k!\, a_k^{\omega,n} t^k \leqslant \sup_{k\geqslant 0} k!\, b_k^{\omega,n} t^k, \qquad t>0.$$

Conversely, for every $t>0$ we have

$$\sup_{k\geqslant 0} k!\, b_k^{\omega,n} t^k \leqslant \sup_{k\geqslant 0}\left(\frac{\sup_{m\geqslant 0} m!\, a_m^{\omega,n} t^m}{k!\, t^k}\right) t^k = \sup_{m\geqslant 0} m!\, a_m^{\omega,n} t^m.$$

Hence

$$\sup_{k\geqslant 0} k!\, a_k^{\omega,n} t^k = \sup_{k\geqslant 0} k!\, b_k^{\omega,n} t^k, \qquad t>0.$$

By (5.1) it follows that

$$\sup_{k\geqslant 0} k!\, b_k^{\omega,n} t^{k+1} \leqslant \omega_n^*(t) \leqslant 3\sup_{k\geqslant 0} k!\, b_k^{\omega,n} (3t)^{k+1}, \qquad t>0. \tag{5.2}$$

The consideration of the numbers $b_k^{\omega,n}$ is motivated by the following regularity property (see [45], p. 17): for every integer $k\geqslant 1$ we have

$$(k!\, b_k^{\omega,n})^2 = \inf_{t>0} \frac{(\sup_{m\geqslant 0} m!\, a_m^{\omega,n} t^m)^2}{t^{2k}} \geqslant \inf_{t>0} \frac{\sup_{m\geqslant 0} m!\, a_m^{\omega,n} t^m}{t^{k-1}} \cdot \inf_{s>0} \frac{\sup_{m\geqslant 0} m!\, a_m^{\omega,n} s^m}{s^{k+1}}$$
$$= [(k-1)!\, b_{k-1}^{\omega,n}][(k+1)!\, b_{k+1}^{\omega,n}].$$

Hence the sequence

$$\left\{k\frac{b_k^{\omega,n}}{b_{k-1}^{\omega,n}}\right\}_{k\geqslant 1}$$

is decreasing. Moreover, it converges to 0. Indeed, if we suppose that

$$\lim_{k\to\infty} k\frac{b_k^{\omega,n}}{b_{k-1}^{\omega,n}} = \varepsilon_0 > 0,$$

then we successively get

$$\lim_{k\to\infty} (k!\, b_k^{\omega,n})^{1/k} = \varepsilon_0, \qquad \lim_{k\to\infty} k!\, b_k^{\omega,n} (2/\varepsilon_0)^k = +\infty,$$

in contradiction with the fact that

$$\sup_{k\geqslant 0} k!\, b_k^{\omega,n}\left(\frac{2}{\varepsilon_0}\right)^k = \sup_{k>0} k!\, a_k^{\omega,n}\left(\frac{2}{\varepsilon_0}\right)^k \leqslant \frac{\varepsilon_0}{2}\, \omega_n^*\left(\frac{2}{\varepsilon_0}\right) < +\infty.$$

We shall examine further the possible relations between $a_k^{\omega,n}$ and $b_k^{\omega,n}$.

LEMMA 5.3. *The following statements concerning ω are equivalent*:

(i) *there exist* $0 < r_1 \leqslant r_2/2 \leqslant r_3/3 \leqslant \ldots, r_1 < +\infty$, $\sum_{k=1}^{\infty} r_k < +\infty$ *such that* $\mathscr{D}_\omega$ *coincides with* $\mathscr{D}_{\omega\{r_k\}}$ *as topological vector spaces*;

(ii) *there exist* $L>0$, *integer* $m\geqslant 1$ *and* $c>0$ *such that*

$$b_k^{\omega,1} \leqslant cL^k a_k^{\omega,m}, \qquad k\geqslant 0;$$

(iii) *there exist* $L>0$, *integer* $m\geqslant 1$ *and* $c>0$ *such that*

$$b_k^{\omega,n} \leqslant c(Ln)^k a_k^{\omega,mn}, \qquad n\geqslant 1,\ k\geqslant 0.$$

Proof. It is clear that (iii) ⇒ (ii).

(ii) ⇒ (i). As $1 = b_0^{\omega,1} \leqslant cL^0 a_0^{\omega,n} = c$, it follows that $c\geqslant 1$. Let us put

$$r_k = \frac{b_{k-1}^{\omega,1}}{b_k^{\omega,1}}, \qquad k\geqslant 1.$$

As the sequence $\left\{k\dfrac{b_k^{\omega,n}}{b_{k-1}^{\omega,n}}\right\}_{k\geqslant 1}$ is decreasing, we have $0 < r_1 \leqslant r_2/2 \leqslant r_3/3 \leqslant \ldots \leqslant \ldots, r_1 < +\infty$. Using (ii), the inequality of T. Carleman (see [45], Lemma 1.8.VI, or [48], Ch. XVI, § 4, 5, 6) and Corollary 2.9, we get

$$\sum_{k=1}^{\infty}\frac{1}{r_k} \leqslant \sum_{k=1}^{\infty}\left(\frac{1}{r_1\ldots r_k}\right)^{1/k} = \sum_{k=1}^{\infty}(b_k^{\omega,1})^{1/k} \leqslant cL\sum_{k=1}^{\infty}(a_k^{\omega,m})^{1/k}$$

$$= cL\sum_{k=1}^{\infty}\left(\frac{a_1^{\omega,m}}{a_0^{\omega,m}}\cdots\frac{a_k^{\omega,m}}{a_{k-1}^{\omega,m}}\right)^{1/k} \leqslant cLe\sum_{k=1}^{\infty}\frac{a_k^{\omega,m}}{a_{k-1}^{\omega,m}} < +\infty.$$

For each $t\in \boldsymbol{R}$,

$$|\omega(t)| \leqslant \sqrt{2}\sup_{k\geqslant 0} a_k^{\omega,1}\left|\sqrt{2}t\right|^k \leqslant \sqrt{2}\sup_{k\geqslant 0} b_k^{\omega,1}\left|\sqrt{2}t\right|^k \leqslant \sqrt{2}\left|\omega_{\{r_k\}_{k\geqslant 1}}(\sqrt{2}t)\right|.$$

On the other hand, by Lemma 1.7, for every $t\in\boldsymbol{R}$ we have

$$|\omega_{\{r_k\}}(t)| \leqslant 3\max\{1, (\sup_{k\geqslant 1} b_k^{\omega,1}|4t|)^2\} \leqslant 3\max\{1, c^2(\sup_{k\geqslant 1} a_k^{\omega,m}|4Lt|^k)^2\} \leqslant 3c^2|\omega(4Lt)^{2m}|.$$

By Theorem 3.20, it follows that $\mathscr{D}_\omega$ and $\mathscr{D}_{\omega_{\{r_k\}}}$ coincide as topological vector spaces.

(i) $\Rightarrow$ (iii). By Theorem 3.20 there exist $L', L'' > 0$, integers $n', n'' \geqslant 1$ and $c', c'' > 0$, such that

$$|\omega(t)| \leqslant c'|\omega_{\{r_k\}}(L't)^{n'}|, \quad t\in\boldsymbol{R},$$
$$|\omega_{\{r_k\}}(t)| \leqslant c''|\omega(L''t)^{n''}|, \quad t\in\boldsymbol{R}.$$

Let $n\geqslant 1$ be a fixed integer and consider the sequence $0 < s_1 \leqslant s_2/2 \leqslant s_3/3 \leqslant \ldots \leqslant \ldots, \sum\limits_{k=1}^{\infty} 1/s_k < +\infty$, defined by

$$s_k = \frac{k}{p}r_p, \quad 2nn'(p-1) < k < 2nn'p.$$

By Lemma 1.7, for each $t>0$ we have

$$\sup_{k\geqslant 0} a_k^{\omega,n}t^k \leqslant |\omega(t)^n| \leqslant c'|\omega_{\{r_k\}}(L't)^{nn'}|$$

$$\leqslant 3c'\max\left\{1, \left(\sup_{k\geqslant 1}\frac{(4L't)^k}{r_1\ldots r_k}\right)^{2nn'}\right\} \leqslant 3c'\max\left\{1, \sup_{k\geqslant 1}\frac{(8nn'L't)^k}{s_1\ldots s_k}\right\}.$$

Hence, for each $k\geqslant 1$

$$a_k^{\omega,n} \leqslant \frac{\sup\limits_{q\geqslant 0} a_q^{\omega,n}\left(\dfrac{s_k}{8nn'L'}\right)^q}{\left(\dfrac{s_k}{8nn'L'}\right)^k} \leqslant 3c'\max\left\{\left(\frac{8nn'L'}{s_k}\right)^k, \frac{\sup\limits_{q\geqslant 1}\dfrac{(s_k)^q}{s_1\ldots s_q}}{\left(\dfrac{s_k}{8nn'L'}\right)^k}\right\}$$

$$= 3c'\max\left\{\left(\frac{8nn'L'}{s_k}\right)^k, \frac{(8nn'L')^k}{s_1\ldots s_k}\right\} = 3c'\frac{(8nn'L')^k}{s_1\ldots s_k},$$

so that

$$b_k^{\omega,n} \leqslant \frac{\sup\limits_{q\geqslant 0} q!\,a_q^{\omega,n}\left(\dfrac{s_k}{8nn'L'k}\right)^q}{k!\left(\dfrac{s_k}{8nn'L'k}\right)^k} \leqslant \max\left\{\frac{1}{k!}\left(\frac{8nn'L'k}{s_k}\right)^k, 3c'\frac{\sup\limits_{q\geqslant 1}\dfrac{(s_k/k)^q}{s_1\ldots s_q}\,\big/\,1\ldots q}{k!\left(\dfrac{s_k}{8nn'L'k}\right)^k}\right\}$$

$$= \max\left\{\frac{1}{k!}\left(\frac{8nn'L'k}{s_k}\right)^k, 3c'\frac{(8nn'L')^k}{s_1\ldots s_k}\right\} = 3c'\frac{(8nn'L')^k}{s_1\ldots s_k}.$$

It follows that for each $t>0$

$$\sup_{k\geqslant 0} b_k^{\omega,n}t^k \leqslant 3c'|\omega_{\{s_k\}}(8nn'L't)| \leqslant 3c'|\omega_{\{r_k\}}(8nn'L't)^{2nn'}|$$

$$\leqslant 3c'c''|\omega(8nn'L'L''t)^{2nn'n''}| \leqslant 3\sqrt{2}\,c'c''\sup_{k\geqslant 0} a_k^{\omega,2nn'n''}(8\sqrt{2}\,nn'L'L''t)^k.$$

In conclusion, putting $L = 8\sqrt{2}n'L'L''$, $m = 2n'n''$ and $c = 3\sqrt{2}c'c''$, we have for each integer $n\geqslant 1$

$$\sup_{k\geqslant 0} b_k^{\omega,n}t^k \leqslant c\sup_{k\geqslant 0} a_k^{\omega,mn}(Lnt)^k, \quad t>0.$$

Hence for each $n, k\geqslant 1$

$$b_k^{\omega,n} \leqslant \frac{\sup\limits_{q\geqslant 0} b_q^{\omega,n}\left(\dfrac{a_k^{\omega,mn}}{Lna_{k-1}^{\omega,mn}}\right)^q}{\left(\dfrac{a_k^{\omega,mn}}{Lna_{k-1}^{\omega,mn}}\right)^k} \leqslant c\frac{\sup\limits_{q\geqslant 0} a_q^{\omega,mn}\left(\dfrac{a_k^{\omega,mn}}{a_{k-1}^{\omega,mn}}\right)^q}{\left(\dfrac{a_k^{\omega,mn}}{Lna_{k-1}^{\omega,mn}}\right)^k} = c(Ln)^k a_k^{\omega,mn}. \ \blacksquare$$

After the above preliminaries concerning the regularity of ω, we begin the proof of the existence of the boundary values of the functions from $\mathscr{H}_\omega(X)$.

DEFINITION XXV. We say that ω is *regular* if it satisfies the equivalent conditions of Lemma 5.3.

LEMMA 5.4. *Let $s_0>0$, $\delta>0$ and let Φ be an X-valued analytic function on $\{z\in\boldsymbol{C};\ |\mathrm{Re}z| < s_0+\delta, 0<\mathrm{Im}z<2\delta\}$ such that for some $L>0$ and integer $n\geqslant 1$*

$$\|\Phi(z)\| \leqslant \omega_n^*\left(\frac{L}{\mathrm{Im}z}\right), \quad z\in\boldsymbol{C},\ |\mathrm{Re}z| < s_0+\delta, 0<\mathrm{Im}z<2\delta.$$

Then there exists a sequence $\{\Phi_k\}_{k\geqslant 0}$ of X-valued continuous functions on $\{z\in\boldsymbol{C};\ |\mathrm{Re}z|\leqslant s_0,\ 0<\mathrm{Im}z\leqslant\delta\}$ which are analytical on $\{z\in\boldsymbol{C};\ |\mathrm{Re}z|\leqslant s_0,\ 0<\mathrm{Im}z\leqslant\delta\}$ and constants $M>0$ and $c>0$ such that for z in the region $\{z\in\boldsymbol{C};\ |\mathrm{Re}z|\leqslant s_0, 0<\mathrm{Im}z<\delta\}$ we have

$$\|\Phi_k(z)\| \leqslant ck!\,b_k^{\omega,n}\left(\frac{M}{\mathrm{Im}z}\right)^{k+2}, \quad k\geqslant 0,$$

$$\Phi(z) = \sum_{k=0}^{\infty}\Phi_k(z).$$

Proof. It is well known that

$$\Theta_1\colon \{v \in \boldsymbol{C};\ |v| < 1\} \ni v \mapsto i\frac{1+v}{1-v}$$

is a conformal mapping of the unit circle $\{v \in \boldsymbol{C}; |v| < 1\}$ onto the half plane $\{w \in \boldsymbol{C}; \operatorname{Im} w > 0\}$.

Further, if $0 < \varkappa < 1$, $x > 0$ are such that

$$x\int_0^1 \frac{dt}{(1-t^2)(1-\varkappa^2t^2)} = s_0+\delta, \quad x\int_0^{\varkappa^{-1}} \frac{dt}{(t^2-1)(1-\varkappa^2t^2)} = 2\delta$$

then

$$\Theta_2\colon \{w \in \boldsymbol{C};\ \operatorname{Im} w > 0\} \ni w \mapsto \int_0^w \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-\varkappa^2\zeta^2)}}$$

is a conformal mapping of the half plane $\{w \in \boldsymbol{C}; \operatorname{Im} w > 0\}$ onto the rectangle $\{z \in \boldsymbol{C}; |\operatorname{Re} z| < s_0+\delta, 0 < \operatorname{Im} z < 2\delta\}$, if we agree that $\sqrt{1-\zeta^2}$ and $\sqrt{1-\varkappa^2\zeta^2}$ have positive real parts (see for example [1], Ch. 6, 2.3). Hence $\Theta = \Theta_2 \circ \Theta_1$ is a conformal mapping of $\{v \in \boldsymbol{C}; |v| < 1\}$ onto $\{z \in \boldsymbol{C};\ |\operatorname{Re} z| < s_0+\delta, 0 < \operatorname{Im} z < 2\delta\}$.

A direct computation (see for details [35], p. 25–27) shows that there exist $d > 0$, $c_1 > 0$, $c_2 > 0$ such that

$$1-|v| \leqslant 4\operatorname{Im}\Theta_1(v),$$

$$\operatorname{Im}\Theta_1(v) \leqslant d(1-|v|), \quad \operatorname{Im}\Theta(v) \leqslant \delta$$

and

$$\operatorname{Im} v \leqslant c_1 \operatorname{Im}\Theta_2(w), \quad \operatorname{Im}\Theta_2(w) \leqslant \delta,$$

$$\operatorname{Im}\Theta_2(w) \leqslant c_2 \operatorname{Im} w, \quad |\operatorname{Re}\Theta_2(w)| \leqslant s_0, \quad \operatorname{Im}\Theta_2(w) \leqslant \delta.$$

Hence

$$(5.3)\qquad 1-|v| \leqslant (4c_1+\delta^{-1})\operatorname{Im}\Theta(v),$$

$$(5.4)\qquad \operatorname{Im}\Theta(v) \leqslant c_2 d(1-|v|), \quad |\operatorname{Re}\Theta(v)| \leqslant s_0, \quad \operatorname{Im}\Theta(v) \leqslant \delta.$$

Next we consider the X-valued analytic function $f = \Phi \circ \Theta$ on $\{v \in \boldsymbol{C}; |v| < 1\}$. Since for all $z \in \boldsymbol{C}$ with $|\operatorname{Re} z| < s_0+\delta, 0 < \operatorname{Im} z < 2\delta$, we have

$$\|\Phi(z)\| \leqslant 3\sup_{k\geqslant 0} k!\,b_k^{\omega,n}\left(\frac{3L}{\operatorname{Im} z}\right)^{k+1} \leqslant 3\sum_{k=1}^{\infty}(k-1)!\,b_{k-1}^{\omega,n}\left(\frac{3L}{\operatorname{Im} z}\right)^{k},$$

by (5.3), we get for all $v \in \boldsymbol{C}$, $|v| < 1$,

$$\|f(v)\| \leqslant 3\sum_{k=1}^{\infty}(k-1)!\,b_{k-1}^{\omega,n}[3L(4c_1+\delta^{-1})]^k(1-|v|)^{-k}.$$

By Theorem 1.12 there exists a sequence $\{f_k\}_{k\geqslant 1}$ of X-valued analytic functions on $\{v \in \boldsymbol{C}; |v| < 1\}$ such that writing

$$a = 3e\,\frac{b_1^{\omega,n}[3L(4c_1+\delta^{-1})]}{b_0^{\omega,n}}, \quad N = 4e\,\frac{b_0^{\omega,n}+b_1^{\omega,n}[3L(4c_1+\delta^{-1})]}{b_1^{\omega,n}}$$

we have

$$\|f_k(v)\| \leqslant a(k-1)!\,b_{k-1}^{\omega,n}N^k(1-|v|)^{-k-1}, \quad |v| < 1,\ k \geqslant 1,$$

$$f(v) = \sum_{k=1}^{\infty} f_k(v), \quad |v| < 1.$$

For each integer $k \geqslant 0$ we denote by Φ_k the restriction of $f_{k+1}\circ\Theta^{-1}$ to $\{z \in \boldsymbol{C}; |\operatorname{Re} z| \leqslant s_0, 0 < \operatorname{Im} z \leqslant \delta\}$. Then, on this rectangle, by (5.4) we get

$$\|\Phi_k(z)\| \leqslant \frac{a}{N}\,k!\,b_k^{\omega,n}\left(\frac{Nc_2 d}{\operatorname{Im} z}\right)^{k+2}, \quad k \geqslant 0,$$

$$\Phi(z) = \sum_{k=0}^{\infty}\Phi_k(z). \quad \blacksquare$$

The following lemma is essentially proved in [35], p. 28–29:

LEMMA 5.5. *Let $s_0 > 0$, $\delta > 0$, let $k \geqslant 0$ be an integer and let Φ be an X-valued continuous function on $\{z \in \boldsymbol{C}; |\operatorname{Re} z| \leqslant s_0, 0 < \operatorname{Im} z \leqslant \delta\}$ which is analytical on $\{z \in \boldsymbol{C}; |\operatorname{Re} z| \leqslant s_0, 0 < \operatorname{Im} z < \delta\}$ and such that*

$$\|\Phi(z)\| \leqslant \left(\frac{1}{\operatorname{Im} z}\right)^{k+2}, \quad z \in \boldsymbol{C},\ |\operatorname{Re} z| \leqslant s_0,\ 0 < \operatorname{Im} z < \delta.$$

