ultaneously valid for $K_D(z, t)$ for any $k \geqslant 1$. However, we do not know whether properties (A_k) and (B) are valid for certain special classes of Weil analytic polyhedra, e.g. for polyhedra in C^n whose skeleton is totally real or for polyhedra in C^n which are defined by n holomorphic functions.

CONCLUSION. *If we join together the results of* [6], [8], [7] *and this paper we obtain the following picture*:

Conditions (A_∞) *and* (B) *hold for Bergman kernel functions of the following bounded domains*: 1. *Plane domains with C^∞-boundaries*; 2. *Strictly pseudoconvex domains with C^∞-boundaries*; 3. *Pseudoconvex domains with real analytic boundaries*; 4. *Complete circular strictly starlike domains*; 5. *Cartesian products of domains belonging to the union of classes* 1–4.

Thus our approach to the problem of the smooth extension of biholomorphic mappings, based on conditions (A_∞) and (B), seems to be quite universal. However, there exist at least three important classes of domains, for which we have no information about the boundary behaviour of their Bergman kernel functions: pseudoconvex domains with C^∞-boundaries, strictly pseudoconvex domains with boundaries of class C^k, $2 \leqslant k < \infty$, and analytic polyhedra (see Remark 4). It is a difficult and important problem to study the boundary behaviour of the Bergman kernel function in these cases.

Added in proof. In a paper: E. Ligocka, *The Hölder continuity of the Bergman projection and proper holomorphic mappings* (Studia Math., to appear) it was proved that if D is a strictly pseudoconvex domain with a boundary of class C^{K+4} then conditions (A_K) and (B) are valid for D.

References

[1] S. R. Bell, *Non-vanishing of the Bergman kernel function at boundary points of certain domains in C^n*, Math. Ann. (to appear).
[2] J. Burghea, *The Bergman projection for plane regions*, Studia Math. (to appear).
[3] G. Folland, *Introduction to partial differential equations*, Mathematical Notes, Princeton University Press, 1976.
[4] G. Folland and J. J. Kohn, *The Neumann problem for the Cauchy–Riemann complex*, Ann. of Math. Studies, Princeton University Press, 1972.
[5] G. M. Goluzin, *Geometric theory of functions of a complex variable* (Russian), Moscow 1966.
[6] E. Ligocka, *How to prove Fefferman's theorem without use of differential geometry*, Ann. Polon. Math. (to appear).
[7] —, *Some remarks on extension of biholomorphic mappings*, Proceedings of the Conference on Analytic Functions, Kozubnik 1979, Lecture Notes in Mathematics 798, Springer-Verlag, Berlin–Heidelberg–New York 1980.
[8] S. Webster, *Biholomorphic mappings and the Bergman kernel off the diagonal*, Inventiones Math. (to appear).

Presented to the Semester
COMPLEX ANALYSIS
February 15–May 30, 1979

COMPLEX ANALYSIS
BANACH CENTER PUBLICATIONS, VOLUME 11
PWN-POLISH SCIENTIFIC PUBLISHERS
WARSAW 1983

ANALYTIC FUNCTIONALS ON THE SPHERE AND THEIR FOURIER–BOREL TRANSFORMATIONS

MITSUO MORIMOTO

Department of Mathematics, Sophia University, Tokyo, Japan

0. Introduction

Let S^{n-1} be the unit sphere in $\boldsymbol{R}^n$. The space $\mathscr{B}(S^{n-1})$ of hyperfunctions on S^{n-1} is, by definition, the dual space of the space $\mathscr{A}(S^{n-1})$ of real-analytic functions on S^{n-1}. For a hyperfunction $T \in \mathscr{B}(S^{n-1})$, Hashizume–Kowata–Minemura–Okamoto [2] defined the transformation

$$\mathscr{P}_\lambda\colon\ T \in \mathscr{B}(S^{n-1}) \mapsto \mathscr{P}_\lambda T(\xi) = \big\langle T_\omega, \exp(i\lambda\langle \xi, \omega\rangle)\big\rangle, \tag{0.1}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \boldsymbol{R}^n$ and $\lambda \neq 0$ is a fixed complex number. They showed that the image of $\mathscr{B}(S^{n-1})$ under the transformation $\mathscr{P}_\lambda$ is strictly contained in the space $C_\lambda^\infty(\boldsymbol{R}^n)$ of C^∞-functions f on $\boldsymbol{R}^n$ which satisfy the differential equation

