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OPTIMAL CHOICE PROBLEM WITH BACKWARD SOLICITATION

0. Introduction. This paper deals with a generalized choosing problem which originated as the beauty contest problem, the secretary problem or the dowry problem. The basic statement of each of these original problems is that an investigator is to view a group of N objects sequentially. Each of the objects has some characteristic. Let $x_1, x_2, \ldots, x_N$ denote characteristics of these objects, assuming that the N values are different so that the investigator observes sequentially $X_1, X_2, \ldots, X_N$, where $X_1, X_2, \ldots, X_N$ is a permutation of $x_1, x_2, \ldots, x_N$. We assume that all permutations are equally likely.

Let z_k denote the absolute rank of the object with characteristic x_k, i.e.

$$z_k = \operatorname{card}\{1 \leqslant i \leqslant N\colon\ x_i \leqslant x_k\}.$$

If the object is viewed, the investigator must either accept or reject it; and, once rejected, the object cannot be reconsidered. The investigator can make only one choice, and the payoff function has the value 1 if he chooses the absolutely first object and 0 otherwise. Thus, his objective is to use a stopping policy which maximizes the probability of choosing the best object. These kind of problems have been discussed by Gilbert and Mosteller [2] and others referred to in that article. The optimal procedure for such a problem is to reject the first s^*-1 objects, where s^* is the value of s which maximizes

$$P(s, N) = \frac{s-1}{N} \sum_{k=s}^{N} \frac{1}{k-1},$$

and to accept the first relatively best thereafter, i.e. to accept the object s', where $s' \geqslant s^*$ and s' is the smallest s for which $s \geqslant s^*$ and $X_s < X_i$ for $i = 1, 2, \ldots, s^*-1$. It has been shown that s^*/N tends to e^{-1} as N tends to infinity, as does the probability of choosing the best object.

In paper [2], the problem of choosing the object with the absolute rank 1 or 2 is also investigated. It has been shown that the optimal strategy for this problem is given by two constants s_1 and s_2 in the following way: the first s_1-1 objects are rejected, next the first one which is the best is chosen, but beginning with the s_2-th object the second one best among the objects examined so far is also chosen. The constants s_1 and s_2 are chosen to maximize the probability of choosing the first or the second best, i.e.

$$P(s_1, s_2; N) = \frac{2(s_1-1)}{N}\sum_{i=s_1}^{s_2}\frac{1}{i-1}+\frac{(s_1-1)(s_1-s_2)}{N(N-1)}+ \\ +\frac{2(s_1-1)(s_2-2)}{N}\left(\frac{1}{s_2-1}-\frac{1}{N-1}\right).$$

If $N\to\infty$, then $s_1/N\to a$, where a is the solution of the equation $e^{x-1} = 3x/2$, $s_2/N\to\beta = 2/3$ and

$$P(s_1, s_2; N)\to 2a\ln\frac{\beta}{a}+a(a-\beta)+2a(1-\beta).$$

Other versions of the secretary problem were considered by Chow et al. [1], Guseĭn-Zade [3], and Mucci [5], [6]. Yang [7] has generalized the classical secretary problem in which the absolutely first object should be chosen to admit a stochastically successful procurement of previous interviewees, but each has a certain probability of refusing the offer. Recently, Karni and Schwartz [4] have dealt with optimal stopping rules for a sampling process with uncertain recall when the probability of refusing is exponential. In this paper we deal with the secretary problem in which the absolutely first or second object should be chosen and it is possible to solicit previously investigated objects with some probability of success.

Let

$$Y_j(k) = \operatorname{card}\{1\leqslant i\leqslant k:\ X_i\leqslant X_j\}.$$

The random variable $Y_j(k)$ will be called the *current relative rank* of the j-th examined object at the moment k ($j = 1, 2, \ldots, k$). The object with the relative rank $Y_j(k) = l$ is called the *l-th candidate* as long as any other object has the current relative rank equal to l.

Let t_k^l denote the relative (to k) position of the current l-th candidate, i.e.

$$t_k^l = k-S_l[Y_1(k), Y_2(k), \ldots, Y_k(k)],$$

where $S_l[Y_1(k), Y_2(k), \ldots, Y_k(k)] = j$ iff $Y_j(k) = l$ and we assume that $t_k^l = \infty$ if an unsuccessful attempt has been made to procure the $(k-t_k^l)$-th object. Suppose we are in the situation described by the coordinates (s, t_s^1, t_s^2). If an attempt is made to procure the first (second) candidate, this attempt will be successful with probability $p(t_s^1)$ $(p(t_s^2))$. We suppose that $p(0) = 1$, $p(\infty) = 0$ and $p(k)$ is non-increasing in k. The assumption $p(\infty) = 0$ implies that an object which has been unavailable remains unavailable thereafter. In any situation we are able to attempt to procure the first or the second candidate or to obtain the next object without solicitation of the current first or the second one. If at the moment s an attempt is made to procure the first (second) candidate and she refuses the offer, then we can immediately try to choose the second (first) one. The procedure ends if an object is chosen or at the moment N.

1. Basic recursive formulae. Suppose the observations $X_1, X_2, \ldots, X_s$ have been made and $t_s^1 = k$, $t_s^2 = l$. So the investigator is in the situation (s, k, l). We assume that the probability of obtaining the required object in this situation is: $u_f(s, k, l)$ if the next object without solicitation of the current candidates is taken, $u_b^1(s, k, l)$ if the current first candidate is solicited, $u_b^2(s, k, l)$ if the current second candidate is solicited.