Then there exists an X-valued continuous function Ψ on $\{z \in \boldsymbol{C}; |\operatorname{Re} z| \leqslant s_0, 0 < \operatorname{Im} z \leqslant \delta\}$ which is analytical on $\{z \in \boldsymbol{C}; |\operatorname{Re} z| < s_0, 0 < \operatorname{Im} z < \delta\}$ and such that

$$\|\Psi(s+i\varepsilon_1)-\Psi(s+i\varepsilon_2)\| \leqslant \frac{1}{(k+1)!}\left[2\left(1+\frac{s_0}{\delta}\right)^{k+2}\varepsilon_2+\sqrt{2\delta\varepsilon_2}\right],$$

$$-s_0 \leqslant s \leqslant s_0,\ 0 < \varepsilon_1 \leqslant \varepsilon_2 \leqslant \delta,$$

$$\Psi^{(k+3)}(z) = \Phi(z), \quad z \in \boldsymbol{C},\ |\operatorname{Re} z| < s_0,\ 0 < \operatorname{Im} z < \delta.$$

Proof. For each $z \in \boldsymbol{C}$, $|\operatorname{Re} z| \leqslant s_0$, $0 < \operatorname{Im} z \leqslant \delta$, we denote by Γ_z the curve obtained by the union of the segments $[i\delta, \operatorname{Re} z+i\delta]$ and $[\operatorname{Re} z+i\delta, z]$. Define a sequence $\{\Psi_j\}_{j\geqslant 0}$ of X-valued continuous functions on $\{z \in \boldsymbol{C}; |\operatorname{Re} z| \leqslant s_0, 0 < \operatorname{Im} z \leqslant \delta\}$ which are analytical on $\{z \in \boldsymbol{C}; |\operatorname{Re} z| < s_0, 0 < \operatorname{Im} z < \delta\}$ by the formulas

$$\Psi_0 = \Phi,$$

$$\Psi_j(z) = \int_{\Gamma_z}\Psi_{j-1}(\zeta)\,d\zeta, \quad j \geqslant 1.$$

Clearly, $\Psi_j^{(j)} = \Phi$ on $\{z \in C; |\mathrm{Re}z| < s_0, 0 < \mathrm{Im}z < \delta\}$ for all $j \geqslant 0$. By induction it is easy to verify that for $1 \leqslant j \leqslant k+1$

$$\|\Psi_j(z)\| \leqslant \left(\frac{1}{\delta}\right)^{k+2} \sum_{m=1}^{j} \frac{1}{m!(j-m)!} |\mathrm{Re}z|^m (\delta - \mathrm{Im}z)^{j-m} +$$

$$+ \frac{(k+1-j)!}{(k+1)!} \left(\frac{1}{\mathrm{Im}z}\right)^{k+2-j} - \frac{1}{(k+1)!} \sum_{m=1}^{j} \frac{(k+1-m)!}{(j-m)!} \left(\frac{1}{\delta}\right)^{k+2-m} (\delta - \mathrm{Im}z)^{j-m}.$$

Using the above inequality for $j = k+2$, we get

$$\|\Psi_{k+2}(z)\| \leqslant \left(\frac{1}{\delta}\right)^{k+2} \sum_{m=1}^{k+2} \frac{1}{m!(k+2-m)!} |\mathrm{Re}z|^m (\delta - \mathrm{Im}z)^{k+2-m} +$$

$$+ \frac{1}{(k+1)!} \ln \frac{\delta}{\mathrm{Im}z} - \frac{1}{(k+1)!} \sum_{m=1}^{k+1} \frac{1}{k+2-m} \left(\frac{1}{\delta}\right)^{k+2-m} (\delta - \mathrm{Im}z)^{k+2-m}$$

$$\leqslant \frac{1}{(k+1)!} \left(1 + \frac{s_0}{\delta}\right)^{k+2} + \ln \frac{\delta}{\mathrm{Im}z}.$$

Hence for $-s_0 \leqslant s \leqslant s_0$ and $0 < \varepsilon_1 \leqslant \varepsilon_2 \leqslant \delta$ we have

$$\|\Psi_{k+3}(s+i\varepsilon_1) - \Psi_{k+3}(s+i\varepsilon_2)\| \leqslant \int_{\varepsilon_1}^{\varepsilon_2} \|\Psi_{k+2}(s+ir)\| dr$$

$$\leqslant \frac{1}{(k+1)!} \left[\left(1 + \frac{s_0}{\delta}\right)^{k+2} (\varepsilon_2 - \varepsilon_1) + \varepsilon_2 - \varepsilon_1 + \varepsilon_2 \ln \frac{\delta}{\varepsilon_2} - \varepsilon_1 \ln \frac{\delta}{\varepsilon_1}\right]$$

$$\leqslant \frac{1}{(k+1)!} \left[2\left(1 + \frac{s_0}{\delta}\right)^{k+2} \varepsilon_2 + \sqrt{2\delta\varepsilon_2}\right].$$

Taking $\Psi = \Psi_{k+3}$, we obtain the desired result. ∎

We can now give the following

THEOREM 5.6. *Assume that ω is regular. Then for each $\Phi \in \mathscr{H}_\omega(X)$ the limit*

$$\lim_{0<\varepsilon\to 0} (\Phi(\cdot + i\varepsilon) - \Phi(\cdot - i\varepsilon))$$

exists in $\mathscr{L}(\mathscr{D}_\omega; X)$.

Proof. Let $s_0 > 0$. There exist $\delta > 0$ and $L > 0$, integer $n \geqslant 1$, $c > 0$, such that in $\{z \in C; |\mathrm{Re}z| < s_0 + \delta, 0 \neq |\mathrm{Im}z| < 2\delta\}$

$$\|\Phi(z)\| \leqslant c\omega_n^*\left(\frac{L}{|\mathrm{Im}z|}\right).$$

By Lemma 5.4 and 5.5, there exists a sequence $\{\Psi_k\}_{k\geqslant 0}$ of X-valued continuous functions on $\{z \in C; |\mathrm{Re}z| \leqslant s_0, 0 \neq |\mathrm{Im}z| \leqslant \delta\}$ which are analytical on the interior of their domains and constants $M > 0$, $N > 0$, $d > 0$, such that for $-s_0 < s < s_0$ and $0 < \varepsilon_1 < \varepsilon_2$ or $-\delta < \varepsilon_2 < \varepsilon_1 < 0$, we have

$$\|\Psi_k(s+i\varepsilon_1) - \Psi_k(s+i\varepsilon_2)\| \leqslant d b_k^{\omega,n} (MN)^{k+2} \sqrt{|\varepsilon_2|}, \quad k \geqslant 0,$$

and for $-s_0 < s < s_0$ and $0 \neq |\varepsilon| < \delta$

$$\|\Psi_k^{(k+3)}(s+i\varepsilon)\| \leqslant d k! \, b_k^{\omega,n} \left(\frac{M}{|\varepsilon|}\right)^{k+2}, \quad k \geqslant 0,$$

$$\Phi(s+i\varepsilon) = \sum_{k=0}^{\infty} \Psi_k^{(k+3)}(s+i\varepsilon).$$

Finally, since ω is assumed to be regular, there exist an $L_n > 0$, an integer $m_n \geqslant 1$ and a $c_0 > 0$ such that

$$b_k^{\omega,n} \leqslant c_0 L_n^k a_k^{\omega,n}, \quad k \geqslant 0.$$

Now let $\varphi \in \mathscr{D}_\omega[-s_0, s_0]$. For $0 < \varepsilon_1 \leqslant \varepsilon_2 < \delta$ or $-\delta < \varepsilon_2 \leqslant \varepsilon_1 < 0$, we have

$$\left\| \int_{-\infty}^{+\infty} \varphi(s)\Phi(s+i\varepsilon_1)ds - \int_{-\infty}^{+\infty} \varphi(s)\Phi(s+i\varepsilon_2)ds \right\|$$

$$= \left\| \sum_{k=0}^{\infty} \int_{-\infty}^{+\infty} \varphi(s)(\Psi_k^{(k+3)}(s+i\varepsilon_1) - \Psi_k^{(k+3)}(s+i\varepsilon_2))ds \right\|$$

$$= \left\| \sum_{k=0}^{\infty} (-1)^{k+3} \int_{-\infty}^{+\infty} \varphi^{(k+3)}(s)(\Psi_k(s+i\varepsilon_1) - \Psi_k(s+i\varepsilon_2))ds \right\|$$

$$\leqslant \sum_{k=0}^{\infty} 2s_0 \sup_{-s_0 \leqslant s \leqslant s_0} |\varphi^{(k+3)}(s)| d b_k^{\omega,n} (MN)^{k+2} \sqrt{|\varepsilon_2|}$$

$$\leqslant \sum_{k=0}^{\infty} 2s_0 d c_0 (MN)^2 \sqrt{|\varepsilon_2|} \sup_{-s_0 \leqslant s \leqslant s_0} |\varphi^{(k+3)}(s)| a_k^{\omega,m_n} (L_m MN)^k$$

$$\leqslant s_0 d c_0 (2MN)^2 \sqrt{|\varepsilon_2|}\, r^{\omega}_{2L_n MN, m_n}(\varphi''') \leqslant s_0 d c_0 (2MN)^2 \sqrt{|\varepsilon_2|}\, q^{\omega}_{2L_n MN, m_n}(\varphi''')$$

$$\leqslant \frac{s_0 d c_0 t_1^3}{2L_n^2 MN} \sqrt{|\varepsilon_2|}\, q^{\omega}_{2L_n MN, m_n+3}(\varphi).$$

Hence for $0 < \varepsilon_1 \leqslant \varepsilon_2 < \delta$,

$$\left\| \int_{-\infty}^{+\infty} \varphi(s)(\Phi(s+i\varepsilon_1) - \Phi(s-i\varepsilon_1))ds - \int_{-\infty}^{+\infty} \varphi(s)(\Phi(s+i\varepsilon_2) - \Phi(s-i\varepsilon_2))ds \right\|$$

$$\leqslant \frac{s_0 d c_0 t_1^3}{L_n^3 MN} \sqrt{|\varepsilon_2|}\, q^{\omega}_{2L_0 MN, m_n+3}(\varphi).$$

Consequently, $\lim\limits_{0<\varepsilon\to 0} (\Phi(\cdot + i\varepsilon) - \Phi(\cdot - i\varepsilon))$ exists in $\mathscr{L}(\mathscr{D}_\omega; X)$. ∎

We remark that for the proof of Theorem 5.6 we adapted some ideas of J. Körner from [35]. Next we intend to characterize those functions from $\mathscr{H}_\omega(X)$,

ω being regular, whose boundary values in $\mathscr{L}(\mathscr{D}_\omega; X)$ are zero. For this purpose we need some lemmas:

LEMMA 5.7. *Let $f(\zeta) = \sum_{k=0}^{\infty} c_k \zeta^k$ be an entire function of exponential type* 0. *Then $g(\zeta) = \sum_{k=0}^{\infty} |c_k| \zeta^k$ is also of exponential type* 0 *and for each $z \in \boldsymbol{C}$, $r > 0$ and X-valued analytic function Φ on some open neighbourhood of $\{\zeta \in \boldsymbol{C};\ |z-\zeta| \leqslant r\}$ we have*

$$\sum_{k=0}^{\infty} |c_k|\, ||\Phi^{(k)}(z)|| \leqslant r g_{\mathrm{Borel}}\left(\frac{1}{r}\right) \sup_{|z-\zeta|=r} ||\Phi(\zeta)||.$$

Proof. Let $\varepsilon > 0$. Then there exists a $c_\varepsilon > 0$ such that $|f(\zeta)| \leqslant c_\varepsilon e^{\varepsilon|\zeta|}$ for all $\zeta \in \boldsymbol{C}$. By the Cauchy integral formula, for each integer $k \geqslant 0$ and $\varrho \geqslant 0$

$$|c_k| \leqslant \frac{1}{\varrho^k} \sup_{|\zeta|=\varrho} |f(\zeta)| \leqslant c_\varepsilon \frac{e^{\varepsilon\varrho}}{\varrho^k}.$$

Taking $\varrho = k/\varepsilon$ and using the Stirling formula, we get

$$|c_k| \leqslant c_\varepsilon \left(\frac{e}{k}\right)^k \varepsilon^k \leqslant c_\varepsilon \sqrt{2\pi k}\, \frac{\varepsilon^k}{k!} \leqslant c_\varepsilon \sqrt{2\pi}\, \frac{(2\varepsilon)^k}{k!}.$$

Hence $|g(\zeta)| \leqslant c_\varepsilon \sqrt{2\pi}\, e^{2\varepsilon|\zeta|}$ for all $\zeta \in \boldsymbol{C}$.

Further, by the Cauchy integral formula, we have

$$||\Phi^{(k)}(z)|| \leqslant \frac{k!}{r^k} \sup_{|z-\zeta|=r} ||\Phi(\zeta)||;$$

hence

$$\sum_{k=0}^{\infty} |c_k|\, ||\Phi^{(k)}(z)|| \leqslant \sum_{k=0}^{\infty} k!\, |c_k| \frac{1}{r^k} \sup_{|z-\zeta|=r} ||\Phi(\zeta)|| = r g_{\mathrm{Borel}}\left(\frac{1}{r}\right) \sup_{|z-\zeta|=r} ||\Phi(\zeta)||. \ \blacksquare$$

By Lemma 5.3 if $f(\zeta) = \sum_{k=0}^{\infty} c_k \zeta^k$ is an entire function of exponential type 0 and Φ is an X-valued analytic function on some open subset Ω of $\boldsymbol{C}$, then we can define another X-valued analytic function $f(D)\Phi$ on Ω by

$$\big(f(D)\Phi\big)(z) = \sum_{k=0}^{\infty} c_k \Phi^{(k)}(z).$$

Moreover, the "differential operator of infinite order" $f(D)$ is continuous on the topology of uniform convergence on compact subsets of Ω.

If f defines an ω-ultradifferential operator with constant coefficients, then the ω-ultradifferential operator $f(D)$ and the "differential operator of infinite order" $f(D)$ defined above coincide on real analytic functions. More precisely, we have

LEMMA 5.8. *Let $-\infty \leqslant a < b \leqslant +\infty$, let Φ be an X-valued analytic function on some open neighbourhood of (a, b) in $\boldsymbol{C}$ and let f be an entire function of exponential type* 0 *such that for a certain $L > 0$, integer $n \geqslant 1$ and $c > 0$*

$$|f(it)| \leqslant c|\omega(Lt)^n|, \quad t \in \boldsymbol{R}.$$

Then for any $\varphi \in \mathscr{D}_\omega(a, b)$ we have

$$\int_a^b \varphi(s)\big(f(D)\Phi\big)(s)\,ds = \int_a^b \big(f(-D)\varphi\big)(s) F(s)\,ds. \tag{5.5}$$

Proof. By Corollary 2.6 it is enough to prove (5.5) for those $\varphi \in \mathscr{D}_\omega$ whose support is contained in a certain $(s_0-\varepsilon, s_0+\varepsilon)$, where $\{\zeta \in \boldsymbol{C};\ |s_0-\zeta| < \varepsilon\}$ is contained in the domain of Φ. In addition, if $\Phi(\zeta) = \sum_{k=0}^{\infty} x_k(\zeta-s_0)^k$ is the power series expansion of Φ in $\{\zeta \in \boldsymbol{C};\ |s_0-\zeta| < \varepsilon\}$, then it is enough to prove (5.5) for Φ of the form $\zeta \to (\zeta-s_0)^k x_k$, $k \geqslant 0$.

Hence if $f(\zeta) = \sum_{m=0}^{\infty} c_m \zeta^m$ is the power series expansion of f, we have to show that for each $\varphi \in \mathscr{D}_\omega$ and $k \geqslant 0$ we have

$$\int_{-\infty}^{+\infty} \varphi(s)\left(\sum_{m=0}^{k} c_m \frac{k!}{(k-m)!} s^{k-m}\right) ds = \int_{-\infty}^{+\infty} \big(F(-D)\varphi\big)(s)\, s^k\,ds. \tag{5.6}$$

But

$$\widehat{\varphi(s) \sum_{m=0}^{k} c_m \frac{k!}{(k-m)!} s^{k-m}} = \sum_{m=0}^{k} i^{k-m} c_m \frac{k!}{(m-k)!} \hat{\varphi}^{(k-m)}(t)$$

and

$$\widehat{\big(f(-D)\varphi\big)(s)s^k} = \sum_{m=0}^{k} i^{k-m} \binom{k}{m} f^{(m)}(-it)\hat{\varphi}^{(k-m)}(t)$$

and it is easy to see that these two Fourier transforms are equal in $t = 0$, which proves (5.6). $\blacksquare$

Next we shall prove a vector version of a classical result of P. Painlevé (see [53], § 2 and [46], § 3).

LEMMA 5.9. *Let $-\infty \leqslant a < b \leqslant +\infty$, let $\delta > 0$ and let Φ be a bounded X-valued analytic function on $\{z \in \boldsymbol{C};\ a < \operatorname{Re} z < b,\ 0 \neq |\operatorname{Im} z| < \delta\}$ such that for each compact $\boldsymbol{K} \subset (a, b)$ and each $x^* \in X^*$*

$$\lim_{0<\varepsilon\to 0}\ \sup_{s \in K} |\langle x^*, \Phi(s+i\varepsilon) - \Phi(s-i\varepsilon)\rangle| = 0.$$

Then Φ has an X-valued analytic extension on $\{z \in \boldsymbol{C};\ a < \operatorname{Re} z < b,\ |\operatorname{Im} z| < \delta\}$.

Proof. Let $a < c < d < b$ be arbitrary and denote by Γ the counter clockwise oriented closed curve obtained by the union of the segments

$$\left[c-i\frac{\delta}{2}, d-i\frac{\delta}{2}\right], \quad \left[d-i\frac{\delta}{2}, d+i\frac{\delta}{2}\right], \quad \left[d+i\frac{\delta}{2}, c+i\frac{\delta}{2}\right], \quad \left[c+i\frac{\delta}{2}, c-i\frac{\delta}{2}\right];$$

consider the X-valued function Ψ on $\{z \in \boldsymbol{C};\, c < \operatorname{Re} z < d,\, |\operatorname{Im} z| < \delta/2\}$ defined by

$$\Psi(z) = \frac{1}{2\pi i} \int_{\Gamma_m} \frac{1}{\zeta - z} \Phi(\zeta)\, d\zeta .$$

Let $x^* \in X^*$; then there exists $\{\varepsilon_k\}_{k \geqslant 1} \subset (0, \delta/2)$ such that $\varepsilon_k \to 0$ and

$$\lim_{k\to\infty} \sup_{s \in K} |\langle x^*, \Phi(s+i\varepsilon_k) - \Phi(s-i\varepsilon_k)\rangle| = 0 .$$

For each k we denote by Γ_k^1 the closed curve obtained by the union of the segments

$$[c+i\varepsilon_k, d+i\varepsilon_k], \quad \left[d+i\varepsilon_k, d+i\frac{\delta}{2}\right], \quad \left[d+i\frac{\delta}{2}, c+i\frac{\delta}{2}\right], \quad \left[c+i\frac{\delta}{2}, c+i\varepsilon_k\right]$$

and by Γ_k^2 the union of the segments

$$[d-i\varepsilon_k, c-i\varepsilon_k], \quad \left[c-i\varepsilon_k, c-i\frac{\delta}{2}\right], \quad \left[c-i\frac{\delta}{2}, d-i\frac{\delta}{2}\right], \quad \left[d-i\frac{\delta}{2}, d-i\varepsilon_k\right].$$

Then by the Cauchy integral formula, for all $z \in \boldsymbol{C}$ with $c < \operatorname{Re} z < d$, $\varepsilon_k < |\operatorname{Im} z| < \delta/2$, we have

$$\langle x^*, \Phi(z)\rangle = \frac{1}{2\pi i} \int_{\Gamma_k^1} \frac{1}{\zeta - z} \langle x^*, \Phi(\zeta)\rangle d\zeta + \frac{1}{2\pi i} \int_{\Gamma_k^2} \frac{1}{\zeta - z} \langle x^*, \Phi(\zeta)\rangle d\zeta .$$

On the other hand, by the choice of $\{\varepsilon_k\}_{k \geqslant 1}$, for $z \in \boldsymbol{C}$ with $c < \operatorname{Re} z < d$, $0 \neq |\operatorname{Im} z| < \delta/2$, we have

$$\langle x^*, \Psi(z)\rangle = \lim_{k\to\infty} \left(\frac{1}{2\pi i} \int_{\Gamma_k^1} \frac{1}{\zeta - z} \langle x^*, \Phi(\zeta)\rangle d\zeta + \frac{1}{2\pi i} \int_{\Gamma_k^2} \frac{1}{\zeta - z} \langle x^*, \Phi(\zeta)\rangle d\zeta .\right.$$

Consequently, $\langle x^*, \Phi(\cdot)\rangle$ and $\langle x^*, \Psi(\cdot)\rangle$ are equal on the intersection of their domains.