$$(\Delta_\xi + \lambda^2) f(\xi) = 0, \tag{0.2}$$

where $\Delta_\xi = \dfrac{\partial^2}{\partial \xi_1^2} + \dfrac{\partial^2}{\partial \xi_2^2} + \ldots + \dfrac{\partial^2}{\partial \xi_n^2}$ is the Laplacian. They constructed a space $\tilde{\mathscr{B}}(S^{n-1})$ which contains strictly $\mathscr{B}(S^{n-1})$, using the sequence of spherical harmonic functions and claimed the transformation $\mathscr{P}_\lambda$ maps $\tilde{\mathscr{B}}(S^{n-1})$ onto $C_\lambda^\infty(\boldsymbol{R}^n)$. But the meaning of $\tilde{\mathscr{B}}(S^{n-1})$ was obscure for us. In the case $n = 2$, Helgason [4] showed that $\tilde{\mathscr{B}}(S^{n-1})$ is the space of "entire functionals". (See our previous paper [9] for the details of the case $n = 2$.) Our aim in this paper is to extend Helgason's result in the case of general n. The space $\tilde{\mathscr{B}}(S^{n-1})$ turns out to be the dual space of the space $\operatorname{Exp}(\tilde{S}^{n-1})$ of the holomorphic functions of exponential type on the complex sphere $\tilde{S}^{n-1} = \{z \in \boldsymbol{C}^n;\ z_1^2 + z_2^2 + \ldots + z_n^2 = 1\}$.

We will consider the following spaces of functions or functionals on the sphere S^{n-1}: $L^2(S^{n-1})$ is the space of L^2 functions on S^{n-1}, $C^\infty(S^{n-1})$ is the space of C^∞-functions on S^{n-1} and $\mathscr{A}(S^{n-1})$ is the space of real-analytic functions on S^{n-1}. $\mathscr{O}(\tilde{S}^{n-1})$ is the space of holomorphic functions on the complex sphere $\tilde{S}^{n-1}$ and $\operatorname{Exp}(\tilde{S}^{n-1})$ is the subspace of $\mathscr{O}(\tilde{S}^{n-1})$ of holomorphic functions of exponential type. By the restriction of variables, the spaces $\mathscr{O}(\tilde{S}^{n-1})$ and $\operatorname{Exp}(\tilde{S}^{n-1})$ are considered as subspaces of $\mathscr{A}(S^{n-1})$. The dual space of $C^\infty(S^{n-1})$ is the space $\mathscr{D}'(S^{n-1})$ of distributions on S^{n-1} and the dual space of $\mathscr{A}(S^{n-1})$ is the space $\mathscr{B}(S^{n-1})$ of hyperfunctions on S^{n-1}. $\mathscr{O}'(\tilde{S}^{n-1})$ and $\mathrm{Exp}'(\tilde{S}^{n-1})$ are the dual spaces of $\mathscr{O}(\tilde{S}^{n-1})$ and $\mathrm{Exp}(\tilde{S}^{n-1})$ respectively and their elements will be called analytic functionals.

Using the bilinear form

$$(f, g) = \int_{S^{n-1}} f(\omega)g(\omega)d\Omega_n(\omega), \tag{0.3}$$

we can consider a function f as a functional $g \mapsto (f, g)$. Thus we have a chain of spaces of functions or functionals on S^{n-1}:

$$\begin{aligned}\mathrm{Exp}(\tilde{S}^{n-1}) \subset \mathscr{O}(\tilde{S}^{n-1}) \subset \mathscr{A}(S^{n-1}) \subset C^\infty(S^{n-1}) \subset L^2(S^{n-1})\\ \subset \mathscr{D}'(S^{n-1}) \subset \mathscr{A}'(S^{n-1}) \subset \mathscr{O}'(\tilde{S}^{n-1}) \subset \mathrm{Exp}'(\tilde{S}^{n-1}).\end{aligned} \tag{0.4}$$