Let

$$u(s, k, l) = \max\{u_f(s, k, l),\ u_b^1(s, k, l),\ u_b^2(s, k, l)\}. \tag{1}$$

We adopt the following strategy: if $u(s, k, l) = u_f(s, k, l)$, then the next object without solicitation is taken; if $u(s, k, l) = u_b^t(s, k, l)$, then the t-th candidate is solicited, $t = 1, 2$. (It is obvious that only the first or second candidate is able to be the absolutely first or second object.)

We can write the following simple recursive formulae for all s, k, l:

$$u_f(s, k, l) = \frac{1}{s+1}\,[u(s+1, k+1, 0) + u(s+1, 0, k+1)] + \frac{s-1}{s+1}\, u(s+1, k+1, l+1), \tag{2}$$

$$\begin{aligned} u_b^1(s, k, l) &= p(k)[g_1(s, 1) + g_2(s, 1)] + \bigl(1 - p(k)\bigr)u(s, \infty, l) \\ &= p(k)\,\frac{s(2N-s-1)}{N(N-1)} + \bigl(1-p(k)\bigr)u(s, \infty, l), \end{aligned} \tag{3}$$

$$\begin{aligned} u_b^2(s, k, l) &= p(l)g_2(s, 2) + \bigl(1-p(l)\bigr)u(s, k, \infty) \\ &= p(l)\,\frac{s(s-1)}{N(N-1)} + \bigl(1-p(l)\bigr)u(s, k, \infty), \end{aligned} \tag{4}$$

$$u(N, k, l) = 1 - \bigl(1-p(k)\bigr)\bigl(1-p(l)\bigr). \tag{5}$$

Here $g_a(s, j)$ is the probability that the current relative j-th object is the absolutely a-th one at the moment s:

$$g_a(s, j) = \frac{\binom{a-1}{j-1}\binom{N-a}{s-j}}{\binom{N}{s}}, \qquad \begin{array}{l} a = 1, 2, \ldots, N;\ s = 1, 2, \ldots, N; \\ j = 1, 2, \ldots, \min(a, s). \end{array}$$

These equations are a consequence of the assumption that all permutations of the characteristics x_i $(i = 1, 2, \ldots, N)$ are equally likely.

We can obtain (2) in the following way. If we have the situation (s, k, l) and the next object is taken for investigation without solicitation of any current candidate, then this object has the current relative rank $Y_{s+1}(s+1)$. The random variable $Y_{s+1}(s+1)$ has the uniform distribution. If $Y_{s+1}(s+1) = 1$, then the new situation is $(s+1, 0, k+1)$; if $Y_{s+1}(s+1) = 2$, then the new situation is $(s+1, k+1, 0)$; if $Y_{s+1}(s+1) \neq 1, 2$, then the new situation is $(s+1, k+1, l+1)$. These facts imply (2).

Formulae (3), (4) and (5) are obtained by similar arguments.

If $t_s^1 = \infty$ and $t_s^2 = \infty$, we have by induction, using (2) and (5),

$$u(s, \infty, \infty) = s(s-1) \sum_{j=s+1}^{N} \frac{u(j, 0, \infty) + u(j, \infty, 0)}{j(j-1)(j-2)}. \tag{6}$$

For $s = N-1$, by (2) we have

$$u(N-1, \infty, \infty) = \frac{1}{N}[u(N, 0, \infty) + u(N, \infty, 0)] + \frac{N-2}{N} u(N, \infty, \infty)$$

$$\overset{(5)}{=} \frac{1}{N}[u(N, 0, \infty) + u(N, \infty, 0)].$$

Then (6) is proved for $s = N-1$.

Let us assume that (6) is fulfilled for $t = N-1, N-2, \ldots, s+1$. Then, using (2) and the induction assumption, we have

$$\begin{aligned} u(s, \infty, \infty) &= \frac{1}{s+1}[u(s+1, 0, \infty) + u(s+1, \infty, 0)] + \frac{s-1}{s+1} u(s+1, \infty, \infty) \\ &= \frac{1}{s+1}[u(s+1, 0, \infty) + u(s+1, \infty, 0)] + \\ &\qquad + \frac{s-1}{s+1}(s+1)s \sum_{j=s+2}^{N} \frac{u(j, 0, \infty) + u(j, \infty, 0)}{j(j-1)(j-2)} \\ &= s(s-1) \sum_{j=s+1}^{N} \frac{u(j, 0, \infty) + u(j, \infty, 0)}{j(j-1)(j-2)}. \end{aligned}$$

The system of equations (1)-(5) can be solved recursively to yield the best strategy if the set of values $p(k)$ for $k = 1, 2, \ldots, N-1$ is given. Some general remarks about the optimal strategy are made in the next section.

2. Analysis of recursive formulae. For all proofs we assume $N \geqslant 3$.

THEOREM 1. *There exists an s_2 such that if $s < s_2$, then $u(s, \infty, l) = u_f(s, \infty, l)$ for all l.*

Proof. Write

$$B(s,\infty,l) = u(s,\infty,l) - u_b^2(s,\infty,l).$$

Then, for finite l,

$$u_f(s,\infty,l) - u_b^2(s,\infty,l) = \frac{s-1}{s+1}B(s+1,\infty,l+1) + \frac{1}{s+1}[u(s+1,0,\infty)+u(s+1,\infty,0)] + \frac{s(s-1)}{N(N-1)}p(l+1) + \frac{s-1}{s+1}\big(1-p(l+1)\big)u(s+1,\infty,\infty) - p(l)\frac{s(s-1)}{N(N-1)} - \big(1-p(l)\big)u(s,\infty,\infty).$$