Since $x^* \in X^*$ was arbitrary, it follows that Φ and Ψ are equal on the intersection of their domains, that is Φ can be extended analytically to $\{z \in \boldsymbol{C};\, a < \operatorname{Re} z < b,\, 0 \neq |\operatorname{Im} z| < \delta\} \cup (c, d)$. ∎

Now we can give an ω-ultradistributional version of Painlevé's theorem (cf. [53[, Theorem B, [46], Theorem 1, and [35], p. 43–45).

THEOREM 5.10. *Assume that ω is regular. Let $\Phi \in \mathscr{H}_\omega(X)$ and let $-\infty \leqslant a < b \leqslant +\infty$ be such that for each $\varphi \in \mathscr{D}_\omega(a, b)$ and each $x^* \in X^*$*

$$\lim_{0<\varepsilon\to 0} \int_a^b \varphi(s) \langle x^*, \Phi(s+i\varepsilon) - \Phi(s-i\varepsilon)\rangle ds = 0 .$$

Then Φ has an X-valued analytic extension on $(\boldsymbol{C} \setminus \boldsymbol{R}) \cup (a, b)$.

Proof. By Lemma 5.9 and by the uniform boundedness principle (see [20], Theorem II, 3.20), it is sufficient to consider the case $X = \boldsymbol{C}$. Let $a < c < d < b$. By Theorem 5.6 the limits $\lim_{0<\varepsilon\to 0} \Phi(\cdot + i\varepsilon)$ and $\lim_{0<\varepsilon\to 0} \Phi(\cdot - i\varepsilon)$ exist in the strong topology of $\mathscr{D}'_\omega[c, d]$; hence $\{\Phi(\cdot \pm i\varepsilon)\}_{0<\varepsilon<1}$ is a bounded subset of $\mathscr{D}'_\omega[c, d]$ in this topology.

Since $\mathscr{D}_\omega[c, d]$ is Montel, by our hypothesis on Φ and by [22], § 22, 2.4, we have

$$\lim_{0<\varepsilon\to 0} \big(\Phi(\cdot + i\varepsilon) - \Phi(\cdot - i\varepsilon)\big) = 0$$

in the strong topology of $\mathscr{D}'_\omega[c, d]$. Further, since $\mathscr{D}_\omega[c, d]$ is a nuclear Fréchet space, by [22], § 2.6, 2.4 (or by [29], Ch. III, Theorem 1.1) and by [22], § 2.5, 2.7 and 2.10, there exist $L > 0$, integer $n \geqslant 1$, $\gamma > 0$ and $\alpha\colon (0, 1) \to (0, +\infty)$, $\lim_{0<\varepsilon\to 0} \alpha(\varepsilon) = 0$, such that

$$\left|\int_c^d \varphi(s) \big(\Phi(s \pm i\varepsilon)\big) ds\right| \leqslant \gamma q^\omega_{L,n}(\varphi),$$

$$\left|\int_c^d \varphi(s) \big(\Phi(s+i\varepsilon) - \Phi(s-i\varepsilon)\big) ds\right| \leqslant \alpha(\varepsilon) q^\omega_{L,n}(\varphi) \tag{5.7}$$

for $\varphi \in \mathscr{D}_\omega[c, d]$, $\varepsilon \in (0, 1)$.

Denote by $\mathscr{D}_{\omega,L,n}(c, d)$ the vector space of all continuous complex functions with a compact support φ on $\boldsymbol{R}$ such that $\operatorname{supp}\varphi \subset (c, d)$ and $q^\omega_{L,n}(\varphi) < +\infty$. For each $\varphi \in \mathscr{D}_{\omega,L,n}(c, d)$ there exists a sequence $\{\varphi_k\}_{k\geqslant 1}$ in $\mathscr{D}_\omega(c, d) \subset \mathscr{D}_{\omega,L,n}(c, d)$ such that $\lim_{k\to\infty} q^\omega_{L,n}(\varphi - \varphi_k) = 0$.

Indeed, if $\theta > 0$ is such that $\operatorname{supp}\varphi \subset [a+\theta, d-\theta]$, then by Theorem 2.5 there exists a $\psi \in \mathscr{D}_\omega$, $\operatorname{supp}\psi \subset (-\theta, \theta)$, $\int_{-\infty}^{+\infty} \psi(s)\, ds = 1$, and defining for every $k \geqslant 1$ the function

$$\psi_k(s) = k^2 \psi(k^2 s), \quad s \in \boldsymbol{R},$$

we obtain

$$\psi_k \in \mathscr{D}_\omega(-\theta/k^2, \theta/k^2) \quad \text{and} \quad \varphi * \psi_k \in \mathscr{D}_\omega(c, d).$$

Moreover

$$\begin{aligned} q^\omega_{L,n}(\varphi - \varphi * \psi_k) &= \int_{-\infty}^{+\infty} |\hat\varphi(t)\big(1 - 2\pi\hat\psi_k(t)\big)\omega(Lt)^n|\, dt \\ &= 2\pi \int_{-\infty}^{+\infty} |\hat\varphi(t)\omega(Lt)^n| |\hat\psi(0) - \hat\psi(t/k^2)|\, dt \\ &\leqslant 2\pi q^\omega_{L,n}(\varphi) \sup_{|r|\leqslant 1/k} |\hat\psi(0) - \hat\psi(r)| + 4\sup_{r\in R} |\hat\psi(r)| \int_{|t|\geqslant k} |\hat\varphi(t)\omega(Lt)^n|\, dt; \end{aligned}$$

hence $\lim_{k\to+\infty} q^\omega_{L,n}(\varphi - \varphi * \psi_k) = 0$.

It follows that (5.7) can be extended to every $\varphi \in \mathscr{D}_{\omega,L,n}(c, d)$. Let $\theta > 0$ be such that $c+\theta < b-\theta$. Define the function E on $\boldsymbol{R}$ by

$$E(s) = \int_{-\infty}^{+\infty} \omega\big(\sqrt{2}\,Lt\big)^{-n-2} e^{ist}\,dt, \quad s \in \boldsymbol{R}.$$

Then E is a bounded continuous function on $\boldsymbol{R}$, in particular $E \in \mathscr{D}'_\omega$, and it is easy to verify that

$$\omega\big(-i\sqrt{2}\,LD\big)^{n+2} E = 2\pi\delta_0,$$

where δ_0 is the Dirac measure concentrated at 0.

On the other hand, the restriction of E to $\boldsymbol{R}\setminus\{0\}$ can be extended analytically to some complex neighbourhood of $\boldsymbol{R}\setminus\{0\}$ by the formula

$$E(z) = z^{-1}\int_{-\infty}^{+\infty} \omega\big(\sqrt{2}\,Lz^{-1}t\big)^{-n-2} e^{it}\,dt.$$

By Theorem 2.5 there exists a $\psi \in \mathscr{D}_\omega$ such that $\operatorname{supp}\psi \subset (-\theta, \theta)$ and $\psi(s) = 1$ for s in some neighbourhood of 0. Using the arguments from the proof of Corollary 2.2, one can easily verify that ψE is a continuous function, $\operatorname{supp}\psi E \subset (-\theta, \theta)$ and

$$q^\omega_{L,n}(\psi E) \leqslant q^\omega_{\sqrt{2}L,n}(\psi)\int_{-\infty}^{+\infty} \big|\omega\big(\sqrt{2}\,Lt\big)^{-2}\big|\,dt < +\infty.$$

Further, by Lemma 2.10 and Proposition 2.9, $(1-\psi)E \in \mathscr{E}_\omega$. Applying the ω-ultradifferential operator $\omega\big(i\sqrt{2}\,LD\big)^{n+2}$ to both sides of the equality $E = \psi E + (1-\psi)E$, we get

(5.8) $$2\pi\delta_0 = \omega\big(-i\sqrt{2}\,LD\big)^{n+2}(\psi E) + \omega\big(-i\sqrt{2}\,LD\big)^{n+1}\big((1-\psi)E\big);$$

hence the support of $\varphi = \omega\big(-i\sqrt{2}\,LD\big)^{n+1}\big((1-\psi)E\big) \in \mathscr{E}_\omega$ is contained in $(-\theta, \theta)$, that is

$$\varphi \in \mathscr{D}_\omega(-\theta, \theta).$$

If $s \in (c+\theta, d-\theta)$ and $\varepsilon \in \boldsymbol{R}$, $0 \neq |\varepsilon| < 1$, then $r \to \Phi(s+r+i\varepsilon)$ belongs to $\mathscr{E}_\omega(-\theta, \theta)$, so that (5.6) gives

$$\begin{aligned} 2\pi\Phi(s+i\varepsilon) &= \big[\omega\big(-i\sqrt{2}\,LD\big)^{n+2}(\psi E)\big]\,[\Phi(s+\cdot+i\varepsilon)] + \int_{-\theta}^{+\theta} \varphi(r)\Phi(s+r+i\varepsilon)\,dr \\ &= \int_{-\theta}^{+\theta} (\psi E)(r)\big[\omega\big(-i\sqrt{2}\,LD\big)^{n+2}\Phi(s+\cdot+i\varepsilon)\big](r)\,dr + \int_{-\theta}^{+\theta} \varphi(r)\Phi(s+r+i\varepsilon)\,dr. \end{aligned}$$

Define on $\{z \in \boldsymbol{C};\, c+\theta < \operatorname{Re} z < d-\theta,\, 0 \neq |\operatorname{Im} z| < 1\}$ an analytic function Ψ by the formula

$$\Psi(z) = \int_{-\infty}^{+\infty} (\psi E)(r)\Phi(r+z)\,dr.$$

Then by (5.7), for $s \in (c+\theta, b-\theta)$ and $\varepsilon \in (0, 1)$, we have

$$|\Psi(s \pm i\varepsilon)| = \Big|\int_c^d (\psi E)(r-s)\Phi(r \pm i\varepsilon)\,dr\Big| \leqslant \gamma q^\omega_{L,n}(\psi E)$$

and

$$\begin{aligned} |\Psi(s+i\varepsilon) - \Psi(s-i\varepsilon)| &= \Big|\int_c^d (\psi E)(r-s)\big(\Phi(r+i\varepsilon) - \Phi(r-i\varepsilon)\big)\,dr\Big| \\ &\leqslant \alpha(\varepsilon) q^\omega_{L,n}\big((\psi E)(\cdot - s)\big) = \alpha(\varepsilon) q^\omega_{L,n}(\psi E); \end{aligned}$$

hence by Lemma 5.9, Ψ has an analytic extension on $\{z \in \boldsymbol{C};\, c+\theta < \operatorname{Re} z < d-\theta,\, |\operatorname{Im} z| < 1\}$. It follows that $\omega\big(i\sqrt{2}\,LD\big)^{n+2}\Psi$ also has an analytic extension Φ_1 on $\{z \in \boldsymbol{C};\, c+\theta < \operatorname{Re} z < d-\theta,\, |\operatorname{Im} z| < 1\}$, and using Lemma 5.8 we get

$$\int_{-\theta}^{+\theta} (\psi E)(r)\big[\omega\big(i\sqrt{2}\,LD\big)^{n+2}\Phi(s+\cdot+i\varepsilon)\big](r)\,dr = \Phi_1(s+i\varepsilon),$$

for every $s \in (c+\theta, d-\theta)$ and $\varepsilon \in \boldsymbol{R}$, $0 \neq |\varepsilon| < 1$.

Further, we define on $\{z \in \boldsymbol{C};\, c+\theta < \operatorname{Re} z < d-\theta,\, 0 \neq |\operatorname{Im} z| < 1\}$ another analytic function Θ by the formula

$$\Theta(z) = \int_{-\theta}^{+\theta} \varphi(r)\Phi(r+z)\,dr.$$

By (5.7) and by Lemma 5.9, we infer as before that Θ has an analytic extension Φ_2 on $\{z \in \boldsymbol{C};\, c+\theta < \operatorname{Re} z < d-\theta;\, |\operatorname{Im} z| < 1\}$.

Since $\Phi(z) = \dfrac{1}{2\pi}\big(\Phi_1(z) + \Phi_2(z)\big)$ on $\{z \in \boldsymbol{C};\; c+\theta < \operatorname{Re} z < d-\theta,\; 0 \neq |\operatorname{Im} z| < 1\}$, we conclude that Φ can be extended analytically to $\{z \in \boldsymbol{C};\, a < \operatorname{Re} z < b,\, 0 \neq |\operatorname{Im} z| < +\infty\} \cup (c+\theta, d-\theta)$.

Finally, since $\theta > 0$, $c+\theta < d-\theta$ and $a < c < d < b$ are arbitrary, it follows that Φ can be extended analytically to $(\boldsymbol{C}\setminus\boldsymbol{R}) \cup (a, b)$. ∎

Recall that following M. Sato (see [54], [33]) an X-valued hyperfunction on $\boldsymbol{R}$ is an equivalence class of X-valued analytic functions on $\boldsymbol{C}\setminus\boldsymbol{R}$, where two X-valued analytic functions on $\boldsymbol{C}\setminus\boldsymbol{R}$ are equivalent if their difference has an entire extension. The equivalence class of an X-valued analytic function Φ on $\boldsymbol{C}\setminus\boldsymbol{R}$ will be denoted by $[\Phi]$ and Φ will be called a *defining function* for $[\Phi]$. If Ω is the greatest open subset of $\boldsymbol{C}$ on which Φ can be extended analytically, then $\boldsymbol{C}\setminus\Omega \subset \boldsymbol{R}$ is called the *support of the hyperfunction* $[\Phi]$.

The X-valued hyperfunctions on $\boldsymbol{R}$ form a linear space with the operations $[\Phi_1]+[\Phi_2] = [\Phi_1+\Phi_2]$, $\lambda[\Phi] = [\lambda\Phi]$, $\lambda \in \boldsymbol{C}$. Assume that ω is regular. By Theorems 5.2 and 5.10, every $F \in \mathscr{L}(\mathscr{D}_\omega; X)$ defines a unique X-valued hyperfunction $j(F)$ on $\boldsymbol{R}$ such that every defining function Φ for $j(F)$ belongs to $\mathscr{H}_\omega(X)$ and

$$\lim_{0<\varepsilon\to 0} \Phi(\cdot+i\varepsilon)-\Phi(\cdot-i\varepsilon) = F$$

in $\mathscr{L}(\mathscr{D}_\omega; X)$. Moreover, the support of $j(F)$ coincides with $\operatorname{supp} F$. Thus j is injective. But by Theorem 5.6 j is also surjective, and so we conclude that j is a linear isomorphism of $\mathscr{L}(\mathscr{D}_\omega; X)$ onto $\{[\Phi]; \Phi \in \mathscr{H}_\omega(X)\}$.

Let us denote by $\mathscr{H}^c_\omega(X)$ the linear space of all functions from $\mathscr{H}_\omega(X)$ which have analytical extensions on $\boldsymbol{C}\setminus$some compact subset of $\boldsymbol{R}$ and which vanish at ∞. Then, associating with each $F \in \mathscr{L}(\mathscr{E}_\omega; X)$ its Cauchy transform, we get a linear isomorphism k of $\mathscr{L}(\mathscr{E}_\omega; X)$ onto $\mathscr{H}^c_\omega(X)$. Clearly

$$j(F) = [k(F)], \quad F \in \mathscr{L}(\mathscr{E}_\omega; X).$$

We shall further give an intrinsic characterization of

$$\bigcup_{\omega_{\{t_k\}}\text{ regular}} \{[\Phi]; \Phi \in \mathscr{H}_{\omega_{\{t_k\}}}(X)\},$$

which can be considered as the union of all X-valued $\omega_{\{t_k\}}$-ultradistributions with regular $\omega_{\{t_k\}}$.