It is known that a function $f \in L^2(S^{n-1})$ can be developped in the series of the spherical harmonic functions:

$$f(\omega) = \sum_{k=0}^{\infty} S_k(\omega),$$

where $S_k(\omega)$ is a spherical harmonic function of degree k. We can characterize the spaces in (0.4) by the behavior of the spherical harmonic development, namely

$$f \in \mathrm{Exp}(\tilde{S}^{n-1}) \Leftrightarrow \limsup_{k\to\infty}(k!\,\|S_k\|)^{1/k} < \infty, \tag{0.5}$$

$$f \in \mathscr{O}(\tilde{S}^{n-1}) \Leftrightarrow \limsup_{k\to\infty}(\|S_k\|)^{1/k} = 0, \tag{0.6}$$

$$f \in \mathscr{A}(S^{n-1}) \Leftrightarrow \limsup_{k\to\infty}(\|S_k\|)^{1/k} < 1, \tag{0.7}$$

$$f \in C^\infty(S^{n-1}) \Leftrightarrow \|S_k\| \text{ is rapidly decreasing as } k \to \infty, \tag{0.8}$$

$$f \in L^2(S^{n-1}) \Leftrightarrow \|S_k\|_{L_2} \in l^2, \tag{0.9}$$

$$f \in \mathscr{D}'(S^{n-1}) \Leftrightarrow \|S_k\| \text{ is slowly increasing as } k \to \infty, \tag{0.10}$$

$$f \in \mathscr{B}(S^{n-1}) \Leftrightarrow \limsup_{k\to\infty}(\|S_k\|)^{1/k} \leqslant 1, \tag{0.11}$$

$$f \in \mathscr{O}'(\tilde{S}^{n-1}) \Leftrightarrow \limsup_{k\to\infty}(\|S_k\|)^{1/k} < \infty, \tag{0.12}$$

$$f \in \mathrm{Exp}'(\tilde{S}^{n-1}) \Leftrightarrow \limsup_{k\to\infty}(\|S_k\|/k!)^{1/k} = 0, \tag{0.13}$$

where $\|S_k\|$ is $\|S_k\|_{L_\infty} = \sup\{|S_k(\omega)|;\, \omega \in S^{n-1}\}$ or $\|S_k\|_{L_2} = \left(\int_{S^{n-1}} |f(\omega)|^2 d\Omega_n(\omega)\right)^{1/2}$. We will state the theorems using the norm $\|\cdot\|_{L_\infty}$ but it is clear from Proposition 1.1 that we may replace the norm $\|\cdot\|_{L_\infty}$ by the norm $\|\cdot\|_{L_2}$.

A part of the above results is not new. The case $L^2(S^{n-1})$ is very classical. The case $C^\infty(S^{n-1})$ is due to Seeley [11]. The case $\mathscr{A}(S^{n-1})$ is due to Seeley [12] and Hashizume–Minemura–Okamoto [3]. I am ignorant of the literature on $\mathscr{D}'(S^{n-1})$ but it seems to me that it is not new. The case $\mathscr{B}(S^{n-1})$ is also due to Hashizume–Minemura–Okamoto [3]. If $n = 2$, that is, if S^{n-1} is the unit circle in the complex plane, all isomorphisms above are known (Morimoto [9]).

The plan of this paper is as follows: In § 1 we recall the definitions and the properties of spherical harmonic functions. In § 2 we recall the results of the spherical harmonic development of $C^\infty(S^{n-1})$ and $\mathscr{A}(S^{n-1})$ and give an idea of proof to (0.8). In § 3 we treat the case of $\mathscr{D}'(S^{n-1})$ and $\mathscr{B}(S^{n-1})$ and prove the equivalence (0.10). § 4 is devoted to recall elementary results on holomorphic functions and analytic functionals on $\boldsymbol{C}^n$. A theorem of Martineau [8] on the Fourier–Borel transformation will be recalled. In § 5 we consider the case $\mathscr{O}(\tilde{S}^{n-1})$ and $\mathrm{Exp}(\tilde{S}^{n-1})$, and more generally $\mathscr{O}(\tilde{S}^{n-1}(r))$ and $\mathscr{O}(\tilde{S}^{n-1}[r])$, and we obtain the results (0.5), (0.6) and (0.7). As an interesting byproduct, we show the space $\mathscr{O}(\tilde{S}^{n-1})$ of holomorphic functions on the complex sphere $\tilde{S}^{n-1}$ is canonically isomorphic to the space $C^\infty_\Delta(\boldsymbol{R}^n)$ of harmonic functions on the whole Euclidean space $\boldsymbol{R}^n$ (Theorem 5.3). In § 6 we consider the case $\mathscr{O}'(\tilde{S}^{n-1})$ and $\mathrm{Exp}'(\tilde{S}^{n-1})$, and more generally $\mathscr{O}'(\tilde{S}^{n-1}(r))$ and $\mathscr{O}'(S^{n-1}[r])$ and prove the equivalences (0.11), (0.12) and (0.13). In the last section, we will consider the transformation $\mathscr{P}_\lambda$ and prove, among others, that the transformation $\mathscr{P}_\lambda$ establishes a linear topological isomorphism of $\mathrm{Exp}'(\tilde{S}^{n-1})$ onto $C^\infty_\lambda(\boldsymbol{R}^n)$.