Using (6) we obtain

$$u_f(s,\infty,l) - u_b^2(s,\infty,l) = \frac{s-1}{s+1}B(s+1,\infty,l+1) + s(s-1)[p(l+1)b(s+1) - p(l)b(s)], \tag{7}$$

where

$$b(s) = \frac{1}{N(N-1)} - \frac{1}{s(s-1)}u(s,\infty,\infty).$$

Let

$$s_2 = \min\{3 \leqslant s \leqslant N : b(s) > 0\}.$$

Since $b(s)$ is an increasing function in s, $b(N) > 0$ for $N \geqslant 3$ and $b(2) < 0$, the required s_2 exists. For $s < s_2$ we have $b(s) \leqslant 0$; since $b(s)$ is increasing in s and $p(l)$ is non-increasing in l, we get

$$b(s+1)p(l+1) - b(s)p(l) \geqslant 0.$$

$B(s+1,\infty,l+1)$ is non-negative, and therefore the right-hand side of (7) is non-negative for all l and $s < s_2$. This conclusion completes the proof.

LEMMA 1. *The inequalities*

$$u(s,k,l) \geqslant u(s,k+1,l), \quad u(s,k,l) \geqslant u(s,\infty,l); \tag{8a}$$

$$u(s,k,l) \geqslant u(s,k,l+1), \quad u(s,k,l) \geqslant u(s,k,\infty); \tag{8b}$$

$$u_f(s,k,l) \geqslant u_f(s,k+1,l), \quad u_f(s,k,l) \geqslant u_f(s,\infty,l); \tag{8c}$$

$$u_f(s,k,l) \geqslant u_f(s,k,l+1), \quad u_f(s,k,l) \geqslant u_f(s,k,\infty) \tag{8d}$$

hold for $s = 1,2,\ldots,N$ *and* $k,l = 0,1,2,\ldots,s-1$.

Proof. The first inequality is true because the sets of strategies in these two cases are the same and the first candidate in the situation (s, k, l) is more available than in the situation $(s, k+1, l)$ (by the monotonicity of $p(k)$). The remaining inequalities hold by the same argument.

THEOREM 2. *There exists an s_1 such that if $s < s_1$, then $u_f(s, k, l) > u_b^1(s, k, l)$.*

Proof. Let us investigate the difference $u_f(s, k, l) - u_b^1(s, k, l)$:

$$u_f(s, k, l) - u_b^1(s, k, l) = u_f(s, k, l) - u(s, \infty, l) - p(k)\left[\frac{s(2N-s-1)}{N(N-1)} - u(s, \infty, l)\right].$$

Write

$$\varphi(s, l) = \frac{s(2N-s-1)}{N(N-1)} - u(s, \infty, l).$$

The function $\varphi(s, \infty)$ is increasing in s, $\varphi(1, \infty) < 0$, and $\varphi(N, \infty) > 0$. Let

$$s_1 = \min\{3 \leqslant s \leqslant N\colon \varphi(s, \infty) \geqslant 0\}.$$

From (8b) we have $\varphi(s, l) \leqslant \varphi(s, \infty) < 0$ for $s < s_1$. Therefore the required inequality holds if

$$u_f(s, k, l) - u(s, \infty, l) > 0. \tag{9}$$

If $u(s, \infty, l) = u_f(s, \infty, l)$, then (9) holds by (8c). If $u(s, \infty, l) = u_b^2(s, \infty, l)$ and $s < s_1$, then

$$u_f(s, k, l) - u_b^2(s, \infty, l) = u_f(s, k, l) - u(s, \infty, \infty) - p(l)\left[\frac{s(s-1)}{N(N-1)} - u(s, \infty, \infty)\right]$$

and (9) holds by (8c), (8d) and by the inequality $s(s-1)b(s) \leqslant \varphi(s, \infty)$. This completes the proof of Theorem 2.

THEOREM 3. *If $u(s, k, l) \neq u_f(s, k, l)$ and $u(s, k, \infty) \neq u_f(s, k, \infty)$, then $u_b^2(s, k, l) \leqslant u_b^1(s, k, l)$.*

Proof. From (3) and (4) it follows that

$$u_b^1(s, k, l) - u_b^2(s, k, l) = p(k)\frac{s(2N-s-1)}{N(N-1)} + \\ + \big(1-p(k)\big)u(s, \infty, l) - p(l)\frac{s(s-1)}{N(N-1)} - \big(1-p(l)\big)u(s, k, \infty)$$

for all s, k, l. If $u(s, \infty, l) = u_b^2(s, \infty, l)$ and $u(s, k, \infty) = u_b^1(s, k, \infty)$, then from (3) and (4) we obtain

$$u_b^1(s, k, l) - u_b^2(s, k, l) = p(k)p(l)\frac{2s(N-s)}{N(N-1)} \geqslant 0.$$

If $u(s, \infty, l) = u_f(s, \infty, l)$ and $u(s, k, \infty) = u_b^1(s, k, \infty)$, then

$$u_b^1(s, k, l) - u_b^2(s, k, l) \geqslant p(k)p(l)\frac{2s(N-s)}{N(N-1)} \geqslant 0,$$

because in this case $u(s, \infty, l) \geqslant u_b^2(s, \infty, l)$. This completes the proof.

Theorems 1-3 imply the following

COROLLARY. *In the optimal procedure for the considered problem we should not solicit any object to some moment s_1 and we should not solicit the second candidate to some moment $s_2 \geqslant s_1$.*

Proof. If $s < s_1$, then $s(s-1)b(s) < \varphi(s, \infty) < 0$ and by (4) we have

$$(10) \qquad \begin{aligned} & u_f(s, k, l) - u_b^2(s, k, l) \\ & \quad = u_f(s, k, l) - u(s, k, \infty) - p(l)\left[\frac{s(s-1)}{N(N-1)} - u(s, k, \infty)\right] \\ & \quad \geqslant u_f(s, k, \infty) - u(s, k, \infty) - p(l)s(s-1)b(s) > 0, \end{aligned}$$

because from Theorem 2 for $s < s_1$ we get $u(s, k, \infty) = u_f(s, k, \infty)$. Consequently, for $s < s_1$ we have $u(s, k, l) = u_f(s, k, l)$.