DEFINITION XXVI. We say that an X-valued analytic function Φ on $\boldsymbol{C}\setminus\boldsymbol{R}$ satisfies the *local Levinson condition* if for each $s_0 \in \boldsymbol{R}$ there exists an $\varepsilon_0 > 0$ such that defining $f^{\Phi}_{[s_0-\varepsilon_0, s_0+\varepsilon_0]}\colon [1, +\infty) \to [0, +\infty)$ by

$$f^{\Phi}_{[s_0-\varepsilon_0, s_0+\varepsilon_0]}(t) = \sup\{\|\Phi(z)\|;\ z \in \boldsymbol{C};\ s_0-\varepsilon_0 \leqslant \operatorname{Re} z \leqslant s_0+\varepsilon_0,\ t^{-1} \leqslant |\operatorname{Im} z| \leqslant 1\}$$

we have

$$\int_1^{+\infty} \frac{\ln_+\ln_+ f^{\Phi}_{[s_0-\varepsilon_0, s_0+\varepsilon_0]}(t)}{t^2}\,dt < +\infty.$$

It is easy to see that *Φ satisfies the local Levinson condition if and only if, for each compact set $K \subset \boldsymbol{R}$, defining $f^{\Phi}_K\colon [1, +\infty) \to [0, +\infty)$ by*

$$f^{\Phi}_K(t) = \sup\{\|\Phi(z)\|;\ z \in \boldsymbol{C},\ \operatorname{Re} z \in K,\ t^{-1} \leqslant |\operatorname{Im} z| \leqslant 1\},$$

we have

$$\int_1^{+\infty} \frac{\ln_+\ln_+ f^{\Phi}_K(t)}{t^2}\,dt < +\infty.$$

THEOREM 5.11. *Let Φ be an X-valued analytic function on $\boldsymbol{C}\setminus\boldsymbol{R}$. Then the following statements are equivalent*:

(i) *Φ satisfies the local Levinson condition*;

(ii) *there exist $t_1, t_2, \ldots > 0$, $t_1 < +\infty$, $\sum_{k=1}^{\infty} 1/t_k < +\infty$ such that $\omega_{\{t_k\}}$ is regular and $\Phi \in \mathscr{H}_\omega(X)$.*

Proof. Assume that (ii) is satisfied. Since $\omega_{\{t_k\}}$ is regular, by Theorem 3.20, there exist $0 < r_1 \leqslant r_2/2 \leqslant r_3/3 \leqslant \ldots$, $r_1 < +\infty$, $\sum_{k=1}^{\infty} 1/r_k < +\infty$, $L_0 > 0$, integer $n_0 \geqslant 1$ and $c_0 > 0$ such that

$$|\omega_{\{t_k\}}(t)| \leqslant c_0|\omega_{\{r_k\}}(L_0 t)^{n_0}|, \quad t \in \boldsymbol{R}.$$

Let $K \subset \boldsymbol{R}$ be a compact set. Since $\Phi \in \mathscr{H}_{\omega_{\{t_k\}}}(X)$, there exist $L > 0$, integer $n \geqslant 1$ and $c > 0$ such that

$$\|\Phi(z)\| \leqslant c(\omega_{\{t_k\}})^*_n\left(\frac{1}{|\operatorname{Im} z|}\right), \quad z \in \boldsymbol{C},\ \operatorname{Re} z \in K,\ 0 \neq |\operatorname{Im} z| < 1.$$

So, defining $f_K\colon [1, +\infty) \to (0, +\infty)$ by

$$f_K(t) = \sup\{\|\Phi(z)\|;\ z \in \boldsymbol{C};\ \operatorname{Re} z \in K,\ t^{-1} < |\operatorname{Im} z| < 1\},$$

we have

$$f_K(t) \leqslant c(\omega_{\{t_k\}})^*_n(Lt) \leqslant \frac{cc_0^n}{L_0}(\omega_{\{r_k\}})^*_{nn_0}(LL_0 t), \quad t > 0.$$

Consider the sequence $0 < s_1 \leqslant s_1/1 \leqslant s_2/2 \leqslant s_3/3 \leqslant \ldots$, $\sum_{k=0}^{\infty} 1/s_k < +\infty$, defined by

$$s_k = \frac{k}{p} r_p, \quad nn_0(p-1) < k \leqslant nn_0 p.$$

Then

$$f_K(t) \leqslant \frac{cc_0^n}{L_0 nn_0}(\omega_{\{s_k\}})^*_1(LL_0 nn_0 t), \quad t > 0;$$

so by Theorem 1.10

$$\int_1^{+\infty} \frac{\ln_+\ln_+ f_K(t)}{t^2}\,dt < +\infty.$$

Conversely, assume that (i) is satisfied. Then by Theorem 1.10 for each integer $m \geqslant 1$ there exist

$$0 < t_{m,1} \leqslant t_{m,2}/2 \leqslant t_{m,3}/3 \leqslant \ldots, \quad t_{m,1} < +\infty, \quad \sum_{k=1}^{\infty} 1/t_{m,k} = 4^{-m}, \quad L_m > 0$$

and $c_m > 0$ such that

$$\|\Phi(z)\| \leqslant c_m \cdot \sup_{k\geqslant 1} \frac{k!}{t_{m,1}\ldots t_{m,n}} \cdot \left(\frac{L_m}{|\operatorname{Im} z|}\right)^{k+1}, \quad z \in \boldsymbol{C},\ |\operatorname{Re} z| \leqslant m,\ 0 < |\operatorname{Im} z| < 1.$$

For each $m \geqslant 1$ we define $\alpha_m\colon [0, +\infty) \to [1, +\infty)$ by

$$\alpha_m(t) = 1+\sum_{k=1}^{\infty} \frac{t^k}{t_{m,1}\ldots t_{m,k}}, \quad t \geqslant 0.$$

By Lemma 1.7, α_m is submultiplicative. Moreover, for each $k \geqslant 1$ we get successively

$$\frac{r^k}{t_{m,1} \dots t_{m,k}} \leqslant \alpha_m(r), \quad r > 0,$$
$$\frac{k!\, t^{k+1}}{t_{m,1} \dots t_{m,k}} \leqslant (\alpha_m)_{\mathrm{Borel}}(t), \quad t > 0;$$

hence

$$\|\Phi(z)\| \leqslant c_m (\alpha_m)_{\mathrm{Borel}}\left(\frac{L_m}{|\mathrm{Im}\, z|}\right), \quad z \in \boldsymbol{C},\ |\mathrm{Re}\, z| \leqslant m,\ 0 < |\mathrm{Im}\, z| < 1.$$

Now for each $m \geqslant 1$

$$1 \leqslant \alpha_m(t) \leqslant (1+2^{-m+1})|\omega_{\{t_{m,k}\}}(2^m t)|, \quad t \geqslant 0;$$

hence we can define a continuous submultiplicative function $\alpha\colon [0, +\infty) \to [1, +\infty)$ by

$$\alpha(t) = \sum_{m=1}^{\infty} \alpha_m(t), \quad t \geqslant 0.$$

Denoting the set

$$\frac{t_{1,1}}{2}, \frac{t_{1,2}}{2}, \dots, \frac{t_{2,1}}{2^2}, \frac{t_{2,2}}{2^2}, \dots, \frac{t_{3,1}}{2^3}, \frac{t_{3,2}}{2^3}, \dots,$$

simply by $r_1, r_2, \dots$, we have

$$\sum_{k=1}^{\infty} \frac{1}{r_k} = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{2^m}{t_{m,k}} = 1 < +\infty$$

and

$$\alpha(t) \leqslant \prod_{m=1}^{\infty} (1+2^{-m+1})|\omega_{\{r_k\}}(t)|, \quad t \geqslant 0,$$

so that by Theorem 1.6

$$\int_1^{+\infty} \frac{\ln \alpha(t)}{t^2}\, dt < +\infty.$$

Finally by Theorem 1.8 there exist $0 < t_1 \leqslant t_2/2 \leqslant t_3/3 \leqslant \dots$, $t_1 < +\infty$, $\sum_{k=1}^{\infty} 1/t_k < +\infty$ and $c > 0$ such that

$$\alpha(r) \leqslant c|\omega_{\{t_k\}}(r)|, \quad r > 0;$$

thus, for each $m \geqslant 1$

$$(\alpha_m)_{\mathrm{Borel}}(t) \leqslant \alpha_{\mathrm{Borel}}(t) \leqslant c|\omega_{\{t_k\}}|_{\mathrm{Borel}}(t) = c(\omega_{\{t_k\}})_1^*(t), \quad t > 0.$$

Consequently, for every $m \geqslant 1$

$$\|\Phi(z)\| \leqslant c_m c(\omega_{\{t_k\}})_1^*\left(\frac{L_m}{|\mathrm{Im}\, z|}\right), \quad z \in \boldsymbol{C},\ |\mathrm{Re}\, z| \leqslant m,\ 0 < |\mathrm{Im}\, z| \leqslant 1,$$

that is $\Phi \in \mathscr{H}_{\omega_{\{t_k\}}}(X)$. ■

We end this section with the examination of the regularity of several classes of functions $\omega_{\{t_k\}}$.

Lemma 5.12. *Let $\{t_k\} \subset (0, +\infty]$, $t_1 < +\infty$, $\sum_{k=1}^{\infty} 1/t_k < +\infty$ be such that $\sum_{k=1}^{\infty} \frac{\ln t_k}{t_k} < +\infty$ and let $\omega = \omega_{\{t_k\}}$. Then, for each integer $n \geqslant 1$ we have*

$$(5.9) \qquad (k!\, c_k^{\omega,n})^2 \geqslant \big((k-1)!\, c_{k-1}^{\omega,n}\big)\big((k+1)!\, c_{k+1}^{\omega,n}\big), \quad k \geqslant 1;$$

$$(5.10) \qquad \sum_{k=1}^{\infty} \frac{c_k^{\omega,n}}{c_{k-1}^{\omega,n}} < +\infty.$$

Proof. Inequality (5.9) is an immediate consequence of Proposition 2.8.

By Corollary 1.9, there exist $0 < s_1 \leqslant s_2 \leqslant \dots$, $\sum_{n=1}^{\infty} 1/s_n < +\infty$ and $c > 0$ such that

$$\omega^n(-it) \leqslant c \sup_{p \geqslant 1} \frac{t^p}{s_1 \dots s_p}, \quad t \in [s_1, +\infty).$$

For each integer $k \geqslant 1$

$$c_k^{\omega,n} \leqslant \frac{\omega^n(-is_k)}{s_k^k} \leqslant \frac{c}{s_k^k} \sup_{p \geqslant 1} \frac{s_k^p}{s_1 \dots s_p} = \frac{c}{s_1 \dots s_p};$$

so, using the inequality of T. Carleman (see [45], Lemma 1.8.VI, or [48], Ch. XVI, §§ 4, 5, 6), we deduce

$$\sum_{k=1}^{\infty} \frac{c_k^{\omega,n}}{c_{k-1}^{\omega,n}} \leqslant \sum_{k=1}^{\infty} \left(\frac{c_1^{\omega,n}}{c_0^{\omega,n}} \dots \frac{c_k^{\omega,n}}{c_{k-1}^{\omega,n}}\right)^{1/k}$$
$$= \sum_{k=1}^{\infty} (c_k^{\omega,n})^{1/k} \leqslant c \sum_{k=1}^{\infty} \frac{1}{(s_1 \dots s_k)^k} \leqslant ce \sum_{k=1}^{\infty} \frac{1}{s_k} < +\infty. \ ■$$

Proposition 5.13. *Let $\{t_k\} \subset (0, +\infty]$, $t_1 < +\infty$, $\sum_{k=1}^{\infty} 1/t_k < +\infty$ be such that $\sum_{k=1}^{\infty} \frac{\ln t_k}{t_k} < +\infty$. Then there exist $0 < r_1 \leqslant r_2/2 \leqslant r_3/3 \leqslant \dots$, $r_1 < +\infty$, $\sum_{k=1}^{\infty} 1/r_k < +\infty$ such that $\mathscr{D}_{\omega_{\{r_k\}}} \subset \mathscr{D}_{\omega_{\{t_k\}}}$.*

Proof. We denote $\omega_{\{t_k\}}$ simply by ω. Putting

$$r_k = \frac{c_{k-1}^{\omega,n}}{c_k^{\omega,n}}, \qquad k \geqslant 1,$$

by Lemma 5.12 we have

$$0 < r_1 \leqslant r_2/2 \leqslant r_3/3 \leqslant \ldots, \quad r_1 < +\infty, \quad \sum_{k=1}^{\infty} 1/r_k < +\infty.$$

For each $t \in \boldsymbol{R}$

$$|\boldsymbol{\omega}(t)| \leqslant \omega(-i|t|) = \sum_{k=0}^{\infty} 2^{-k} c_k^{\omega,1}(2|t|)^k \leqslant 2\max\left\{1, \sup \frac{(2|t|)^k}{r_1 \ldots r_k}\right\} \leqslant 2|\omega_{\{r_k\}}(2t)|,$$

and the statement results from Theorem 3.20. ∎

We remark that by this proposition every $\omega_{\{t_k\}}$-ultradistribution with $\sum_{k=1}^{\infty} \frac{\ln t_k}{t_k} < +\infty$ belongs to the "union" of all $\omega_{\{r_k\}}$-ultradistributions with regular $\omega_{\{r_k\}}$. Note that the converse inclusion is not true. For example, if $t_1 = 1$, $t_2 = 2, \ldots,$ $t_k = k(\ln k)^\varepsilon$ for $k \geqslant 3$ and $1 < \varepsilon \leqslant 2$, then $\omega_{\{t_k\}}$ is regular, but $\sum_{k=1}^{\infty} \frac{\ln t_k}{t_k} = +\infty$.

PROPOSITION 5.14. *Let* $\{t_k\} \subset (0, +\infty]$, $t_1 < +\infty$, $\sum_{k=1}^{\infty} 1/t_k < +\infty$ *be such that* $\omega_{\{t_k\}}$ *satisfies the strong non-quasianalyticity condition. Then there exists an* $0 < r_1 \leqslant r_2/2 \leqslant r_3/3 \leqslant \ldots, r_1 < +\infty$, $\sum_{k=1}^{\infty} 1/r_k < +\infty$ *such that* $\mathscr{D}_{\omega_{\{r_k\}}} = \mathscr{D}_{\omega_{\{t_k\}}}$.

Proof. We denote again $\omega_{\{t_k\}}$ simply by ω and we define

$$r_k = \frac{c_{k-1}^{\omega,1}}{c_k^{\omega,1}}, \qquad k \geqslant 1.$$

As in the proof of Proposition 4.13, we have $0 < r_1 \leqslant r_2/2 \leqslant r_3/3 \leqslant \ldots, r_1 < +\infty$, $\sum_{k=1}^{\infty} 1/r_k < +\infty$ and $\mathscr{D}_{\omega_{\{r_k\}}} \subset \mathscr{D}_{\omega_{\{t_k\}}}$.

On the other hand, there exist an $L_0 > 0$, and integer $n_0 \geqslant 1$ and a $c_0 > 0$ such that

$$\omega(-it) \leqslant c_0|\omega(L_0 t)^{n_0}|, \qquad t \geqslant 0.$$

Hence, using Lemma 1.7, for every $t \in \boldsymbol{R}$ we have

$$|\omega_{\{r_k\}}(t)| \leqslant 3\max\{1, (\sup_{k\geqslant 1} c_k^{\omega,1}(4|t|)^k)^2\} \leqslant 3\omega(-4i|t|)^2 \leqslant 3c_0^2|\omega(4L_0 t)^{2n_0}|.$$

By Theorem 3.20 we conclude that $\mathscr{D}_{\omega_{\{t_k\}}} \subset \mathscr{D}_{\omega_{\{r_k\}}}$. ∎

We note that if $\mathscr{D}_{\omega_{\{r_k\}}} = \mathscr{D}_{\omega_{\{t_k\}}}$ and $\omega_{\{t_k\}}$ satisfies the strong non-quasianalyticity condition, then by Theorem 3.20 and Theorem 1.2, $\omega_{\{r_k\}}$ also satisfies the strong non-quasianalyticity condition.

If $\{t_k/k\}_{k\geqslant 1}$ is increasing, then the strong non-quasianalyticity of $\omega_{\{t_k\}}$ can be characterized in terms of the sequence $\{t_k\}$. More precisely, we have

PROPOSITION 5.15. *Let* $0 < t_1 \leqslant t_2/2 \leqslant t_3/3 \leqslant \ldots$, $t_1 < +\infty$, $\sum_{k=1}^{\infty} 1/t_k < +\infty$. *Then the following statements are equivalent:*

(i) $\omega_{\{t_k\}}$ *satisfies the strong non-quasianalyticity condition;*

(ii) *there exist* $L > 0$, *integer* $n \geqslant 1$ *and* $d > 0$ *such that*

$$\omega_{\{t_k\}}(-it) \leqslant d\max\left\{1, \left(\sup_{k\geqslant 1} \frac{(Lt)^k}{t_1 \ldots t_k}\right)^n\right\}, \qquad t > 0;$$

(iii) *there exist* $c > 0$ *such that*

$$\sum_{p=k}^{\infty} \frac{1}{t_p} \leqslant c\frac{k}{t_k}\left(1 + \ln\frac{t_k}{(t_1 \ldots t_k)^{1/k}}\right), \qquad k \geqslant 1.$$

Proof. We shall denote $\omega_{\{t_k\}}$ simply by ω.

For each $\lambda > 0$ let $n(\lambda)$ be the number of all t_k with $t_k \leqslant \lambda$ and write also

$$N(t) = \ln\max\left\{1, \sup_{k\geqslant 1} \frac{t^k}{t_1 \ldots t_k}\right\}, \qquad t > 0.$$

Recall that (see [45], 1.8, or [34], §§ 3, 4, or the proof of Lemma 1.7)

$$\lim_{\lambda\to+\infty} \frac{n(\lambda)}{\lambda} = 0,$$

$$N(t) = \int_0^t \frac{n(\lambda)}{\lambda} d\lambda, \qquad t > 0.$$

Integrating by parts and using the above relations, we get for all $t > 0$

$$\ln\omega(-it) = \int_0^{+\infty} \ln\left(1 + \frac{t}{\lambda}\right) dn(\lambda) = t\int_0^{+\infty} \frac{n(\lambda)}{\lambda(\lambda+t)} d\lambda.$$

Using the inequalities

$$0 \leqslant t\int_0^t \frac{n(\lambda)}{\lambda(\lambda+t)} d\lambda \leqslant \int_0^t \frac{n(\lambda)}{\lambda} d\lambda = N(t),$$

$$\frac{1}{2} t\int_t^{+\infty} \frac{n(\lambda)}{\lambda^2} d\lambda \leqslant t\int_t^{+\infty} \frac{n(\lambda)}{\lambda(\lambda+t)} d\lambda \leqslant t\int_t^{+\infty} \frac{n(\lambda)}{\lambda^2} d\lambda,$$

we obtain

$$\frac{1}{2} t\int_t^{+\infty} \frac{n(\lambda)}{\lambda^2} d\lambda \leqslant \ln\omega(-it) \leqslant N(t) + t\int_t^{+\infty} \frac{n(\lambda)}{\lambda^2} d\lambda, \qquad t > 0. \tag{5.11}$$

On the other hand, again by partial integration, we obtain for all $t > 0$

$$t\int_0^t \frac{n(\lambda)}{\lambda^2}d\lambda = n(t)+t\int_0^t \frac{1}{\lambda}\,dn(\lambda) = n(t)+t\sum_{t_p>t}\frac{1}{t_p}.$$

Since

$$n(t) = n(t)\int_t^{et}\frac{d\lambda}{\lambda} \leqslant \int_t^{et}\frac{n(\lambda)}{\lambda}d\lambda \leqslant \int_0^{et}\frac{n(\lambda)}{\lambda}d\lambda = N(et),$$

it follows that

$$t\sum_{t_p>t}\frac{1}{t_p} \leqslant t\int_t^{+\infty}\frac{n(\lambda)}{\lambda^2}d \leqslant N\lambda(et)+t\sum_{t_p>t}\frac{1}{t_p}, \qquad t>0. \tag{5.12}$$

From (5.11) and (5.12) we deduce

$$\frac{1}{2}t\sum_{t_p>t}\frac{1}{t_p} \leqslant \ln\omega(-it) \leqslant 2N(et)+t\sum_{t_p>t}\frac{1}{t_p}, \qquad t>0. \tag{5.13}$$

We shall prove further the implication (ii) ⇒ (iii).

Let q be the integer with $q-1 < L \leqslant q$. Since

$$\omega(-it) \leqslant \prod_{p=1}^{\infty}\left(1+\frac{L^{-1}t}{t_p}\right)^q \leqslant d^q\max\left\{1,\left(\sup_{p\geqslant 1}\frac{t^p}{t_1\ldots t_p}\right)^{nq}\right\}, \qquad t>0,$$

by (5.13) it follows that for each integer $k \geqslant 1$ and $0 < t < t_k$, we have

$$t\sum_{p=k}^{\infty}\frac{1}{t_p} \leqslant 2\mathrm{l}(\mathrm{n}\omega-it)$$

$$\leqslant 2q\ln d+2\max\left\{0,\ nq\ln\sup_{p\geqslant 1}\frac{t_k^p}{t_1\ldots t_p}\right\}$$

$$2q\ln q+2nq = k\ln\frac{t_k}{(t_1\ldots t_k)^{1/k}}$$

$$\leqslant 2nq(1+\ln d)k\left(1+\ln\frac{t_k}{(t_1\ldots t_k)^{1/k}}\right).$$

Denoting $c = 2nq(1+\ln d)$ and letting $t \to t_k$, we obtain (iii).

Next we prove that (iii) ⇒ (ii).