This paper was written during my stay in France in the academic year 1978/79. I am grateful to Sophia University for a one year's sabbatical leave of absence. I am much obliged to the Departments of Mathematics of University of Nancy I and University of Lyon I for their kind hospitality. An outline of this paper was lectured in Semester on Complex Analysis at Stefan Banach International Mathematical Center in Warsaw, where I had very stimulating discussions with Professor Józef Siciak and I could improve my original results and my arguments using the Lie norm. I would like to thank him for the hospitality and helpful discussions.

1. Spherical harmonic functions

Let us recall the definitions and the results on the spherical harmonic functions. For the proof we refer the reader to Müller [10], Chapter IV of Stein–Weiss [14] or Chapter IX of Vilenkin [15]. Let us consider the unit sphere S^{n-1} in $\boldsymbol{R}^n$:

$$S^{n-1} = \{x \in \boldsymbol{R}^n;\ \|x\| = 1\}, \tag{1.1}$$

where $\|x\| = (x_1^2+x_2^2+\ldots+x_n^2)^{1/2}$ is the Euclidean norm of x. A general point of S^{n-1} will be denoted by ω, τ, etc. We will write the inner product of $\boldsymbol{R}^n$ by $\langle x, y\rangle = x_1y_1+x_2y_2+\ldots+x_ny_n$. We will denote by $d\Omega_n$ the non-normalized invariant measure on S^{n-1} induced by the Lebesgue measure on $\boldsymbol{R}^n$. The volume of S^{n-1} will be denoted by

$$\Omega_n = 2\pi^{n/2}\Gamma(n/2)^{-1}. \tag{1.2}$$

Let us denote by $L^2(S^{n-1})$ the Hilbert space of square summable functions on S^{n-1} with the following inner product and norm:

$$(f, g)_{L_2} = \int_{S^{n-1}} f(\omega)\overline{g(\omega)}d\Omega_n(\omega), \quad \|f\|_{L_2} = \sqrt{(f,f)_{L_2}}. \tag{1.3}$$

Let us denote by H^k the space of all homogeneous polynomials of n variables and of degree k with complex coefficients. We denote by H^k_Δ the subspace of H^k of harmonic polynomials. The space $\mathscr{H}^k(S^{n-1})$ of *spherical harmonic functions* of degree k is, by definition, the image of H^k_Δ by the restriction of variables to S^{n-1}: $\mathscr{H}^k(S^{n-1}) = H^k_\Delta|_{S^{n-1}}$. We enumerate the properties of the spherical harmonic functions, which will be useful in the sequel.

LEMMA 1.1. *The restriction mapping: $H^k_\Delta \to \mathscr{H}^k(S^{n-1})$ is a linear isomorphism; that is, for every spherical harmonic function $S_k \in \mathscr{H}^k(S^{n-1})$ there exists a unique harmonic homogeneous polynomial $\tilde{S}_k \in H^k_\Delta$ such that $\tilde{S}_k(\omega) = S_k(\omega)$ for $\omega \in S^{n-1}$. We will say $\tilde{S}_k$ is corresponding to S_k.*

LEMMA 1.2. *Let us denote $N(n;k) = \dim \mathscr{H}^k(S^{n-1}) = \dim H^k_\Delta$. Then we know*

$$N(n;k) = \frac{2k+n-2}{k}\binom{k+n-3}{k-1} = O(k^{n-2}). \tag{1.4}$$

LEMMA 1.3. *We have the following linear isomorphisms:*

$$\bigoplus_{k=0}^{k_0} H^k|_{S^{n-1}} = \bigoplus_{k=0}^{k_0} H^k_\Delta|_{S^{n-1}} = \bigoplus_{k=0}^{k_0} \mathscr{H}^k(S^{n-1}).$$