Let $s_1 \leqslant s < s_2$. If $u(s, k, \infty) = u_f(s, k, \infty)$, then from (10) and Lemma 1 we obtain

$$u_f(s, k, l) - u_b^2(s, k, l) \geqslant u_f(s, k, l) - u_f(s, k, \infty) - p(l)s(s-1)b(s) > 0.$$

If $u(s, k, \infty) = u_b^1(s, k, \infty)$ and $u(s, k, l) \neq u_f(s, k, l)$, then from Theorem 3 we infer that $u_b^1(s, k, l) \geqslant u_b^2(s, k, l)$ and $u_f(s, k, l) > u_b^2(s, k, l)$. Consequently, for $s < s_2$ we have $u(s, k, l) \geqslant u_b^2(s, k, l)$. This completes the proof of the Corollary.

Write

$$(11) \qquad c(s) = \sum_{j=s}^{N} \frac{2(N-1)}{(j-1)(j-2)} = \frac{2(N-s+1)}{s-2}, \qquad c(N+1) = 0.$$

Let $s_0 = \min\{s\colon c(s+2) > 1\}$.

THEOREM 4. *Let $p(k) > 0$ for each finite k. If there exists a τ such that τ is the smallest $s > \max(s_0, s_2)$ satisfying*

$$(12) \qquad \frac{p(k+1)}{p(k)} \leqslant \frac{1-c(s+1)}{1-c(s+2)} \quad \textit{for all } k < \infty,$$

then solicitation to the first candidate should be made; if she refuses, then solicitation to the second candidate should be made whenever the game does not stop before the moment τ.

Proof. Since $[1-c(s+1)]/[1-c(s+2)]$ is an increasing function for $s > s_0$, inequality (12) is fulfilled for $s \geqslant \tau$. We can immediately verify that for all $k < \infty$ and all l the inequality

$$u_f(N-1, k, l) \leqslant u_b^1(N-1, k, l)$$

holds. Write

$$u_b(s, k, l) = p(k)\frac{s(2N-s-1)}{N(N-1)} + \big(1-p(k)\big)p(l)\frac{s(s-1)}{N(N-1)} + \\ + \big(1-p(k)\big)\big(1-p(l)\big)u(s, \infty, \infty). \tag{13}$$

$u_b(s, k, l)$ is the probability that we obtain the required object if in the position (s, k, l) we solicit the first candidate; if she refuses, we solicit the second one. For the proof of Theorem 4 we have to show that $u_f(s, k, l) \leqslant u_b(s, k, l)$ for $s \geqslant \tau$. We prove this inequality by backward induction. For $s = N-1$, from (13), (2), (10) and (12) we obtain immediately

$$\begin{aligned} u_f(N-1, k, l) - u_b(N-1, k, l) \\ &= \frac{2}{N}\big(1-p(k)\big) + \frac{N-4}{N}\big(1-p(k)\big)\big(1-p(l)\big) - \\ &\quad - \frac{N-2}{N}\big(1-p(k+1)\big)\big(1-p(l+1)\big) \\ &\leqslant \big(1-p(k)\big)\frac{N-2}{N}\Big[\big(1-c(N+1)\big)p(l+1) - \big(1-c(N)\big)p(l)\Big] \leqslant 0. \end{aligned}$$

Let us assume for induction that

$$u_f(s, k, l) \leqslant u_b(s, k, l) \\ \text{for all } k, l \text{ and } s = N-1, N-2, \ldots, t+1. \tag{14}$$

From (2) and (13) we have

$$\begin{aligned} u_f(t, k, l) - u_b(t, k, l) &= \frac{1}{t+1}[u(t+1, 0, k+1) + u(t+1, k+1, 0)] + \\ &+ \frac{t-1}{t+1}u(t+1, k+1, l+1) - p(k)\frac{t(2N-t-1)}{N(N-1)} - \\ &- \big(1-p(k)\big)p(l)\frac{t(t-1)}{N(N-1)} - \big(1-p(k)\big)\big(1-p(l)\big)u(t, \infty, \infty). \end{aligned}$$

By (14) we obtain

$$u_f(t,k,l)-u_b(t,k,l) = \frac{2}{N}+\frac{2t(N-t-1)}{N(N-1)}p(k+1)-$$

$$-\frac{t(t-1)}{N(N-1)}\big(1-p(k+1)\big)\big(1-p(l+1)\big)\left[1-\frac{N(N-1)}{(t+1)t}u(t+1,\infty,\infty)\right]-$$

$$-p(k)\frac{2t(N-t)}{N(N-1)}+\frac{t(t-1)}{N(N-1)}\big(1-p(k)\big)\big(1-p(l)\big)\times$$

$$\times\left[1-\frac{N(N-1)}{t(t-1)}u(t,\infty,\infty)\right].$$

For $t > \max(s_0, s_2)$

$$\frac{2t(N-t-1)\big(1-c(t+1)\big)}{N(N-1)\big(1-c(t+2)\big)}-\frac{2t(N-t)}{N(N-1)} \leqslant -\frac{2}{N}.$$