Defining β: $(0,+\infty) \to [1,+\infty)$ by

$$\beta(t) = \sup_{s>t}\left(1+\sum_{k=1}^{\infty}\frac{s^k}{t_1\ldots t_k}\right)^{t/s}, \qquad t>0,$$

we infer, by Lemma 1.7, that $t \to \dfrac{\ln\beta(t)}{t}$ is decreasing and

$$N(t) \leqslant \ln\beta(t) \leqslant \ln 4+2N(2t), \qquad t>0.$$

Consequently, for each $t > 0$, writing $k = n(t)+1$, we have

$$t\sum_{t_p>t}\frac{1}{t_p} = \frac{t}{t_k}t_k\sum_{p=k}^{\infty}\frac{1}{t_p} \leqslant \frac{t}{t_k}ck\left(1+\ln\frac{t_k}{(t_1\ldots t_k)^{1/k}}\right)$$

$$= \frac{t}{t_k}ck+ct\frac{N(t_k)}{t_k} \leqslant ck+ct\frac{\ln\beta(t_k)}{t_k}$$

$$\leqslant c(n(t)+1)+c\ln\beta(t) \leqslant c(N(et)+1)+c(\ln 4+2N(2t))$$

$$\leqslant 3cN(et)+3c.$$

By (5.13), it is clear that

$$\ln\omega(-it) \leqslant (3c+2)N(et)+3c, \qquad t>0;$$

so (ii) holds with $L = e$, $n \geqslant 3c+2$ and $d = e^{3c}$.

Finally, (i) ⇒ (ii) is an immediate consequence of Lemma 1.7. ∎

Concerning the implication (iii) ⇒ (ii), we note that if there exists a $c > 0$ such that

$$\sum_{p=k}^{\infty}\frac{1}{t_p} \leqslant c\frac{k}{t_k}, \qquad k \geqslant 1,$$

then (ii) holds under the assumption that $0 < t_1 \leqslant t_2 \leqslant t_3 \leqslant \ldots$ (cf. [34], Proposition 4.6). Indeed, for every $t > 0$, writing $k = n(t)+1$ and using (5.13), we get

$$\ln\omega(-it) \leqslant 2N(et)+t_k\sum_{p=k}^{\infty}\frac{1}{t_p} \leqslant 2N(et)+ck \leqslant (c+2)N(et)+c.$$

We end our considerations of strong non-quasianalyticity with some examples.

Firstly, if $t_k = k^{\varepsilon}$, $\varepsilon > 1$ (the Gevrey sequence), then $\omega_{\{t_k\}}$ satisfies the strong non-quasianalyticity condition. Indeed,

$$\sum_{p=k}^{\infty}\frac{1}{t_p} \leqslant \int_{k-1}^{+\infty}\frac{d\lambda}{\lambda^{\varepsilon}} = \frac{1}{(\varepsilon-1)(k-1)^{\varepsilon-1}} \leqslant \frac{2^{\varepsilon}}{\varepsilon-1}\frac{k}{t_k}, \qquad k \geqslant 2.$$

Secondly, if $t_1 = 1$, $t_2 = 2$ and $t_k = k(\ln k)^{\varepsilon}$ for $k \geqslant 3$, $\varepsilon > 1$, then $\omega_{\{t_k\}}$ does not satisfy the strong non-quasianalyticity condition.

Indeed, for $k \geqslant 3$, we have

$$t_k\sum_{p=k}^{\infty}\frac{1}{t_p} \geqslant k(\ln k)^{\varepsilon}\int_k^{+\infty}\frac{d\lambda}{\lambda(\ln\lambda)^{\varepsilon}} = \frac{k\ln k}{\varepsilon-1},$$

$$\frac{t_k}{(t_1\ldots t_k)^{1/k}} \leqslant \frac{k(\ln k)^{\varepsilon}}{(k!)^{1/k}} \leqslant e(\ln k)^{\varepsilon},$$

so

$$\lim_{k\to\infty}\frac{t_k\sum_{p=k}^{\infty}1/t_p}{k\left(1+\ln\frac{t_k}{(t_1\ldots t_k)^{1/k}}\right)}\geqslant\lim_{k\to\infty}\frac{\ln k}{(\varepsilon-1)(2+\varepsilon\ln\ln k)}=+\infty.$$

Since for $\varepsilon>2$ we have

$$\sum_{k=1}^{\infty}\frac{\ln t_k}{t_k}<+\infty,$$

it follows that the above condition does not imply strong non-quasianalyticity, even if $0<t_1\leqslant t_2/2\leqslant t_3/3\leqslant\ldots$

6. ω-self-adjoint operators

Also in this section $0<t_1\leqslant t_2\leqslant t_3\leqslant\ldots,\ t_1<+\infty,\ \sum_{k=1}^{\infty}1/t_k<+\infty$, will be fixed and $\omega_{\{t_k\}}$ will be denoted simply by ω. We also fix a Banach space X.

The following definitions are inspired by [24], [44] and [13]:

DEFINITION XXVII. By a *spectral ω-ultradistribution* we mean any $E\in\mathscr{L}(\mathscr{D}_\omega;\mathscr{L}(X))$ satisfying the conditions

(i) $E(\varphi\psi)=E(\varphi)E(\psi),\ \varphi,\psi\in\mathscr{D}_\omega$;

(ii) $\{x\in X;\ E(\varphi)x=0$ for all $\varphi\in\mathscr{D}_\omega\}=\{0\}$;

(iii) $\bigcup_{\varphi\in\mathscr{D}_\omega}E(\varphi)X$ is dense in X.

We say that a closed linear operator T in X is *ω-self-adjoint* if the spectrum $\sigma(T)$ of T is contained in $\boldsymbol{R}$ and there exists a spectral ω-ultradistribution $E\in\mathscr{L}(\mathscr{D}_\omega;\mathscr{L}(X))$ such that

(iv) defining $\varphi_1\in\mathscr{E}_\omega$ by $\varphi_1(s)=s,\ s\in\boldsymbol{R}$, we have for every $\varphi\in\mathscr{D}_\omega$

$$E(\varphi)X\subset D(T),\quad E(\varphi)T\subset TE(\varphi)=E(\varphi_1\varphi).$$

We remark that, if T is ω-self-adjoint, then by (iv), $D(T)\supset\bigcup_{\varphi\in\mathscr{D}_\omega}E(\varphi)X$, and so by (iii) T is densely defined.

It is easy to see that in the above definition (iv) can be replaced by

(iv') for every $\varphi\in\mathscr{D}_\omega$ and $z\in\boldsymbol{C}\setminus\boldsymbol{R}$ we have

$$E(\varphi)R(z;T)=R(z;T)E(\varphi)=E\left(\frac{\varphi}{z-\cdot}\right).$$

PROPOSITION 6.1. *Let T be an ω-self-adjoint operator. Then there exists a unique spectral ω-ultradistribution E satisfying condition* (iv).

Proof. Let E_1,E_2 be a spectral ω-ultradistributions satisfying (iv). Then we have for every $\psi,\theta\in\mathscr{D}_\omega$

$$E_1(\psi\varphi_1)E_2(\theta)=E_1(\psi)TE_2(\theta)=E_1(\psi)E_2(\varphi_1\theta).$$

By induction, it follows for every $\psi,\theta\in\mathscr{D}_\omega$ and integer $m\geqslant0$ that

$$E_1(\psi\varphi_1^m)E_2(\theta)=E_1(\psi)E_2(\varphi_1^m\theta).$$

Using Theorem 2.13, we deduce

$$E_1(\psi\varphi)E_2(\theta)=E_1(\psi)E_2(\varphi\theta),\qquad\varphi\in\mathscr{E}_\omega,\ \psi,\theta\in\mathscr{D}_\omega;$$

so by condition (i) we have

$$E_1(\psi)E_1(\varphi)E_2(\theta)=E_1(\psi)E_2(\varphi)E_2(\theta),\qquad\varphi,\psi,\theta\in\mathscr{D}_\omega.$$

Now, by conditions (ii) and (iii) we conclude that

$$E_1(\varphi)=E_2(\varphi),\qquad\varphi\in\mathscr{D}_\omega.\ \blacksquare$$

By Proposition 6.1, we can give the following

DEFINITION XXVIII. For every ω-self-adjoint operator T the spectral ω-ultradistribution satisfying condition (iv) is called the *spectral ω-ultradistribution associated with T.*

Further, we point out how the spectral ω-ultradistribution associated with some ω-self-adjoint operator T can be obtained from T:

PROPOSITION 6.2. *Let T be an ω-self-adjoint operator and E the spectral ω-ultradistribution associated with T. Then, for each $\psi\in\mathscr{D}_\omega$ the function*

$$\boldsymbol{C}\setminus\boldsymbol{R}\ni z\mapsto E(\psi)R(z;T)\in\mathscr{L}(X)$$

belongs to $\mathscr{H}_\omega(\mathscr{L}(X))$ and we have in $\mathscr{L}(\mathscr{D}_\omega;\mathscr{L}(X))$

$$E(\psi)E=\lim_{0<\varepsilon\to0}\frac{1}{2\pi i}\big(E(\psi)R(\cdot-i\varepsilon;T)-E(\psi)R(\cdot+i\varepsilon;T)\big).$$

Proof. Let us consider $F\in\mathscr{L}(\mathscr{D}_\omega;\mathscr{L}(X))$ defined by

$$F(\varphi)=E(\psi)E(\varphi)=E(\psi\varphi),\qquad\varphi\in\mathscr{D}_\omega.$$

Then $\operatorname{supp}F\subset\operatorname{supp}\psi$ is compact and the Cauchy transform Φ of F is defined by

$$\Phi(z)=\frac{1}{2\pi i}F\left(\frac{1}{\cdot-z}\right)=\frac{1}{2\pi i}E\left(\frac{\psi}{\cdot-z}\right)=-\frac{1}{2\pi i}E(\psi)R(z;T),\qquad z\in\boldsymbol{C}\setminus\operatorname{supp}F.$$

Thus our statement directly results from Proposition 5.1. ∎

COROLLARY 6.3. *If T is an ω-self-adjoint operator and E is the spectral ω-ultradistribution associated with T, then*

$$\operatorname{supp}E\subset\sigma(T).$$

Proof. Let $\varphi\in\mathscr{D}_\omega$, $\sigma(T)\cap\operatorname{supp}\varphi=\emptyset$. For every $\psi\in\mathscr{D}_\omega$ the $\mathscr{L}(X)$-valued function $z\mapsto E(\psi)R(z;T)$ is analytical on $\boldsymbol{C}\setminus\sigma(T)$, and so by Proposition 6.2 we have $E(\psi)E(\varphi)=0$. Using condition (ii) from Definition XXVII, we infer that $E(\varphi)=0$. ∎

The boundedness of ω-self-adjoint operators can be characterized in terms of the associated spectral ω-ultradistributions by:

Corollary 6.4. *Let T be an ω-self-adjoint operator and E the spectral ω-ultradistribution associated with T. Then the following statements are equivalent*:

(i) $T \in \mathscr{L}(X)$;

(ii) $\sigma(T)$ *is compact*;

(iii) $\operatorname{supp} E$ *is compact.*

Moreover, if the above conditions are satisfied, then E can be considered as an element of $\mathscr{L}(\mathscr{E}_\omega; \mathscr{L}(X))$, and defining $\varphi_0, \varphi_1 \in \mathscr{E}_\omega$ by

$$\varphi_0(s) = 1, \quad \varphi_1(s) = s, \quad s \in \boldsymbol{R},$$

we have

$$E(\varphi_0) = I_X, \quad E(\varphi_1) = T. \tag{6.1}$$

Proof. It is clear that (i) $\Rightarrow$ (ii). By Corollary 6.3, we have also the implication (ii) $\Rightarrow$ (iii).

Assume further that (iii) is satisfied. Then we can consider that $E \in \mathscr{L}(\mathscr{E}_\omega; \mathscr{L}(X))$ and then, for every $\varphi, \psi \in \mathscr{E}_\omega$, we have

$$E(\varphi\psi) = E(\varphi)E(\psi).$$

Hence

$$E(\varphi_0)E(\varphi)x = E(\varphi_0\varphi)x = E(\varphi)x, \quad \varphi \in \mathscr{D}_\omega,\ x \in X,$$

so that

$$E(\varphi_0) = I_X.$$

Moreover, from

$$E(\varphi_1)E(\varphi)x = E(\varphi_1\varphi)x = TE(\varphi)x, \quad \varphi \in \mathscr{D}_\omega,\ x \in X,$$

we get

$$E(\varphi_1) = T.$$

In particular, $T \in \mathscr{L}(X)$. ■

It is easy to see that $T \in \mathscr{L}(X)$ *is ω-self-adjoint if and only if there exists an* $E \in \mathscr{L}(\mathscr{E}_\omega; \mathscr{L}(X))$ *satisfying condition* (i) *from Definition* XXVII *and* (6.1). In this case E is the spectral ω-ultradistribution associated with T.

We remark that $\mathscr{E}_\omega$ is an admissible algebra in the sense of I. Colojoară and C. Foiaş (see [16], Ch. 3, Definition 1.2). Thus the above statement is equivalent to the following one:

$T \in \mathscr{L}(X)$ is ω-self-adjoint if and only if it is $\mathscr{E}_\omega$-self-adjoint (see [16], Ch. 5, Definition 4.1).

Using our results from § 5, we shall next give a sufficient condition in order that an operator T be ω-self-adjoint. For this purpose we first prove some preliminary results:

Lemma 6.5. *Let us assume that ω is regular and let T be a closed linear operator in X such that $\sigma(T) \subset \boldsymbol{R}$ and*

$$\boldsymbol{C}\setminus\boldsymbol{R} \ni z \mapsto R(z; T) \in \mathscr{L}(X)$$

belongs to $\mathscr{H}_\omega(\mathscr{L}(X))$. Then

$$E = \lim_{0<\varepsilon\to 0} \frac{1}{2\pi i}\big(R(\cdot - i\varepsilon; T) - R(\cdot + i\varepsilon; T)\big)$$

exists in $\mathscr{L}(\mathscr{D}_\omega; \mathscr{L}(X))$ and we have:

$$E(\varphi\psi) = E(\varphi)E(\psi), \quad \varphi, \psi \in \mathscr{D}_\omega; \tag{6.2}$$

$$E(\varphi)R(z; T) = R(z; T)E(\varphi) = E\left(\frac{\varphi}{z - \cdot}\right), \quad \varphi \in \mathscr{D}_\omega,\ z \in \boldsymbol{C}\setminus\boldsymbol{R}; \tag{6.3}$$

$$\operatorname{supp} E = \sigma(T). \tag{6.4}$$

Proof. By Theorem 5.6, the limits

$$E_- = \lim_{0<\varepsilon\to 0} \frac{1}{2\pi i} R(\cdot - i\varepsilon; T),$$

$$E_+ = \lim_{0<\varepsilon\to 0} \frac{1}{2\pi i} R(\cdot + i\varepsilon; T)$$

exist in $\mathscr{L}(\mathscr{D}_\omega; \mathscr{L}(X))$; hence also the limit

$$E = \lim_{0<\varepsilon\to 0} \frac{1}{2\pi i}\big(R(\cdot - i\varepsilon; T) - R(\cdot + i\varepsilon; T)\big)$$

exists in $\mathscr{L}(\mathscr{D}_\omega; \mathscr{L}(X))$ and we have

$$E = E_- - E_+.$$

By Theorem 5.10, it follows immediately that (6.4) holds.

Let us now prove (6.3). Let $\varphi \in \mathscr{D}_\omega$ and $z \in \boldsymbol{C}\setminus\boldsymbol{R}$; we write $\psi = \varphi/(z - \cdot) \in \mathscr{D}_\omega$. Using the resolvent equation, for every $\varepsilon > 0$ and $s \in \boldsymbol{R}$ we get

$$\begin{aligned}
&R(s - i\varepsilon; T) - R(s + i\varepsilon; T) \\
&\quad = [R(s - i\varepsilon; T) - R(z; T)] + [R(z; T) - R(s + i\varepsilon; T)] \\
&\quad = (z - s + i\varepsilon)R(z; T)R(s - i\varepsilon; T) + (s + i\varepsilon - z)R(z; T)R(s + i\varepsilon; T) \\
&\quad = R(z; T)\big[(z - s)\big(R(s - i\varepsilon; T) - R(s + i\varepsilon; T)\big)\big] + \\
&\qquad + i\varepsilon R(z; T)\big(R(s - i\varepsilon; T) + R(s + i\varepsilon; T)\big).
\end{aligned}$$

Since

$$\lim_{0<\varepsilon\to 0} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \psi(s)\big(R(s - i\varepsilon; T) + R(s + i\varepsilon; T)\big)\,ds$$

exists in $\mathscr{L}(X)$ (actually it is $E_-(\psi) + E_+(\psi)$), we deduce

$$\begin{aligned}
E(\psi) &= R(z; T)\left[\lim_{0<\varepsilon\to 0} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \psi(s)(z - s)\big(R(s - i\varepsilon; T) - R(s + i\varepsilon; T)\big)\,ds\right. \\
&= R(z; T)E(\varphi).
\end{aligned}$$

By the definition of E, $R(z;T)$ commutes with $E(\varphi)$, so that we obtain

$$E(\psi) = R(z;T)E(\varphi) = E(\varphi)R(z;T).$$

Finally, let $\varphi, \psi \in \mathscr{D}_\omega$. We define $F \in \mathscr{L}(\mathscr{D}_\omega; \mathscr{L}(X))$ by the formula

$$F(\theta) = E(\varphi\theta), \quad \theta \in \mathscr{D}_\omega.$$

Since $\operatorname{supp} F \subset \operatorname{supp} \varphi$ is compact, using Proposition 5.1 and (6.3), we get

$$\begin{aligned} E(\varphi\psi) &= F(\psi) \\ &= \lim_{0<\varepsilon\to 0} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \psi(s)\left[F\left(\frac{1}{s-i\varepsilon-\cdot}\right) - F\left(\frac{1}{s+i\varepsilon-\cdot}\right)\right] ds \\ &= \lim_{0<\varepsilon\to 0} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \psi(s)\left[E\left(\frac{\varphi}{s-i\varepsilon-\cdot}\right) - E\left(\frac{\varphi}{s+i\varepsilon-\cdot}\right)\right] ds \\ &= E(\varphi)\left[\lim_{0<\varepsilon\to 0} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \psi(s)\big(R(s-i\varepsilon;T) - R(s+i\varepsilon;T)\big) ds\right] \\ &= E(\varphi)E(\psi). \end{aligned}$$

Thus also (6.2) is proved. ■

LEMMA 6.6. *Let T be a closed linear operator in X for which $\sigma(T) \subset \boldsymbol{R}$ and for which there exist two increasing functions*

$$f\colon [0, +\infty) \to [1, +\infty),$$

$$\overline{\lim_{t\to+\infty}} \frac{\ln f(t)}{t} \leqslant 0,$$

and

$$g\colon [0, +\infty) \to [e, +\infty),$$

$$\int_1^{+\infty} \frac{\ln\ln g(t)}{t^2} dt < +\infty,$$

such that

$$\|R(z;T)\| \leqslant f(|\operatorname{Re} z|)\, g\left(\frac{1}{|\operatorname{Im} z|}\right), \quad z \in \boldsymbol{C}\setminus\boldsymbol{R}.$$

Then, for every $x \in X$ such that

$$\boldsymbol{C}\setminus\boldsymbol{R} \ni z \mapsto R(z;T)x \in X$$

has an entire extension, we have $x = 0$.

Proof. Let $x^* \in X^*$ be arbitrary.