LEMMA 1.4. *The spaces $\mathscr{H}^k(S^{n-1})$ and $\mathscr{H}^j(S^{n-1})$, $k \neq j$, are orthogonal with respect to the inner product $(\ ,\ )_{L_2}$.*

Let us denote by $P_k(n;t)$ the Legendre polynomial of degree k and of dimension n. We know that $S(\omega) = P_k(n; \langle \omega, \tau_0\rangle)$ is the unique element of $\mathscr{H}^k(S^{n-1})$ which satisfies the following two conditions:

(1) $S(\tau_0) = 1$,

(2) $S(A\omega) = S(\omega)$ for every rotation A such that $A\tau_0 = \tau_0$.

Remark. $P_k(2;t) = \cos(k\cos^{-1}t)$ is the Chebyshev polynomial; $P_k(3;t)$ is the usual Legendre polynomial up to a constant; in general,

$$P_k(n;t) = \frac{k!\,(n-3)!}{(k+n-3)!} C_k^{(n-2)/2}(t),$$

where $C_k^{(n-2)/2}(t)$ is the Gegenbauer polynomial. (See for example, Vilenkin [15], p. 459.)

LEMMA 1.5. *Let $\|\ \|_{L_\infty}$ be the supremum norm on S^{n-1}. Then we have*

$$\|P_k(n; \langle\cdot, \tau_0\rangle)\|_{L_2} = \sqrt{\frac{\Omega_n}{N(n;k)}}, \quad \|P_k(n;\langle\cdot,\tau_0\rangle)\|_{L_\infty} = 1. \tag{1.5}$$

DEFINITION 1.1. For a continuous function f on S^{n-1} (or $f \in L^2(S^{n-1})$), we define

$$S_k(f;\omega) = \frac{N(n;k)}{\Omega_n}\int_{S^{n-1}} f(\tau)P_k(n;\langle\omega,\tau\rangle)d\Omega_n(\tau). \tag{1.6}$$

In view of the following theorem, we will call $S_k(f;\omega)$ the *k-th spherical harmonic component* of f.

THEOREM 1.1. *If f is continuous on S^{n-1} or more generally if f is of the class $L^2(S^{n-1})$, then $S_k(f;\omega) \in \mathscr{H}^k(S^{n-1})$ and*

$$f(\omega) = \sum_{k=0}^{\infty} S_k(f;\omega) \tag{1.7}$$

in the sense of $L^2(S^{n-1})$. The Hilbert space $L^2(S^{n-1})$ is the direct sum of the spaces $\mathscr{H}^k(S^{n-1})$:

$$L^2(S^{n-1}) = \bigoplus_{k=0}^{\infty} \mathscr{H}^k(S^{n-1}).$$

The application $f \mapsto S_k(f;\omega)$ is the orthogonal projection of $L^2(S^{n-1})$ onto $\mathscr{H}^k(S^{n-1})$. We have the Bessel equality:

$$\|f\|_{L_2}^2 = \sum_{k=0}^{\infty} \|S_k(f;\cdot)\|_{L_2}^2.$$

Especially we have

$$\|S_k(f;\cdot)\|_{L_2} \leqslant \|f\|_{L_2}. \tag{1.8}$$

PROPOSITION 1.1. (i) *If $S_k \in \mathscr{H}^k(S^{n-1})$, we have*

$$S_k(\omega) = S_k(S_k;\omega) = \frac{N(n;k)}{\Omega_n}\int_{S^{n-1}} S_k(\tau)P_k(n;\langle\omega,\tau\rangle)d\Omega_n(\tau), \tag{1.9}$$

$$\|S_k\|_{L_\infty} \leqslant \sqrt{\frac{N(n;k)}{\Omega_n}}\|S_k\|_{L_2} \quad \text{and} \quad \|S_k\|_{L_2} \leqslant \sqrt{\Omega_n}\|S_k\|_{L_\infty}. \tag{1.10}$$

(ii) *If $f \in L^2(S^{n-1})$, then we have*

$$\|S_k(f;\cdot)\|_{L_\infty} \leqslant \sqrt{\frac{N(n;k)}{\Omega_n}}\|f\|_{L_2}. \tag{1.11}$$