Indeed, the left-hand side of this inequality is an increasing function in t and for $t = N-1$ this function is equal to $-2/N$. Therefore, by the monotonicity of $p(k)$, by (11), (12) and the induction assumption that $u(s,k,l) = u_b(s,k,l)$ we get

$$u_f(t,k,l)-u_b(t,k,l)$$

$$\leqslant \frac{2}{N}+\frac{2t(N-t-1)\big(1-c(t+1)\big)}{N(N-1)\big(1-c(t+2)\big)}p(k)-p(k)\frac{2t(N-t)}{N(N-1)}+$$

$$+\frac{t(t-1)}{N(N-1)}\big(1-p(k)\big)\Big[\big(1-c(t+1)\big)\big(1-p(l)\big)-\big(1-p(l+1)\big)\big(1-c(t+2)\big)\Big]$$

$$\leqslant \big(1-p(k)\big)\left\{\frac{2}{N}+\frac{t(t-1)}{N(N-1)}\Big[\big(1-c(t+1)\big)-\big(1-c(t+2)\big)\Big]+\right.$$

$$\left.+\frac{t(t-1)}{N(N-1)}\Big[p(l+1)\big(1-c(t+2)\big)-p(l)\big(1-c(t+1)\big)\Big]\right\}$$

$$= \big(1-p(k)\big)\frac{t(t-1)}{N(N-1)}\Big[p(l+1)\big(1-c(t+2)\big)-p(l)\big(1-c(t+1)\big)\Big].$$

The induction stops at $t = \tau$. Thus Theorem 4 is proved.

THEOREM 5. *If*

$$\frac{p(l+1)}{p(l)} > \frac{N-4}{N-2} \quad \textit{for all } l<\infty \tag{15}$$

and the first candidate has refused the offer, then the optimal strategy is to attempt no procurement of the second one until the N objects are investigated.

Proof. It suffices to show that $u_f(s, \infty, l) > u_b^2(s, \infty, l)$ for all $l < \infty$ and $s = 3, 4, \ldots, N-1$. From (15), (2) and (4) we obtain

$$u_f(N-1, \infty, l) - u_b^2(N-1, \infty, l) = \frac{N-2}{N}\, p(l+1) - \frac{N-4}{N}\, p(l) > 0.$$

Let us assume for backward induction that $u_f(s, \infty, l) > u_b^2(s, \infty, l)$ for all $l < \infty$ and $s = N-1, N-2, \ldots, t+1$. Equation (2) and the above assumption give us

$$u_f(t, \infty, l) - u_b^2(t, \infty, l)$$

$$= \sum_{j=t+1}^{N} \frac{t(t-1)}{j(j-1)(j-2)} [u(j, 0, \infty) + u(j, \infty, 0)] + \frac{t(t-1)}{N(N-1)}\, p(l+N-t) -$$

$$-p(l)\, \frac{t(t-1)}{N(N-1)} - \big(1-p(l)\big) \sum_{j=t+1}^{N} \frac{t(t-1)}{j(j-1)(j-2)} [u(j, 0, \infty) + u(j, \infty, 0)]$$

$$= -p(l)\, t(t-1) \left[\frac{1}{N(N-1)} - \sum_{j=t+1}^{N} \frac{u(j, 0, \infty) + u(j, \infty, 0)}{j(j-1)(j-2)} \right] +$$

$$+ \frac{t(t-1)}{N(N-1)}\, p(l+N-t).$$

Write

$$f(t) = \frac{u_f(t, \infty, l) - u_b^2(t, \infty, l)}{[t(t-1)/N(N-1)]\, p(l)}.$$

Then

$$f(t) = \frac{p(l+N-t)}{p(l)} - 1 + N(N-1) \sum_{j=t+1}^{N} \frac{u(j, 0, \infty) + u(j, \infty, 0)}{j(j-1)(j-2)}.$$

$f(t)$ is a decreasing function in t. We have

$$f(t+1) - f(t) = \frac{p(l+N-t-1)}{p(l)} - \frac{p(l+N-t)}{p(l)} -$$

$$- \frac{N(N-1)}{t(t-1)(t+1)} [u(t+1, 0, \infty) + u(t+1, \infty, 0)].$$

By (15) and by the inequality

$$u(t+1, 0, \infty) + u(t+1, \infty, 0) \geqslant \frac{2(t+1)}{N}$$

we obtain

$$f(t+1) - f(t) = \frac{p(l+N-t-1)}{p(l)} \left(1 - \frac{N-4}{N-2}\right) - \frac{2(N-1)}{t(t-1)}.$$

Since $p(l)$ is non-increasing, we have

$$f(t+1)-f(t) \leqslant \frac{2}{N-2} - \frac{2(N-1)}{t(t-1)} \leqslant 0. \tag{16}$$

Formula (16) implies the following inequalities: $f(3) \geqslant f(4) \geqslant \ldots \ldots \geqslant f(N-1) > 0$. Thus Theorem 5 is proved.

3. Example. We deal with the case $p(k) = p = \text{const}$ for all $k \neq 0$ and $k \neq \infty$.

THEOREM 6. *Under the above assumption the optimal strategy is to proceed to stage s_1-1 without solicitation; solicit the first candidate as she appears among the remaining $N-s_1-1$ objects, but beginning with the s_2-th object, as the second candidate appears, solicit the first candidate; if she refuses, choose the second one. If the game does not stop before stage N, solicit the first candidate; if she refuses, solicit the second one. The values of s_1 and s_2 $(s_1 \leqslant s_2)$ are those which maximize*

$$\begin{aligned} P(s_1, s_2; N) &= \frac{s_1-1}{N(N-1)} \sum_{j=s_1}^{s_2-1} \frac{2N-j-1}{j-1} + \frac{(s_1-1)(s_2-2)}{N(N-1)} \sum_{j=s_2}^{N} \frac{2N-j-1}{(j-1)(j-2)} + \\ &+ \frac{(s_1-1)(s_2-2)}{N(N-1)} \sum_{j=s_2}^{N} \left[p\frac{2N-j-1}{(j-1)(j-2)} + (1-p)\frac{1}{j-2}\right] + \\ &+ \frac{(s_1-1)(s_2-2)}{N(N-1)} [1-(1-p)^2] \end{aligned} \tag{17}$$

and the probability of obtaining the required object is $P(s_1, s_2; N)$.