By our hypothesis on x, the function $\boldsymbol{C}\setminus\boldsymbol{R} \ni z \mapsto \langle R(z;T)x, x^*\rangle$ has an entire extension Φ. Since f is increasing, we have

$$\left|\frac{\Phi(z)}{f(n)(1+\|x\|)(1+\|x^*\|)}\right| \leqslant g\left(\frac{1}{|\operatorname{Im} z|}\right), \quad n \geqslant 2 \text{ integer}, |\operatorname{Re} z| < n, 0 \neq |\operatorname{Im} z| < 1.$$

By Corollary 1.13 there exists a constant $c = c_{g,1} > 0$, depending only on g, such that

$$\left|\frac{\Phi(z)}{f(n)(1+\|x\|)(1+\|x^*\|)}\right| \leqslant c, \quad n \geqslant 2 \text{ integer}, n-2 \leqslant |\operatorname{Re} z| < n-1, |\operatorname{Im} z| < 1.$$

Using again the fact that f is increasing, we infer that

$$|\Phi(z)| \leqslant c(1+\|x\|)(1+\|x^*\|) f(|\operatorname{Re} z|+2), \quad |\operatorname{Im} z| < 1.$$

Since

$$|\Phi(z)| \leqslant g(1) f(|\operatorname{Re} z|), \quad |\operatorname{Im} z| \geqslant 1,$$

we deduce that Φ is of exponential type 0 and, in addition, it is bounded on the imaginary axis. By the Phragmén–Lindelöf principle [see [38], Ch. I, Theorem 22) it follows that Φ is bounded, and so by the Liouville theorem it is constant. Consequently, $\Phi' \equiv 0$, and hence

$$\langle R(z;T)^2 x, x^*\rangle = 0, \quad z \in \boldsymbol{C}\setminus\boldsymbol{R}.$$

Since $x^* \in X^*$ is arbitrary, we conclude that

$$R(z;T)^2 x = 0, \quad z \in \boldsymbol{C}\setminus\boldsymbol{R};$$

so, by the injectivity of $R(z;T)$, $x = 0$. ■

We can now give the following

THEOREM 6.7. *Let us assume that ω is regular. Let T be a densely defined closed linear operator in X, $\sigma(T) \subset \boldsymbol{R}$, for which there exist an increasing function*

$$f\colon [0, +\infty) \to [1, +\infty),$$

$$\overline{\lim_{t\to+\infty}} \frac{\ln f(t)}{t} \leqslant 0,$$

and $L > 0$, $c > 0$, and integer $n \geqslant 1$, such that

$$\|R(z;T)\| \leqslant f(|\operatorname{Re} z|)\left(c + \omega_n^*\left(\frac{L}{|\operatorname{Im} z|}\right)\right), \quad z \in \boldsymbol{C}\setminus\boldsymbol{R}.$$

Then T is ω-self-adjoint, the spectral ω-ultradistribution E associated with T is defined by the formula

$$E = \lim_{0<\varepsilon\to 0} \frac{1}{2\pi i}\big(R(\cdot - i\varepsilon;T) - R(\cdot + i\varepsilon;T)\big), \tag{6.5}$$

where the limit exists in $\mathscr{L}(\mathscr{D}_\omega; \mathscr{L}(X))$ and $\operatorname{supp} E = \sigma(T)$.

Proof. Clearly, the function $\boldsymbol{C}\setminus\boldsymbol{R} \ni z \mapsto R(z;T) \in \mathscr{L}(X)$ belongs to $\mathscr{H}_\omega(\mathscr{L}(X))$; so by Lemma 6.5 the limit in (6.5) exists in $\mathscr{L}(\mathscr{D}_\omega; \mathscr{L}(X))$ and (6.2), (6.3), (6.4) hold.

Let $x \in X$ be such that $E(\varphi)x = 0$ for every $\varphi \in \mathscr{D}_\omega$. Then, by Theorem 5.10, the mapping $\boldsymbol{C}\setminus\boldsymbol{R} \ni z \mapsto R(z;T)x \in X$ can be extended to an X-valued entire function. By Theorem 1.10 and Lemma 6.6, it follows that $x = 0$.

Next let $x^* \in X^*$ be such that

$$\langle E(\varphi)x, x^*\rangle = 0, \quad \varphi \in \mathcal{D}_\omega, \ x \in X.$$

Then $E(\varphi)^* x^* = 0$ for every $\varphi \in \mathcal{D}_\omega$ and, using Theorem 5.10, Theorem 1.10, and Lemma 6.6 as before, we deduce that $x^* = 0$. Thus, by the Hahn–Banach theorem, we conclude that $\bigcup_{\varphi \in \mathcal{D}_\omega} E(\varphi)X$ is dense in X. ∎

The result from the above theorem extends Lemma 2.3 from [44] and at the same time it gives an answer to the question raised in Remark 2.5 in the same work.

In the case of bounded operators we can prove a more complete result, extending [62], Satz 1 and Satz 2, [31], Corollary 2.11, [16], Ch. 5, Theorem 4.3 and [13], Theorem 3.1:

THEOREM 6.8. *Let us assume that ω is regular and let $T \in \mathcal{L}(X)$. Then the following statements are equivalent*:

(i) *T is ω-self-adjoint;*

(ii) *there exist $L > 0$, integer $n \geqslant 1$ and $c > 0$ such that*

$$\|e^{itT}\| \leqslant c|\omega(Lt)|^n, \quad t \in \mathbf{R};$$

(iii) *$\sigma(T) \subset \mathbf{R}$ and*

$$\mathbf{C}\setminus\mathbf{R} \ni z \mapsto R(z;T) \in \mathcal{L}(X)$$

belongs to $\mathcal{H}_\omega(\mathcal{L}(X))$.

Moreover, if the above statements hold, then the spectral ω-ultradistribution E associated with T is defined by the formula

$$E = \lim_{0<\varepsilon\to 0} \frac{1}{2\pi i}\big(R(\cdot - i\varepsilon; T) - R(\cdot + i\varepsilon; T)\big),$$

where the limit exists in $\mathcal{L}(\mathcal{D}_\omega; \mathcal{L}(X))$ and $\operatorname{supp} E = \sigma(T)$.

Proof. Firstly we assume that (i) holds and we denote by E the spectral ω-ultradistribution associated with T. By Corollary 6.4, $E \in \mathcal{L}(\mathcal{E}_\omega; \mathcal{L}(X))$ and, defining $\varphi_1 \in \mathcal{E}_\omega$ by $\varphi_1(s) = s$, $s \in \mathbf{R}$, we have

$$E(\varphi_1^m) = T^m, \quad m \geqslant 0 \text{ integer}.$$

Hence,

$$e^{itT} = \sum_{m=0}^{\infty} \frac{(it)^m}{m!} T^m = E\Big(\sum_{m=0}^{\infty} \frac{(it)^m}{m!}\varphi_1^m\Big) = E(e^{it\cdot}), \quad t \in \mathbf{R}.$$

On the other hand, by the continuity of E, there exist compact $K \subset \mathbf{R}$, $L > 0$, integer $n \geqslant 1$ and $c > 0$ such that

$$\|E(\varphi)\| \leqslant c r_{L,n}^{\omega,K}(\varphi), \quad \varphi \in \mathcal{E}_\omega.$$

Consequently, we have for each $t \in \mathbf{R}$

$$\|e^{itT}\| = \|E(e^{it\cdot})\| \leqslant c \sup_{k\geqslant 0} L^k a_k^{\omega,n}|t|^k \leqslant c|\omega(Lt)|^n.$$

Next we suppose that (ii) is satisfied. Then we can define an analytic function

$$R\colon \mathbf{C}\setminus\mathbf{R} \to \mathcal{L}(X)$$

by the formulas

$$R(z) = \begin{cases} i\displaystyle\int_0^{+\infty} e^{-itz} e^{itT}\,dt, & \operatorname{Im} z < 0,\\ -i\displaystyle\int_0^{+\infty} e^{itz} e^{-itT}\,dt, & \operatorname{Im} z > 0\end{cases}$$

and we have

$$(6.6) \qquad \|R(z)\| \leqslant \int_0^{+\infty} e^{-t|\operatorname{Im} z|} c|\omega(Lt)|^n\,dt = \frac{c}{L}\omega_n^*\Big(\frac{L}{|\operatorname{Im} z|}\Big), \quad z \in \mathbf{C}\setminus\mathbf{R}.$$

Using the identity

$$e^{itT} = \sum_{m=0}^{\infty} \frac{(it)^m}{m!} T^m, \quad t \in \mathbf{R},$$

we can easily see that

$$R(z) = \sum_{m=0}^{\infty} z^{-m-1} T^m = R|(z;T), \quad \operatorname{Im} z| > \|T\|.$$

Consequently, $\sigma(T) \subset \mathbf{R}$ and

$$R(z;T) = R(z), \quad z \in \mathbf{C}\setminus\mathbf{R},$$

and so (6.6) shows that $\mathbf{C}\setminus\mathbf{R} \ni z \mapsto R(z;T) \in \mathcal{L}(X)$ belongs to $\mathcal{H}_\omega(\mathcal{L}(X))$.

Finally, let us assume that (iii) holds. Since $\sigma(T)$ is compact and $\lim_{|z|\to+\infty} \|R(z;T)\| = 0$, by Theorem 6.7, (i) results as well as the formula for E from the last part of the theorem and the equality $\operatorname{supp} E = \sigma(T)$. ∎

We now give a quite general sufficient condition for a densely defined closed linear operator T with a real spectrum to be $\omega_{\{r_k\}}$-self-adjoint for some regular $\omega_{\{r_k\}}$.

DEFINITION XXIX. We say that an analytic function Φ on $\mathbf{C}\setminus\mathbf{R}$ with values in some Banach space *satisfies the* 0 *exponential type Levinson condition* if there exists an increasing function

$$f\colon [0, +\infty) \to [1, +\infty),$$

$$\lim_{t\to+\infty} \frac{\ln f(t)}{t} \leqslant 0,$$

such that, defining $g_f^\Phi\colon [1, +\infty) \to [0, +\infty)$ by

$$g_f^\Phi(t) = \sup\{f(|\operatorname{Re} z|)^{-1}\|\Phi(z)\|;\ z \in \mathbf{C},\ |\operatorname{Im} z| \geqslant t^{-1}\},$$

we have

$$\int_1^{+\infty} \frac{\ln_+ \ln_+ g_f^{\Phi}(t)}{t^2}\,dt < +\infty.$$

If Φ satisfies the above condition with $f \equiv 1$, then we say simply that *Φ satisfies the Levinson condition.*

It is clear that the 0 exponential type Levinson condition is stronger than the local Levinson condition, defined in Section 5, Definition XXVI.

The following result extends and "explains" [43], Theorem 6:

THEOREM 6.9. *Let T be a densely defined closed linear operator in X, $\sigma(T) \subset \mathbf{R}$, such that*

$$\mathbf{C} \diagdown \mathbf{R} \ni z \mapsto R(z; T) \in \mathscr{L}(X)$$

satisfies the 0 exponential type Levinson condition. Then there exist $0 < r_1 \leqslant r_2/2 \leqslant r_3/3 \leqslant \ldots,\ r_1 < +\infty,\ \sum_{k=1}^{\infty} 1/r_k < +\infty$, such that T is $\omega_{\{r_k\}}$-self-adjoint and, denoting by E the spectral $\omega_{\{r_k\}}$-ultradistribution associated with T, we have $\operatorname{supp} E = \sigma(T)$.

Proof. Since $\mathbf{C} \diagdown \mathbf{R} \ni z \mapsto R(z; T) \in \mathscr{L}(X)$ satisfies the 0 exponential type Levinson condition, there exist increasing functions

$$f\colon [0, +\infty) \to [1, +\infty),$$

$$\varlimsup_{t \to +\infty} \frac{\ln f(t)}{t} \leqslant 0,$$

and

$$g\colon [1, +\infty) \to [2, +\infty),$$

$$\int_1^{+\infty} \frac{\ln \ln g(t)}{t^2}\,dt < +\infty,$$

such that

$$\|R(z; T)\| \leqslant f(|\mathrm{Re}\, z|)\, g\left(\frac{1}{|\mathrm{Im}\, z|}\right), \quad 0 \neq |\mathrm{Im}\, z| < 1,$$

$$\|R(z; T)\| \leqslant f(|\mathrm{Re}\, z|)\, g(1), \qquad |\mathrm{Im}\, z| > 1.$$

By Theorem 1.10 there exist $0 < r_1 \leqslant r_2/2 \leqslant r_3/3 \leqslant \ldots,\ r_1 < +\infty,\ \sum_{k=1}^{\infty} 1/r_k < +\infty$ and c_0 such that

$$g(t) \leqslant c_0 |\omega_{\{r_k\}}|_{\mathrm{Borel}}(t), \qquad t \in [1, +\infty),$$

so that, defining the increasing function $f_0\colon [0, +\infty) \to [1, +\infty)$, by $f_0(t) = \max\{1, c_0\} f(t)$, we have

$$\varlimsup_{t \to +\infty} \frac{\ln f_0(t)}{t} \leqslant 0$$

and

$$\|R(z; T)\| \leqslant f_0(|\mathrm{Re}\, z|)\left(g(1) + |\omega_{\{r_k\}}|_{\mathrm{Borel}}\left(\frac{1}{|\mathrm{Im}\, z|}\right)\right), \quad z \in \mathbf{C} \diagdown \mathbf{R}.$$

Applying Theorem 6.7, we obtain our statement. ∎

In the case of bounded operators we can again give a more complete result:

THEOREM 6.10. *Let $T \in \mathscr{L}(X)$. Then the following statements are equivalent:*

(i) *there exist $0 < r_1 \leqslant r_2/2 \leqslant r_3/3 \leqslant \ldots,\ r_1 < +\infty, \sum_{k=1}^{\infty} 1/r_k < +\infty$, such that T is $\omega_{\{r_k\}}$-self-adjoint;*

(ii) $\displaystyle\int_{-\infty}^{+\infty} \frac{\ln \|e^{itT}\|}{1+t^2}\,dt < +\infty$;

(iii) *$\sigma(T) \subset \mathbf{R}$ and $\mathbf{C} \diagdown \mathbf{R} \ni z \mapsto R(z; T) \in \mathscr{L}(X)$ satisfies the local Levinson condition;*

(iv) *$\sigma(T) \subset \mathbf{R}$ and $\mathbf{C} \diagdown \mathbf{R} \ni z \mapsto R(z; T) \in \mathscr{L}(X)$ satisfies the Levinson condition.*

Proof. It is evident that

$$\|e^{i(t+s)T}\| \leqslant \|e^{itT}\|\,\|e^{isT}\|, \qquad t, s \in \mathbf{R},$$

so that we obtain (i) ⇔ (ii), using the corresponding equivalence from Theorem 6.8 and Theorem 1.8.

Using Theorem 1.8, the implication (ii) ⇒ (iii) from Theorem 6.8 and Theorem 1.10, we obtain the implication (ii) ⇒ (iii).

Since $\sigma(T)$ is compact and $\lim_{|z| \to +\infty} \|R(z; T)\| = 0$, clearly (iii) ⇒ (iv). Finally, by Theorem 6.9 also the implication (iv) ⇒ (i) holds. ∎

7. Final comments

In this section we analyse the connection between the theory of ω-ultradistributions and the existing ultradistributions theories; next we present the advantages of the ω-ultradistribution theory and finally we formulate some open problems.

7.1. We begin by clarifying what we understand by an ultradistribution theory. We note that our concept of ultradistribution is slightly different from that of P. Schapira (see [55]).

Let $\mathfrak{S}$ be a family of parameters. Assume that with every $\sigma \in \mathfrak{S}$ is associated a locally convex topological vector space $\mathscr{D}_\sigma$ of infinitely differentiable functions $\varphi\colon \mathbf{R} \to \mathbf{C}$ with a compact support such that

(7.1) $\mathscr{D}_\sigma$ is the inductive limit of a sequence of Fréchet spaces;

(7.2) the topology of $\mathscr{D}_\sigma$ is stronger than the topology of pointwise convergence;

(7.3) $\mathscr{D}_\sigma$ is an algebra under pointwise multiplication;

(7.4) for every compact $K \subset R$ and open $D \subset K$, there exists a $\varphi \in \mathcal{D}_\sigma$ such that

$$0 \leqslant \varphi \leqslant 1, \quad \varphi(s) = 1 \text{ for } s \in K, \quad \operatorname{supp}\varphi \subset D.$$

Denoting by $\mathcal{E}_\sigma$ the vector space of all functions ψ: $R \to C$ such that

$$\varphi\psi \in \mathcal{D}_\sigma, \quad \varphi \in \mathcal{D}_\sigma,$$

and endowing it with the projective limit topology defined by the linear mappings

$$\mathcal{E}_\sigma \ni \psi \mapsto \varphi\psi \in \mathcal{D}_\sigma, \quad \varphi \in \mathcal{D}_\sigma,$$

we suppose also that

(7.5) $\mathcal{A}$ is a dense linear subspace of $\mathcal{E}_\sigma$, where $\mathcal{A}$ is the linear space of all complex functions on R which can be extended analytically to some complex neighbourhood of R.

If the above assumptions hold, then we say that $\{\mathcal{D}_\sigma\}_{\sigma\in\mathfrak{S}}$ is a theory of ultradistributions.

The elements of the dual $\mathcal{D}'_\sigma$ of $\mathcal{D}_\sigma$ are called *σ-ultradistributions*.

Let $\sigma \in \mathfrak{S}$. By (7.3) and (7.4), for every $F \in \mathcal{D}'_\sigma$ there exists a smallest closed set $S \subset R$ such that

$$\varphi \in \mathcal{D}_\sigma, \quad S \cap \operatorname{supp}\varphi = \emptyset \Rightarrow F(\varphi) = 0.$$

Then S is called the *support of* F and it is denoted by $\operatorname{supp} F$. It is easy to see that the σ-ultradistributions with a compact support can be identified with the elements of the dual $\mathcal{E}'_\sigma$ of $\mathcal{E}_\sigma$. Hence, by (7.5), for every $F \in \mathcal{D}'_\sigma$, $\operatorname{supp} F$ being compact, we can define its Fourier transform $\hat{F}$: $R \to C$ by the formula

$$\hat{F}(t) = \frac{1}{2\pi} F(e^{-it\cdot}), \quad t \in R.$$

For $\sigma \in \mathfrak{S}$ we call a *σ-ulradifferential operator* any linear operator T: $\mathcal{D}_\sigma \to \mathcal{D}_\sigma$ such that

$$\operatorname{supp} T(\varphi) \subset \operatorname{supp}\varphi, \quad \varphi \in \mathcal{D}_\sigma.$$

A σ-ultradistribution theory becomes a good tool in many problems of the analysis if one can prove

(N) *the nuclearity* of the fundamental spaces $\mathcal{D}_\sigma$, $\mathcal{D}'_\sigma$, $\mathcal{E}_\sigma$, $\mathcal{E}'_\sigma$,

and if one can solve conveniently

(PW) *the Paley–Wiener problem for* $\mathcal{D}_\sigma$; to characterize the image of $\mathcal{D}_\sigma$ by the Fourier transformation;

(PW′) *the Paley–Wiener problem for* $\mathcal{E}'_\sigma$; to characterize the image of $\mathcal{E}'_\sigma$ by the Fourier transformation;

(UD) *the problem of the σ-ultradifferential operators*: to construct all σ-ultradifferential operators;

(OP) *the problem of the σ-ultradistributions with one-point support*: to describe all σ-ultradistributions F with $\operatorname{supp} F \subset \{s_0\}$, $s_0 \in R$, in terms of the Dirac distribution δ_{s_0} and σ-ultradifferential operators;

(H) *the problem of the imbedding into the hyperfuctions*: to represent the σ-ultradistributions as boundary values of hyperfunctions in $\mathcal{D}'_\sigma$ and to characterize those hyperfunctions which correspond to σ-ultradistributions.

7.2. Next we show how two ultradistribution theories can be compared.

Let $\{\mathcal{D}_\sigma\}_{\sigma\in\mathfrak{S}}$ and $\{\mathcal{D}_\tau\}_{\tau\in\mathfrak{T}}$ be two ultradistribution theories.

Let $\sigma \in \mathfrak{S}$ and $\tau \in \mathfrak{T}$. Then the inclusions

$$\mathcal{D}_\tau \subset \mathcal{D}_\sigma, \tag{7.6}$$

$$\mathcal{E}_\tau \subset \mathcal{E}_\sigma \tag{7.7}$$

are equivalent. Indeed, if (7.6) holds and $\psi \in \mathcal{E}_\tau$, $\varphi \in \mathcal{D}_\sigma$, then, choosing $\theta \in \mathcal{D}_\tau$ such that $\theta(s) = 1$ for $s \in \operatorname{supp}\varphi$, we have $\varphi\psi = \varphi(\theta\psi)$ and $\theta\psi \in \mathcal{D}_\tau \subset \mathcal{D}_\sigma$, and so $\varphi\psi \in \mathcal{D}_\sigma$.

Conversely, if (7.7) holds and $\varphi \in \mathcal{D}_\tau$ then, since $\mathcal{D}_\tau \subset \mathcal{E}_\tau \subset \mathcal{E}_\sigma$, choosing $\theta \in \mathcal{D}_\sigma$ such that $\theta(s) = 1$ for $s \in \operatorname{supp}\varphi$, we have $\varphi = \varphi\theta \in \mathcal{D}_\sigma$.

Let us assume that (7.6) holds. By (7.2), it follows that the graph of the inclusion $\mathcal{D}_\tau \subset \mathcal{D}_\sigma$ is closed, and so by (7.1) and by the closed graph theorem of D. A. Raikov (see [50]) it is continuous. By the same closed graph theorem, the multiplication in $\mathcal{D}_\sigma$ is separately continuous. Hence, for every $\varphi \in \mathcal{D}_\sigma$, if we choose $\theta \in \mathcal{D}_\tau$ with $\theta(s) = 1$ for $s \in \operatorname{supp}\varphi$, the composition

$$\mathcal{E}_\tau \ni \psi \mapsto \varphi\psi \in \mathcal{D}_\sigma$$

of the mappings $\mathcal{E}_\tau \ni \psi \mapsto \theta\psi \in \mathcal{D}_\tau$, $\mathcal{D}_\tau \subset \mathcal{D}_\sigma$ and $\mathcal{D}_\sigma \ni \varrho \mapsto \varphi\varrho \in \mathcal{D}_\sigma$ is continuous. Consequently, also the inclusion $\mathcal{E}_\tau \subset \mathcal{E}_\sigma$ is continuous.

Finally, if $\mathcal{E}_\tau \subset \mathcal{E}_\sigma$ then by (7.5) $\mathcal{E}_\tau$ is dense in $\mathcal{E}_\sigma$. It follows that also $\mathcal{D}_\tau$ is dense in $\mathcal{D}_\sigma$. Indeed, if $\varphi \in \mathcal{D}_\sigma$ then, choosing $\theta \in \mathcal{D}_\tau$ such that $\theta(s) = 1$ for $s \in \operatorname{supp}\varphi$, and using the continuity of $\mathcal{E}_\sigma \ni \psi \mapsto \theta\psi \in \mathcal{D}_\sigma$ and the density of $\mathcal{E}_\tau$ in $\mathcal{E}_\sigma$, we conclude that $\varphi = \theta\varphi$ belongs to the closure of $\{\theta\psi;\, \psi \in \mathcal{E}_\tau\} \subset \mathcal{D}_\tau$ in $\mathcal{D}_\sigma$.

Thus, *if the equivalent inclusions* (7.6) *and* (7.7) *hold, then they are continuous and have a dense range.* So in this case we can consider

$$\mathcal{D}'_\sigma \subset \mathcal{D}'_\tau, \quad \mathcal{E}'_\sigma \subset \mathcal{E}'_\tau,$$

where the inclusions are continuous and have dense ranges with the $\mathcal{D}_\sigma$-, $\mathcal{D}_\tau$-, $\mathcal{E}_\sigma$-, $\mathcal{E}_\tau$-topology on $\mathcal{D}'_\sigma$, $\mathcal{D}'_\tau$, $\mathcal{E}'_\sigma$, $\mathcal{E}'_\tau$, respectively.

By the above considerations, it is justified to say that *the ultradistribution theory* $\{\mathcal{D}_\tau\}_{\tau\in\mathfrak{T}}$ *is larger than* $\{\mathcal{D}_\sigma\}_{\sigma\in\mathfrak{S}}$ *if for every* $\sigma \in \mathfrak{S}$ *there exists* $\tau \in \mathfrak{T}$ *such that the equivalent inclusions* (7.6) *and* (7.7) *hold.*

If $\{\mathcal{D}_\tau\}_{\tau\in\mathfrak{T}}$ *is larger than* $\{\mathcal{D}_\sigma\}_{\sigma\in\mathfrak{S}}$ *and conversely, then we say that* $\{\mathcal{D}_\tau\}_{\tau\in\mathfrak{T}}$ *and* $\{\mathcal{D}_\sigma\}_{\sigma\in\mathfrak{S}}$ *are equivalent ultradistribution theories.*

It is clear that if $\{\mathcal{D}_\tau\}_{\tau\in\mathfrak{T}}$ is an ultradistribution theory and $\mathfrak{S} \subset \mathfrak{T}$, then also $\{\mathcal{D}_\tau\}_{\tau\in\mathfrak{S}}$ is an ultradistribution theory and $\{\mathcal{D}_\tau\}_{\tau\in\mathfrak{T}}$ is larger than $\{\mathcal{D}_\tau\}_{\tau\in\mathfrak{S}}$.

7.3. We recall the usual ultradistribution theories.

Let $\mathscr{M}$ be the family of all sequences $\{M_p\}_{p\geqslant 0}\subset(0,+\infty)$, $M_0=1$, satisfying the conditions:

(M.1) *logarithmic convexity*: $M_p^2\leqslant M_{p-1}M_{p+1}$, $p\geqslant 1$;

(M.3)′ *non-quasianalyticity*: $\sum_{p=1}^{\infty}M_{p-1}/M_p<+\infty$.

Let $\{M_p\}_{p>0}\in\mathscr{M}$ be fixed. For every $h>0$ and $-\infty<a<b<+\infty$ we denote by $\mathscr{D}_{\{M_p\},h}[a,b]$ the vector space of all infinitely differentiable functions $\varphi\colon \boldsymbol{R}\to\boldsymbol{C}$, $\operatorname{supp}\varphi\subset[a,b]$, such that

$$\|\varphi\|_{\{M_p\},h}=\sup_{p\geqslant 0}\left(\frac{1}{h^pM_p}\sup_{s\in\boldsymbol{R}}|\varphi^{(p)}(s)|\right)<+\infty.$$

If $\mathscr{D}_{\{M_p\},h}[a,b]$ is endowed with the norm $\|\cdot\|_{\{M_p\},h}$, it becomes a Banach space.

Following [51] and [52] (see also [34]) we define

$$\mathscr{D}_{\{M_p\}}=\bigcup_{\substack{h>0\\-\infty<a<b<+\infty}}\mathscr{D}_{\{M_p\},h}[a,b],$$

and we endow it with the inductive limit topology. It is clear that (7.1) and (7.2) are satisfied. By [51], Ch. I, § 2, Lemma 1, (7.3) is satisfied and by [51], Ch. I, § 2, Proposition 3, also (7.4) is satisfied. Finally, by [34], Theorem 7.2, (7.5) holds.

Consequently, $\{\mathscr{D}_{\{M_p\}}\}_{\{M_p\}\in\mathscr{M}}$ is an ultradistribution theory in the sense of 7.1. The elements of $\mathscr{D}'_{\{M_p\}}$ are called *Roumieu ultradistributions of class* $\{M_p\}$.

On the other hand, for every $-\infty<a<b<+\infty$ we consider

$$\mathscr{D}_{(M_p)}[a,b]=\bigcap_{h>0}\mathscr{D}_{\{M_p\},h}[a,b],$$

endowed with the locally convex topology defined by the norms $\|\cdot\|_{\{M_p\},h}$, $h>0$. Then $\mathscr{D}_{(M_p)}[a,b]$ is a Fréchet space. Following [41], Ch. 7 (see also [34]), we define

$$\mathscr{D}_{(M_p)}=\bigcup_{-\infty<a<b<+\infty}\mathscr{D}_{(M_p)}[a,b],$$

and we endow it with the inductive limit topology. Clearly, (7.1) and (7.2) are again satisfied. By [34], Theorem 2.8, we get (7.3), by [34], Lemma 5.1, we get (7.4) and by [34], Theorem 7.2, also (7.5) holds.

Therefore, also $\{\mathscr{D}_{(M_p)}\}_{\{M_p\}\in\mathscr{M}}$ is an ultradistribution theory.

The elements of $\mathscr{D}'_{(M_p)}$ are called *Beurling ultradistributions of class* $\{M_p\}$.

Further, let $\mathfrak{A}$ be the family of all continuous functions $\alpha\colon\boldsymbol{R}\to[0,+\infty)$ such that

$$\alpha(0)=0,\quad \alpha(t+s)\leqslant\alpha(t)+\alpha(s),\quad t,s\in\boldsymbol{R},$$

$$\int_{-\infty}^{+\infty}\frac{\alpha(t)}{1+t^2}\,dt<+\infty,$$

and there exist $a_\alpha\in\boldsymbol{R}$, $b_\alpha>0$ with

$$\alpha(t)\geqslant a_\alpha+b_\alpha\ln(1+|t|),\qquad t\in\boldsymbol{R}.$$

Let $\alpha\in\mathfrak{A}$. For every $-\infty<a<b<+\infty$ we can define the vector space $\mathscr{D}_\alpha[a,b]$ of all continuous functions $\varphi\colon\boldsymbol{R}\to\boldsymbol{C}$, $\operatorname{supp}\varphi\subset[a,b]$, such that for every $\lambda>0$

$$\|\varphi\|_{\alpha,\lambda}=\int_{-\infty}^{+\infty}|\hat{\varphi}(t)|\,e^{\lambda\alpha(t)}dt<+\infty.$$

If $\mathscr{D}_\alpha[a,b]$ is endowed with the locally convex topology defined by the norms $\|\cdot\|_{\alpha,\lambda}$, $\lambda>0$, it becomes a Fréchet space. By [7], Corollary 1.3.21, all functions from $\mathscr{D}_\alpha[a,b]$ are infinitely differentiable. Following [6] (see also [7]), we define

$$\mathscr{D}_\alpha=\bigcup_{-\infty<a<b<+\infty}\mathscr{D}_\alpha[a,b],$$

and we endow it with the inductive limit topology. Conditions (7.1) and (7.2) are obviously satisfied. By [7], Proposition 1.3.5, (7.3) follows and by [7], Theorem 1.3.7, we get (7.4). Finally, by [7], Corollary 1.5.15, $\mathscr{A}\subset\mathscr{E}_\alpha$ and a reasoning similar to that used in the proof of [34], Lemma 7.1, shows us that $\mathscr{A}$ is dense in $\mathscr{E}_\alpha$.

We conclude that $\{\mathscr{D}_\alpha\}_{\alpha\in\mathfrak{A}}$ is an ultradistribution theory. The elements of $\mathscr{D}'$ are called *Beurling ultradistributions of class* α.

Finally, denoting by Ω the family of all entire functions $\omega_{\{t_k\}}$, where $0<t_1\leqslant t_2\leqslant\ldots, t_1<+\infty$, $\sum_{k=1}^{\infty}1/t_k<+\infty$, by Ω_0 the family of all regular $\omega\in\Omega$ and by Ω_{00} the family of all $\omega\in\Omega$ satisfying the strong non-quasianalyticity condition, by Proposition 5.14 we have $\Omega\supset\Omega_0\supset\Omega_{00}$.

By our results from § 2, $\{\mathscr{D}_\omega\}_{\omega\in\Omega}$, $\{\mathscr{D}_\omega\}_{\omega\in\Omega_0}$ *and* $\{\mathscr{D}_\omega\}_{\omega\in\Omega_{00}}$ *are ultradistribution theories, each of them being larger than the next one.*

7.4. Next we shall compare the ultradistribution theories defined above. Obviously

$$\mathscr{D}_{(M_p)}\subset\mathscr{D}_{\{M_p\}},\qquad \{M_p\}_{p\geqslant 0}\in\mathscr{M}. \tag{7.8}$$

Let $\{M_p\}\in\mathscr{M}$. By Lemma 1.5 (ii) there exists an $\{N_p\}\in\mathscr{M}$ such that

$$\lim_{p\to\infty}\frac{M_{p-1}}{M_p}\frac{N_p}{N_{p-1}}=0.$$

Thus, for every $\varepsilon>0$ there exists an integer $p_\varepsilon\geqslant 1$ such that successively we have

$$\frac{M_{p-1}}{M_p}\leqslant\varepsilon\frac{N_{p-1}}{N_p},\qquad p>p_\varepsilon,$$

$$\frac{M_{p_\varepsilon}}{M_p}\leqslant\varepsilon^{p-p_\varepsilon}\frac{N_{p_\varepsilon}}{N_p},\qquad p>p_\varepsilon.$$

So, if $-\infty < a < b < +\infty$, $h > 0$ and $\varphi \in \mathscr{D}_{\{N_p\},h}$ then we have for every $g > 0$ and $0 < \varepsilon \leqslant g/h$

$$\|\varphi\|_{\{M_p\},g} \leqslant \max\left\{\sup_{p\leqslant p_\varepsilon}\left(\frac{1}{g^pM_p}\sup_s|\varphi^{(p)}(s)|\right); \frac{N_{p_\varepsilon}}{M_{p_\varepsilon}\varepsilon^{p_\varepsilon}}\sup_{p>p_\varepsilon}\left(\frac{\varepsilon^p}{g^pN_p}\sup_s|\varphi^{(p)}(s)|\right)\right\}$$

$$\leqslant \max\left\{\sup_{p\leqslant p_\varepsilon}\left(\frac{1}{g^pM_p}\sup_s|\varphi^{(p)}(s)|\right); \frac{N_{p_\varepsilon}}{M_{p_\varepsilon}\varepsilon^{p_\varepsilon}}\|\varphi\|_{\{N_p\},h}\right\} < +\infty.$$

It follows that

$$\mathscr{D}_{\{N_p\},h}[a,b] \subset \mathscr{D}_{(M_p)}[a,b], \qquad -\infty<a<b<+\infty,\ h>0;$$

hence

(7.9) $$\mathscr{D}_{\{N_p\}} \subset \mathscr{D}_{(M_p)}.$$

By (7.8) and (7.9) we conclude that *the ultradistribution theories* $\{\mathscr{D}_{\{M_p\}}\}_{\{M_p\}\in\mathscr{M}}$ *and* $\{\mathscr{D}_{(M_p)}\}_{\{M_p\}\in\mathscr{M}}$ *are equivalent.*

Again let $\{M_p\}_{p\geqslant 0}\in\mathscr{M}$. Denoting

$$m_p = M_p/M_{p-1}, \qquad p\geqslant 1,$$

we have $\omega=\omega_{\{m_p\}}\in\Omega$. For every infinitely differentiable function $\varphi\colon \mathbf{R}\to\mathbf{C}$ with a compact support and $h>0$ we have

$$\|\varphi\|_{\{M_p\},h} = \sup_{p\geqslant0}\left(\frac{1}{h^pm_1\dots m_p}\sup_s|\varphi^{(p)}(s)|\right)$$

$$\leqslant \sup_{p\geqslant0}\left(\left(\frac1h\right)^p a_p^{\omega,1}\sup_s|\varphi^{(p)}(s)|\right) = r^\omega_{1/h,1}(\varphi),$$

so that

$$\mathscr{D}_\omega[a,b]\subset\mathscr{D}_{\{M_p\},h}[a,b], \qquad h>0,\ -\infty<a<b<+\infty.$$

It follows that

$$\mathscr{D}_\omega\subset\mathscr{D}_{(M_p)}.$$

Conversely, let $\omega\in\Omega$. We consider the entire function τ defined by

$$\tau(z)=\prod_{m=1}^\infty \omega\left(\frac{z}{4^m}\right)^{2^m}.$$

Plainly, $\tau\in\Omega$. Denoting

$$M_p=\frac{1}{a_p^{\tau,1}},\qquad p\geqslant0,$$

by Corollary 2.9 we have $\{M_p\}_{p\geqslant0}\in\mathscr{M}$. For every infinitely differentiable function $\varphi\colon\mathbf R\to\mathbf C$ with compact support, $L>0$ and integer $n\geqslant1$ we have

$$r^\omega_{L,n}(\varphi)=\sup_{k\geqslant0}\left(L^ka_k^{\omega,n}\sup_{s\in\mathbf R}|\varphi^{(k)}(s)|\right)$$

$$\leqslant \sup_{k\geqslant0}\left((4^nL)^ka_k^{\tau,1}\sup_{s\in\mathbf R}|\varphi^{(k)}(s)|\right)=\|\varphi\|_{\{M_p\},4^{-n}L^{-1}};$$

so

$$\mathscr{D}_{(M_p)}[a,b]\subset\mathscr{D}_\omega[a,b],\qquad -\infty<a<b<+\infty.$$

Consequently,

$$\mathscr{D}_{(M_p)}\subset\mathscr{D}_\omega.$$

Thus, *the ultradistribution theories* $\{\mathscr{D}_\omega\}_{\omega\in\Omega}$ *and* $\{\mathscr{D}_{(M_p)}\}_{\{M_p\}\in\mathscr{M}}$ *are equivalent.*

We conclude that the Roumieu ultradistribution theory, the Beurling ultradistributions associated with sequences from $\mathscr{M}$ and the ω-ultradistribution theory are all equivalent.

Let $\alpha\in\mathfrak{A}$. By Theorem 1.8 there exist $0<t_1\leqslant t_2/2\leqslant t_3/3\leqslant\dots$, $t_1<+\infty$, $\sum_{k=1}^\infty 1/t_k<+\infty$ and $c>0$ such that, putting $\omega=\omega_{\{t_k\}}\in\Omega_0$, we have

$$e^{\alpha(t)+\alpha(-t)}\leqslant c|\omega(t)|,\qquad t\in\mathbf R.$$

Hence, for every continuous function $\varphi\colon\mathbf R\to\mathbf C$ with a compact support and $\lambda>0$, we have

$$\|\varphi\|_{\alpha,\lambda}\leqslant\int_{-\infty}^{+\infty}|\hat\varphi(t)|(e^{\alpha(t)+\alpha(-t)})^\lambda dt$$

$$\leqslant c\int_{-\infty}^{+\infty}\hat\varphi(t)\omega(t)^{[\lambda]+1}dt = cq^\omega_{1,[\lambda]+1}(\varphi),$$

where $[\lambda]$ denotes the integer part of λ. So

$$\mathscr{D}_\omega[a,b]\subset\mathscr{D}_\alpha[a,b],\qquad -\infty<a<b<+\infty,$$

and we conclude that

$$\mathscr{D}_\omega\subset\mathscr{D}_\alpha.$$

Conversely, let $\omega=\omega_{\{t_k\}}\in\Omega_0$, where $0<t_1\leqslant t_2/2\leqslant t_3/3\leqslant\dots$, $t_1<+\infty$ $\sum_{k=1}^\infty 1/t_k<+\infty$. We define the continuous function $\alpha\colon\mathbf R\to[0,+\infty)$ by

$$\alpha(t)=\ln\left(1+\sum_{k=1}^\infty\frac{|t|^k}{t_1\dots t_k}\right),\qquad t\in\mathbf R.$$

Using Lemma 1.7 and Theorem 1.8 we can easily verify that $\alpha\in\mathfrak A$. Moreover, again by Lemma 1.7,

$$|\omega(t)|\leqslant 3e^{2\alpha(4t)}\leqslant 3e^{8\alpha(t)},\qquad t\in\mathbf R;$$

so, for every continuous function $\varphi\colon\mathbf R\to\mathbf C$ with compact support, $L>0$ and integer $n\geqslant1$, we have

$$q^\omega_{L,n}(\varphi)\leqslant 3^n\int_{-\infty}^{+\infty}|\hat\varphi(t)|e^{8n\alpha(Lt)}dt$$

$$\leqslant 3^n\int_{-\infty}^{+\infty}|\hat\varphi(t)|e^{8n([L]+1)\alpha(t)}dt = 3^n\|\varphi\|_{\alpha,8n([L]+1)}.$$

Consequently,

$$\mathscr{D}_\alpha[a,b] \subset \mathscr{D}_\omega[a,b], \qquad -\infty < a < b < +\infty,$$

and thus

$$\mathscr{D}_\alpha \subset \mathscr{D}_\omega .$$

We conclude that *the ultradistribution theories* $\{\mathscr{D}_\omega\}_{\omega\in\Omega_0}$ *and* $\{\mathscr{D}_\alpha\}_{\alpha\in\mathfrak{A}}$ *are equivalent*. In particular, $\{\mathscr{D}_\omega\}_{\omega\in\Omega}$ is larger than $\{\mathscr{D}_\alpha\}_{\alpha\in\mathfrak{A}}$.