Proof. We have to show that

$$u_f(s, k, l) \geqslant u_b^1(s, k, l) \quad \text{for all } k \neq 0,\ l \neq 0,\ 3 \leqslant s \leqslant N-1, \tag{18}$$

$$u_f(s, 0, l) \geqslant u_b^1(s, 0, l) \quad \text{for all } l \text{ iff } s < s_1, \tag{19}$$

$$u_f(s, k, 0) \geqslant u_b^2(s, k, 0) \quad \text{for all } k \neq 0 \text{ iff } s < s_1, \tag{20}$$

$$u_f(s, k, 0) \leqslant u_b^1(s, k, 0) \quad \text{for all } k \neq \infty \text{ iff } s > s_2\ (s_1 \leqslant s_2), \tag{21}$$

$$u_f(s, \infty, 0) \leqslant u_b^2(s, \infty, 0) \quad \text{for } s \geqslant s_2. \tag{22}$$

First we show that

$$u_f(s, \infty, l) > u_b^2(s, \infty, l) \quad \text{for } l \neq 0,\ 3 \leqslant s \leqslant N-1. \tag{23}$$

For $s = N-1$ we have

$$u_f(N-1, \infty, l) - u_b^2(N-1, \infty, l) = \frac{2}{N}p > 0.$$

We assume for backward induction that $u_f(s, \infty, l) > u_b^2(s, \infty, l)$ for $s = N-1, N-2, \ldots, t+1$. From (2) and (4) we obtain

$$u_f(t, \infty, l) - u_b^2(t, \infty, l) = t(t-1) \sum_{j=t+1}^{N} \frac{u(j, 0, \infty) + u(j, \infty, 0)}{j(j-1)(j-2)} +$$

$$+ \frac{t(t-1)}{N(N-1)} p - p \frac{t(t-1)}{N(N-1)} - (1-p) u(t, \infty, \infty)$$

$$= p\, t(t-1) \sum_{j=t+1}^{N} \frac{u(j, 0, \infty) + u(j, \infty, 0)}{j(j-1)(j-2)}.$$

Inequality (18) is proved for $k = \infty$, $l \neq 0$, $3 \leqslant s \leqslant N-1$.

Let $k \neq 0$, $l \neq 0$, $s = N-1$. We have

$$u_f(N-1, k, l) - u_b^1(N-1, k, l)$$

$$= \frac{2}{N} + \frac{N-2}{N} [1-(1-p)^2] - p - (1-p)\left[\frac{2}{N} + \frac{N-2}{N} p\right] = 0.$$

Let us assume that (18) is valid for $s = N-1, N-2, \ldots, t+1$. From (2), (3) and (23) we obtain

$$u_f(t, k, l) - u_b^1(t, k, l)$$

$$= t(t-1)\left[\sum_{j=t+1}^{N} \frac{u(j, 0, k+j-t)}{j(j-1)(j-2)} + \sum_{j=t+1}^{N} \frac{u(j, k+j-t, 0)}{j(j-1)(j-2)}\right] +$$

$$+ \frac{t(t-1)}{N(N-1)} [1-(1-p)^2] - p \frac{t(2N-t-1)}{N(N-1)} -$$

$$-(1-p) t(t-1) \sum_{j=t+1}^{N} \frac{u(j, 0, \infty) + u(j, \infty, 0)}{j(j-1)(j-2)} - p(1-p) \frac{t(t-1)}{N(N-1)}.$$

By (1) we have

$$u(j, 0, k+j-t) \geqslant u_b^2(j, 0, k+j-t) = p \frac{j(j-1)}{N(N-1)} + (1-p) u(j, 0, \infty),$$

$$u(j, k+j-t, 0) \geqslant u_b^1(j, k+j-t, 0) = p \frac{j(2N-j-1)}{N(N-1)} + (1-p) u(j, \infty, 0)$$

and, consequently,

$$u_f(t, k, l) - u_b^1(t, k, l) \geqslant \frac{t(t-1)}{N(N-1)} p \sum_{j=t+1}^{N} \frac{2(N-1)}{(j-1)(j-2)} - \frac{2(N-t)}{t-1}.$$

Write

$$v(t) = \sum_{j=t+1}^{N} \frac{2(N-1)}{(j-1)(j-2)} - 2\frac{N-t}{t-1} \quad \text{for } 3 \leqslant t \leqslant N-1.$$

The function $v(t)$ is constant and $v(N-1) = 0$ so that $v(t) = 0$ for $3 \leqslant t \leqslant N-1$. Hence $u_f(s, k, l) \geqslant u_b^1(s, k, l)$ for all $k \neq 0$, $l \neq 0$, $3 \leqslant s \leqslant N-1$.

Now we show that (19) is fulfilled. From (2), (3) and (18) we obtain

$$u_f(s, 0, l) - u_b^1(s, 0, l) = s(s-1) \sum_{j=s+1}^{N} \frac{u(j, 0, j-s) + u(j, j-s, 0)}{j(j-1)(j-2)} + \\ + \frac{s(s-1)}{N(N-1)}[1-(1-p)^2] - \frac{s(2N-s-1)}{N(N-1)}.$$

Write

$$g(s) = \frac{u_f(s, 0, l) - u_b^1(s, 0, l)}{s(s-1)/N(N-1)}.$$

Then

$$g(s) = N(N-1) \sum_{j=s+1}^{N} \frac{u(j, 0, j-s) + u(j, j-s, 0)}{j(j-1)(j-2)} + \\ + [1-(1-p)^2] + \frac{2N-s-1}{s-1}.$$

The function $g(s)$ is decreasing for $3 \leqslant s \leqslant N-1$. We have

$$g(s) - g(s+1) = N(N-1)\frac{u(s+1, 0, 1) + u(s+1, 1, 0)}{(s+1)s(s-1)} - \frac{2(N-1)}{s(s-1)}.$$

By (1) we obtain

$$u(s+1, 0, 1) + u(s+1, 1, 0) \geqslant u_b^1(s+1, 0, 1) + u_b^2(s+1, 1, 0) \\ = \frac{2(s+1)(N-1)}{N(N-1)},$$

whence $g(s) - g(s+1) > 0$. Since $g(N-1) = -(1-p)^2 < 0$, there exists an s_1 such that (19) is fulfilled.

For the proof of (20) let us observe that if $s < s_1$, then from (2), (4) and from the proof of (19) we get

$$u_f(s, k, 0) - u_b^2(s, k, 0) = s(s-1) \sum_{j=s+1}^{N} \frac{u(j, 0, j-s) + u(j, j-s, 0)}{j(j-1)(j-2)} + \\ + [1-(1-p)^2]\frac{s(s-1)}{N(N-1)} - \frac{s(s-1)}{N(N-1)} > u_f(s, 0, l) - u_b^1(s, 0, l) > 0.$$

Now we verify (22). From (2), (4) and (18) we obtain

$$u_f(s, \infty, 0)-u_b^2(s, \infty, 0) = s(s-1)\sum_{j=s+1}^{N}\frac{u(j, 0, \infty)+u(j, \infty, 0)}{j(j-1)(j-2)}-\\ -\frac{s(s-1)}{N(N-1)}(1-p).$$

Write

$$w(s) = \frac{u_f(s, \infty, 0)-u_b^2(s, \infty, 0)}{s(s-1)/N(N-1)}.$$

Then

$$w(s) = N(N-1)\sum_{j=s+1}^{N}\frac{u(j, 0, \infty)+u(j, \infty, 0)}{j(j-1)(j-2)}-(1-p).$$

$w(s)$ is a decreasing function in s, $w(N) < 0$, and $w(s_1) \geqslant 0$ by (18)-(21). Hence there exists an s_2' such that $u_f(s, \infty, 0) < u_b^2(s, \infty, 0)$ iff $s \geqslant s_2'$ and $s_2' \geqslant s_1$.

Let us investigate the difference

(24)

$$u_f(s, k, 0)-u_b^1(s, k, 0) = s(s-1)\sum_{j=s+1}^{N}\frac{u(j,0,k+j-s)+u(j,k+j-s,0)}{j(j-1)(j-2)}+\\ +\frac{s(s-1)}{N(N-1)}[1-(1-p)^2]-p\frac{s(2N-s-1)}{N(N-1)}-(1-p)u(s, \infty, 0).$$

If $s \leqslant s_2'$, then $u(s, \infty, 0) = u_f(s, \infty, 0)$ and

$$u_f(s, k, 0)-u_b^1(s, k, 0) = s(s-1)\left[\sum_{j=s+1}^{N}\frac{u(j, 0, k+j-s)-u(j, 0, \infty)}{j(j-1)(j-2)}+\right.\\ \left.+\sum_{j=s+1}^{N}\frac{u(j, k+j-s, 0)-u(j, \infty, 0)}{j(j-1)(j-2)}\right]+\\ +p\left[s(s-1)\sum_{j=s+1}^{N}\frac{u(j, 0, \infty)+u(j, \infty, 0)}{j(j-1)(j-2)}-\frac{2s(N-s)}{N(N-1)}\right]\\ \geqslant s(s-1)\sum_{j=s+1}^{N}\left[\frac{u(j, 0, k+j-s)-u(j, 0, \infty)}{j(j-1)(j-2)}+\right.\\ \left.+\frac{u(j, k+j-s, 0)+u(j, \infty, 0)}{j(j-1)(j-2)}\right]+\\ +p\frac{s(s-1)}{N(N-1)}\left[\sum_{j=s+1}^{N}\frac{2(N-1)}{(j-1)(j-2)}-\frac{2(N-s)}{s-1}\right]\geqslant 0.$$

By (22), for $s \geqslant s_2'$ we have $u(s, \infty, 0) = u_b^2(s, \infty, 0)$ and, by (24),

$$u_f(s, k, 0) - u_b^1(s, k, 0) = s(s-1)\sum_{j=s+1}^{N} \frac{u(j, 0, k+j-s) + u(j, k+j-s, 0)}{j(j-1)(j-2)} - \\ - \frac{s(s-1)}{N(N-1)}(1-p)^2 - p\frac{2s(N-s)}{N(N-1)}.$$

Let

$$h(s) = \frac{u_f(s, k, 0) - u_b^1(s, k, 0)}{s(s-1)/N(N-1)}.$$

For $s \geqslant s_2'$ the function $h(s)$ is decreasing. Indeed,

$$h(s) - h(s+1) = N(N-1)\frac{u(s+1, 0, k+1) + u(s+1, k+1, 0)}{(s+1)(s-1)s} - p\frac{2(N-1)}{s(s-1)}$$
$$\geqslant N(N-1)\frac{u(s+1, 0, k+1) + u(s+1, k+1, 0)}{(s+1)(s-1)s} - \frac{2(N-1)}{s(s-1)}$$
$$= g(s) - g(s+1) \geqslant 0.$$

From (24) we have $h(s_2') \geqslant 0$. Moreover,

$$h(N-1) = (1-p)\left(p - \frac{N-4}{N-2}\right) < 0 \quad \text{iff} \quad p < \frac{N-4}{N-2}.$$

Consequently, if $p < (N-4)/(N-2)$, then there exists an $s_2 \geqslant s_1$ (and $s_2 \geqslant s_2'$) such that (22) is fulfilled.