7.5. It is easy to see that for each $\omega\in\Omega$

$$\mathscr{D}_{\left(\frac{1}{a_p^{\omega,1}}\right)} \supset \mathscr{D}_{\left(\frac{1}{a_p^{\omega,2}}\right)} \supset \dots, \qquad \mathscr{D}_\omega = \bigcap_{n\geqslant 1} \mathscr{D}_{\left(\frac{1}{a_p^{\omega,n}}\right)},$$

and so in a certain sense the ω-ultradistributions are of Beurling type.

We examine further a particular case in which $\mathscr{D}_\omega$ coincides with $\mathscr{D}_{\left(\frac{1}{a_p^{\omega,1}}\right)}$. By [34], we consider the following conditions on $\{M_p\}_{p\geqslant 0}\in\mathscr{M}$:

(M.2) *stability under ultradifferential operators*: there are $A, H>0$ such that

$$M_p \leqslant AH^p \min_{0\leqslant q\leqslant p} M_q M_{p-q}, \qquad p\geqslant 0;$$

(M.3) *strong non-quasianalyticity*: there is $B>0$ such that

$$\sum_{q=p+1}^{\infty} \frac{M_{q-1}}{M_q} \leqslant Bp\frac{M_p}{M_{p+1}}, \qquad p\geqslant 1.$$

We denote by $\mathscr{M}_{00}$ the family of all $\{M_p\}_{p\geqslant 0}\in\mathscr{M}$ satisfying (M.2) and (M.3).

Let $\{M_p\}_{p\geqslant 0}\in\mathscr{M}_{00}$. We define

$$M(t) = \sup_{p\geqslant 0}\frac{t^p}{M_p}, \qquad t>0,$$

$$m_p = \frac{M_p}{M_{p-1}}, \qquad p\geqslant 1, \qquad \omega = \omega_{\{m_p\}}.$$

By [34], Proposition 3.6, (M.2) is equivalent to

$$M(t)^2 \leqslant AM(Ht), \qquad t>0.$$

On the other hand, by our remark after Proposition 5.15,

$$\omega(-it) \leqslant e^B M(t)^{B+2}, \qquad t>0.$$

Hence, choosing some integer $n_0\geqslant 1$ with $B+2\leqslant 2^{n_0}$, we have

$$\omega(-it)^n \leqslant e^{Bn}A^{2^{n_0+n}-1}M(H^{n_0+n}t), \qquad n\geqslant 1 \text{ integer}, \ t>0.$$

Denoting

$$c_n = e^{Bn}A^{2^{n_0+n}-1}, \qquad L_n = H^{n_0+n}, \qquad n\geqslant 1,$$

we get

$$c_k^{\omega,n} \leqslant \frac{\omega(-it)^n}{t^k} \leqslant c_n\frac{M(L_n t)}{t^k}, \qquad n\geqslant 1 \text{ and } k\geqslant 0 \text{ integers}.$$

Since for $t = \frac{1}{L_n}\cdot\frac{M_k}{M_{k-1}}$, we have

$$M(L_n t) = \sup_{p\geqslant 0}\prod_{j=0}^{p}\left(\frac{M_{j-1}}{M_j}\cdot\frac{M_k}{M_{k-1}}\right) = \frac{1}{M_k}(L_n t)^k,$$

we conclude that

$$c_k^{\omega,n} \leqslant c_n\cdot\frac{L_n^k}{M_k}, \qquad n\geqslant 1, \ k\geqslant 0. \tag{7.10}$$

Let $-\infty<a<b<+\infty$. For every infinitely differentiable function $\varphi\colon \boldsymbol{R}\to\boldsymbol{C}$, $\operatorname{supp}\varphi\subset[a,b]$, using (7.10), we deduce, for $h>0$

$$\|\varphi\|_{\{M_p\},h} \leqslant \|\varphi\|_{\{1/a_p^{\omega,1}\},h} = r^{\omega}_{1/h,1}(\varphi) \leqslant \|\varphi\|_{\{1/c_p^{\omega,1}\},h} \leqslant c_1\|\varphi\|_{\{M_p\},h/L_1}$$

and

$$r^{\omega}_{L,n}(\varphi) \leqslant c_n\|\varphi\|_{\{M_p\},h/L_nL}, \qquad L>0, \ n\geqslant 1 \text{ integer}.$$

Hence

$$\mathscr{D}_{(M_p)}[a,b] = \mathscr{D}_{(1/a_p^{\omega,1})}[a,b] = \mathscr{D}_{(1/c_p^{\omega,1})}[a,b] = \mathscr{D}_\omega[a,b].$$

We sum up: *if* $\{M_p\}_{p\geqslant 0}\in\mathscr{M}_{00}$ *and* $\omega=\omega_{\{m_p\}}$, *where* $m_p = M_p/M_{p-1}$, $p\geqslant 1$, *then*

$$\mathscr{D}_{(M_p)} = \mathscr{D}_{(1/a_p^{\omega,1})} = \mathscr{D}_{(1/c_p^{\omega,1})} = \mathscr{D}_\omega .$$

Moreover, by (7.10), *we have*

$$\omega(-it) \leqslant 2c_1|\omega(L_1 t)|, \qquad t>0;$$

so ω *satisfies the strong non-quasianalyticity condition.*

In particular, $\{\mathscr{D}_\omega\}_{\omega\in\Omega_{00}}$ *is larger than* $\{\mathscr{D}_{(M_p)}\}_{\{M_p\}\in\mathscr{M}}$.

We remark that if $\{M_p\}_{p\geqslant 0}\in\mathscr{M}_{00}$ and if we define $\{M_p^*\}$ as in [34], Definition 11.1, then, by Lemma 5.12 and (7.10), we have

$$M_p^* \geqslant \frac{1}{p!}\frac{1}{c_p^{\omega,1}} \geqslant \frac{1}{p!}\cdot\frac{1}{c_1}\cdot\frac{M_p}{L_1^p}, \qquad p\geqslant 0;$$

so

$$M_p \leqslant c_1 L_1^p(p!\,M_p^*), \qquad p\geqslant 0. \tag{7.11}$$

If we use the notation from [34], Definition 3.7, (7.11) means $M_p\subset p!\,M_p^*$. Thus we answer positively the question raised in [34], page 99.

We note also that for $\varepsilon>1$ we have $\{(p!)^\varepsilon\}_{p\geqslant 0}\in\mathscr{M}_{00}$. Indeed, (M.1) is clearly satisfied, (M.2) results from the inequality

$$\frac{p!}{q!\,(p-q)!} = \binom{p}{q} \leqslant \sum_{j=0}^{p}\binom{p}{j} = 2^p, \qquad 0\leqslant q\leqslant p,$$

and (M.3) has been verified at the end of § 5.

The elements of $\mathscr{D}_{\{(p!)^\varepsilon\}}$ and $\mathscr{D}_{((p!)^\varepsilon)}$ are called *Gevrey ultradistributions of Roumieu type* and of *Beurling type*, respectively.

7.6. Next we shall point out those aspects of the theory of ω-ultradistributions which seem us to bring improvements into the above mentioned theories.

The characteristic feature of the theory of ω-ultradistributions is that it is based on the theory of entire functions rather than on the theory of logarithmic convex positive sequences (as the Roumieu and Beurling ultradistributions of class $\{M_p\}$) or on real analysis (as the Beurling ultradistributions of class α). Even when such logarithmic concave sequences as $\{a_k^{\omega,n}\}_{k\geqslant 0}$ or $\{c_k^{\omega,n}\}_{k\geqslant 0}$ appear, they are the coefficients of entire functions. Therefore the deep results of the entire function theory (a good reference for these is [38]) are available. Among them we mention the Phragmén–Lindelöf principle and several results due to S. N. Bernstein.

The "complex analysis feature" of the ω-ultradistribution theory allowed us to solve the problem (PW′) by using "exact" conditions (see Theorem 3.3), rather than "asymptotical" ones, like condition (v) after Theorem 3.3. We note that all the existing solutions of (PW′) for Roumieu or Beurling ultradistributions have an asymptotical character.

Again the "complex analysis feature" of the ω-ultradistribution theory, together with our "exact" solution of the (PW) (see Theorem (2.3), allowed us to construct sufficiently general ω-ultradifferential operators and hence to solve (UD) (see Theorem 2.21). We also mention that the problem (UD) was considered for particular Roumieu and Beurling ultradistributions in [2].

Further, the "exact" solution of (PW′) and the solution of (UD) enable us to solve the problem (OP) (see Theorem 3.5). A partial solution of (OP) for particular Roumieu ultradistributions was given in [51], Ch. III, Th. 4. Moreover, using the weighted approximation theory initiated by S. N. Bernstein, we proved Theorem 3.17 and Corollary 3.18, which complete our solution of (UD) and (OP).

We remark moreover that Theorem 2.21, Theorem 3.5 and Theorem 2.25 give an elegant solution of (UD) and (OP) for the ultradistribution theory $\{\mathscr{D}_\omega\}_{\omega\in\Omega_{00}}$, in particular for $\{\mathscr{D}_{(M_p)}\}_{\{M_p\}\in\mathscr{M}_{00}}$. These solutions seems to be new even for the Gevrey ultradistributions.

7.7. In the present section we show that the ultradistribution theories $\{\mathscr{D}_\omega\}_{\omega\in\Omega}$, $\{\mathscr{D}_\omega\}_{\omega\in\Omega_0}$, $\{\mathscr{D}_\omega\}_{\omega\in\Omega_{00}}$ are mutually non-equivalent.

We remark first that, if the ultradistribution theory $\{\mathscr{D}_\sigma\}_{\sigma\in\mathfrak{S}}$ is larger than $\{\mathscr{D}_\tau\}_{\tau\in\mathfrak{T}}$ then

$$\bigcap_{\tau\in\mathfrak{T}}\mathscr{E}_\tau\subset\bigcap_{\sigma\in\mathfrak{S}}\mathscr{E}_\sigma .$$

We recall that for every logarythmic convec sequence $\{A_p\}_{p\geqslant p_0}\subset(0,+\infty)$, $p_0\geqslant 0$ being integer, the corresponding *Denjoy–Carleman class* $\mathscr{A}_{\{A_p\}}$ is the vector space of all infinitely differentiable functions $\varphi\colon \boldsymbol{R}\to\boldsymbol{C}$ such that for each compact $\boldsymbol{K}\subset\boldsymbol{R}$, there exists an $h>0$ such that

$$\sup_{s\in K}|\varphi^{(p)}(s)|\leqslant h^pA_p,\quad p\geqslant p_0 .$$

By a theorem of Pringsheim (see [45], 4.1.I), $\mathscr{A}_{\{p!\}}=(\mathscr{A}_{\{p^p\}})$ coincides with the vector space $\mathscr{A}$ of all real analytic functions.

Now, by Lemma 2.11, we have $\mathscr{A}\subset\bigcap_{\omega\in\Omega}\mathscr{E}_\omega$. Further, by 7.4, $\{\mathscr{D}_\omega\}_{\omega\in\Omega}$ is larger than $\{\mathscr{D}_{\{M_p\}}\}_{\{M_p\}\in\mathscr{M}}$, so that $\bigcap_{\omega\in\Omega}\mathscr{E}_\omega\subset\bigcap_{\{M_p\}\in\mathscr{M}}\mathscr{E}_{\{M_p\}}$. Finally, by [51], Ch. II, § 1, Lemma 5,

$$\bigcap_{\{M_p\}\in\mathscr{M}}\mathscr{E}_{\{M_p\}}\subset\mathscr{A}_{\{p!\}}=\mathscr{A}.$$

Consequently,

$$\bigcap_{\omega\in\Omega}\mathscr{E}_\omega=\mathscr{A}=\mathscr{A}_{\{p^p\}}. \tag{7.12}$$

On the other hand, by 7.4 $\{\mathscr{D}_\omega\}_{\omega\in\Omega_0}$ is equivalent to $\{\mathscr{D}_\alpha\}_{\alpha\in\mathfrak{A}}$, so $\bigcap_{\omega\in\Omega_0}\mathscr{E}_\omega=\bigcap_{\alpha\in\mathfrak{A}}\mathscr{E}_\alpha$. Using [7], Theorem 1.5.12, we obtain

$$\bigcap_{\omega\in\Omega_0}\mathscr{E}_\omega=\mathscr{A}_{\{(p\ln p)^p\}}. \tag{7.13}$$

We note that (7.12) and (7.13) can also be deduced directly, by using the general result of J. Boman from [8], without making use of Roumieu or Beurling ultradistributions.

By [45], Ch. VI, 6.6. III, there exists a periodic function in $\mathscr{A}_{\{(p\ln p)^p\}}$ which does not belong to $\mathscr{A}_{\{p^p\}}$, so that

$$\bigcap_{\omega\in\Omega}\mathscr{E}_\omega\neq\bigcap_{\omega\in\Omega_0}\mathscr{E}_\omega .$$

Hence *$\{\mathscr{D}_\omega\}_{\omega\in\Omega}$ and $\{\mathscr{D}_\omega\}_{\omega\in\Omega_0}$ are not equivalent.* In particular, *there exists an $\omega\in\Omega$ which is not regular.*

Taking $t_1=1$, $t_2=2$ and $t_k=k(\ln k)^\varepsilon$ for $k\geqslant 3$, $1<\varepsilon\leqslant 2$, we have

$$\sum_{k=1}^{\infty}\frac{\ln t_k}{t_k}=+\infty .$$

If $0<r_1\leqslant r_2\leqslant\ldots, r_1<+\infty$, $\sum_{k=1}^{\infty}1/r_k<+\infty$, are such that $\mathscr{D}_{\omega_{\{r_k\}}}\subset\mathscr{D}_{\omega_{\{t_k\}}}$, then by Theorems 3.20 and 1.2 there exist an $L_0>0$, an integer $n_0\geqslant 1$, and a $c_0>0$ such that

$$|\omega_{\{t_k\}}(z)|\leqslant c_0|\omega_{\{r_k\}}(L_0z)^{n_0}|,\quad \operatorname{Im} z\leqslant 0;$$

so by Corollary 1.9

$$\int_1^{+\infty}\frac{\ln|\omega_{\{r_k\}}(-it)|}{t^2}\,dt=+\infty .$$

Consequently, $\omega_{\{r_k\}}$ does not satisfy the strong non-quasianalyticity condition.

We conclude that *$\{\mathscr{D}_\omega\}_{\omega\in\Omega_0}$ and $\{\mathscr{D}_\omega\}_{\omega\in\Omega_{00}}$ are not equivalent.*

7.8. Similarly to [51] and [7], rapidly decreasing ω-ultradifferentiable functions $\mathscr{S}_\omega$ can be defined and, for regular ω, the tempered ω-ultradistributions $\mathscr{L}(\mathscr{S}_\omega;X)$

can be characterized among the X-valued hyperfunctions. We note that tempered ω-ultradistribution semi-groups are regular and there exists a duality, via Fourier transformation, between the theory of tempered ω-ultradistribution semigroups and tempered ω-self-adjoint operators.

All these topics will be developed in a forthcomming work of the authors.

7.9. Finally we formulate some problems:

(i) *to characterize those functions* $f\colon [1, +\infty) \to (0, +\infty)$ *such that for a certain* $\omega \in \Omega_{00}$ *and* $c > 0$

$$f(t) \leqslant c|\omega(t)|, \quad t \in [1, +\infty).$$

(ii) *can one remove in the solution of* (H) *the regularity assumption on* ω*?*

(iii) *to characterize intrinsically* $\bigcup_{\omega \in \Omega_{00}} \mathscr{D}'_\omega$ *among all hyperfunctions.*

(iv) *to develop, similarly to the case of differential operators, the theory of elliptic, hyperbolic and parabolic* ω*-ultradifferential operators.*

(v) *to develop the whole theory of* ω*-ultradistributions for* $\boldsymbol{R}^n$ *and, further, for suitably defined* ω*-ultradifferentiable manifolds.*

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SZ.-NAGY–FOIAŞ THEORY AND SIMILARITY FOR A CLASS OF TOEPLITZ OPERATORS

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1. Introduction

1.1. In this paper we determine explicitly the Sz.-Nagy–Foiaş characteristic function of a Toeplitz operator of the form $T_{\varphi/\psi}$, where φ and ψ are finite Blaschke products, ψ having one zero. We use this to prove a similarity theorem (Theorem 2, below) for $T_{\varphi/\psi}$. The reason for considering Toeplitz operators of this special form is to compare Theorem 2 with a similarity theorem from [1], restated here as Theorem 1. These two theorems occupied my two lectures to the Spectral Semester at the Stefan Banach Center.

In Section 1.2, we introduce Toeplitz operators and the similarity problem and in Section 1.3, we discuss the Sz.-Nagy–Foiaş characteristic function.

1.2. Let L^2 denote the L^2 space of Lebesgue measure on $[0, 2\pi]$ and H^2 the L^2 closure of the polynomials in e^{it}. For a bounded measurable function F on $[0, 2\pi]$, the Toeplitz operator T_F is defined on H^2 by

$$T_F x = PFx$$

where P is the projection of L^2 on H^2.

If F is reasonably smooth (for example rational), the spectral theory of T_F is well known [3]. The essential spectrum of T_F is the curve Γ: $t \to F(e^{it})$ and for $\lambda \notin \Gamma$, the index of T_F is minus the winding number of Γ around λ. Either the kernel or the cokernel of $T_F - \lambda I$ is always 0, so that the index describes completely the multiplicity of λ as an eigenvalue. Moreover, T_F has no eigenvalues in the essential spectrum [1].

In [1], the following similarity theorem was proved for T_F.

THEOREM 1. *Suppose that $F(z)$ is a rational function with no poles on $|z| = 1$. Suppose that the curve Γ is a simple closed curve of winding number n about its interior points and suppose that $F(z)$ is n-to-1 in some annulus $r < |z| \leqslant 1$. Then T_F is similar to $T_{\tau(z^n)}$ where τ is the Riemann mapping function from the unit disk to the interior of the curve Γ.*

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