The probability $P(s_1, s_2; N)$ of obtaining the required object for fixed s_1 and s_2 is given by the formula

$$(25) \quad P(s_1, s_2; N) = \sum_{j=s_1}^{s_2-1} P\{Y_j(j) = 1 \mid Y_i(i) \neq 1, s_1 \leqslant i < j\}[g_1(j, 1) + \\ + g_2(j, 1)] + \sum_{j=s_2}^{N} P\{Y_j(j) = 1 \mid Y_i(i) \neq 1, s_1 \leqslant i < j; \\ Y_i(i) \neq 2, s_2 \leqslant i < j\}[g_1(j, 1) + g_2(j, 1)] + \\ + \sum_{j=s_2}^{N} P\{Y_j(j) = 2 \mid Y_i(i) \neq 1, s_1 \leqslant i < j;\ Y_i \neq 2, s_2 \leqslant i < j\} \times \\ \times \{p[g_1(j, 1) + g_2(j, 1)] + (1-p)g_2(j, 2)\} + \\ + P\{Y_j(j) \neq 1, s_1 \leqslant j \leqslant N;\ Y_j(j) \neq 2, s_2 \leqslant j \leqslant N\}[1 - (1-p)^2] \\ = \sum_{j=s_1}^{s_2-1} [g_1(j, 1) + g_2(j, 1)]\frac{s_1 - 1}{j(j-1)} +$$

$$+\sum_{j=s_2}^{N}[g_1(j,1)+g_2(j,1)]\frac{(s_1-1)(s_2-2)}{j(j-1)(j-2)}+$$

$$+\sum_{j=s_2}^{N}\{p[g_1(j,1)+g_2(j,1)]+(1-p)g_2(j,2)\}\frac{(s_1-1)(s_2-2)}{j(j-1)(j-2)}+$$

$$+\frac{(s_1-1)(s_2-2)}{N(N-1)}[1-(1-p)^2],$$

where

$$g_a(s,j)=\frac{\binom{a-1}{j-1}\binom{N-a}{s-j}}{\binom{N}{s}}.$$

Formulae (25) and (17) are equivalent.

If $\lim_{N\to\infty} s_1/N = a$ and $\lim_{N\to\infty} s_2/N = \beta$, then

$$\lim_{n\to\infty}P(s_1,s_2;\,N)=2a\ln\frac{a}{\beta}+a(a-\beta)+2a(1-\beta)(1+p)+$$
$$+2pa\beta\ln\frac{1}{\beta}+a\beta[1-(1-p)^2].$$

The solution a^*, β^* of the set of equations

$$\frac{2}{\beta}-2+2p\ln\beta-(1-p)^2=0,$$

$$2(1-p\beta)\ln\beta-2\ln a+2a+2p-2(1+p)\beta-\beta(1-p)^2=0$$

maximizes the asymptotic probability of obtaining the required object. If $p=0$, then the solution of the problem is equivalent to the solution given in [1] and [2] for a similar problem without solicitation.

4. Remark. The method described above may be extended to the problem in which one of the q best applicants is required with possibility of backward solicitation. The calculations in such a problem are more complicated.

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References

[1] Y. S. Chow, S. Moriguti, H. Robbins and S. M. Samuels, *Optimal selection based on relative rank (the "Secretary Problem")*, Israel J. Math. 2 (1964), p. 81-90.

[2] J. P. Gilbert and F. Mosteller, *Recognizing the maximum of a sequence*, J. Amer. Statist. Assoc. 61 (1966), p. 35-73.

[3] S. M. Guseĭn-Zade (С. М. Гусейн-Заде), *Задача выбора и оптимальное правило остановки последовательности независимых испытаний*, Теор. вероятност. и применен. 11 (1966), p. 534-537.

[4] E. Karni and A. Schwartz, *Two theorems on optimal stopping with backward solicitation*, J. Appl. Probability 14 (1977), p. 869-875.

[5] A. G. Mucci, *Differential equations and optimal choice problems*, Ann. Statist. 1 (1973), p. 104-113.

[6] — *On a class of secretary problems*, Ann. Probability 1 (1973), p. 417-427.

[7] M. C. K. Yang, *Recognizing the maximum of a random sequence based on relative rank with backward solicitation*, J. Appl. Probability 11 (1974), p. 504-512.

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OPTYMALNE POSZUKIWANIE NAJLEPSZYCH OBIEKTÓW Z MOŻLIWOŚCIĄ POWROTU

STRESZCZENIE

W pracy rozpatrzono modyfikację „problemu sekretarki", gdy wybiera się jeden z dwu najlepszych obiektów z maksymalnym prawdopodobieństwem i możliwy jest powrót do zbadanych obiektów. Próba wyboru obiektu, który był wcześniej badany, może zakończyć się sukcesem lub porażką. Specjalny przypadek, ze stałym prawdopodobieństwem odmowy, został rozważony dokładnie.