Proof. Because the mapping $f \mapsto S_k(f;\omega)$ is the projection of $L^2(S^{n-1})$ onto its orthogonal component, we have (1.9). The first inequality of (1.10) results from (1.9) and (1.5) by Cauchy's inequality, while the second equality of (1.10) is trivial. (1.11) results from (1.10) and Bessel's equality. ■

Let us denote by Δ_S the Laplace–Beltrami operator on the sphere S^{n-1}. We will use constantly the following results in the sequel:

PROPOSITION 1.2. (i) *If f and g are C^∞ functions on S^{n-1}, we have*

$$\int_{S^{n-1}} \Delta_S f\cdot g\,d\Omega_n = \int_{S^{n-1}} f\cdot \Delta_S g\,d\Omega_n. \tag{1.12}$$

(ii) *$S_k \in \mathscr{H}^k(S^{n-1})$ is an eigenfunction of Δ_S of eigenvalue $-k(k+n-2)$; that is,*

$$\Delta_S S_k(\omega) = -k(k+n-2)S_k(\omega). \tag{1.13}$$

2. C^∞ functions and real-analytic functions on S^{n-1}

Let us recall some results on C^∞ functions and real-analytic functions on S^{n-1}.

THEOREM 2.1 (Seeley [11]). *Suppose $f \in C^\infty(S^{n-1})$. Let us denote by $S_k(f;\omega)$ the k-th spherical harmonic component of f. Then we have*

$$\sup\{k^m \|S_k(f;\cdot)\|_{L_\infty}; k \in \mathbf{Z}_+\} < \infty \quad \text{for any } m \in \mathbf{Z}_+;$$

that is, the sequence $\|S_k(f;\cdot)\|_{L_\infty}$ is rapidly decreasing.

On the contrary, if we are given a sequence $\{S_k \in \mathscr{H}^k(S^{n-1}); k \in \mathbf{Z}_+\}$ such that the sequence $\|S_k(\cdot)\|_{L_\infty}$ is rapidly decreasing, then $f(\omega) = \sum_{k=0}^{\infty} S_k(\omega)$ is a C^∞ function on S^{n-1} and we have

$$S_k(f;\omega) = S_k(\omega) \quad \text{for any } k \in \mathbf{Z}_+. \tag{2.1}$$

Proof. Suppose $f \in C^\infty(S^{n-1})$ and $m \in \mathbf{Z}_+$ is even. We have by Proposition 1.2,

$$\begin{aligned} S_k((-\Delta_S)^{m/2}f;\omega) &= \frac{N(n;k)}{\Omega_n}\int_{S^{n-1}} (-\Delta_S)^{m/2}f(\tau)P_k(n;\langle\omega,\tau\rangle)d\Omega_n(\tau) \\ &= \frac{N(n;k)}{\Omega_n}\int_{S^{n-1}} f(\tau)(-\Delta_S)^{m/2}P_k(n;\langle\omega,\tau\rangle)d\Omega_n(\tau) \\ &= (k(k+n-2))^{m/2}S_k(f;\omega). \end{aligned} \tag{2.2}$$

Consequently, we get by (1.11) the following estimation:

$$k^m|S_k(f;\omega)| \leqslant |S_k((-\Delta_S)^{m/2}f;\omega)| \leqslant \sqrt{\frac{N(n;k)}{\Omega_n}}\|(-\Delta_S)^{m/2}f\|_{L_2}.$$

Thanks to Lemma 1.2, we can conclude that the sequence $\|S_k(f;\cdot)\|_{L_\infty}$ is rapidly decreasing.

Suppose now that $\|S_k(\cdot)\|_{L_\infty}$ is rapidly decreasing. We put $f_N(\omega) = \sum_{k=0}^{N} S_k(\omega)$. Then $f_N(\omega)$ converges uniformly to a continuous function $f(\omega)$ as $N \to \infty$. If we fix an even number $m \in \mathbf{Z}_+$, the function $(-\Delta_S)^{m/2}f_N(\omega) = \sum_{k=0}^{N}(k(k+n-2))^{m/2}S_k(\omega)$ also converges uniformly to a continuous function $g_m(\omega)$ as $N \to \infty$, from which we can conclude that $f \in C^\infty(S^{n-1})$ and $(-\Delta_S)^{m/2}f(\omega) = g_m(\omega)$ for $m = 2, 4, \ldots$ (2.1) results from Lemma 1.4. ∎

DEFINITION 2.1. For $f \in C^\infty(S^{n-1})$ and $m \in \mathbf{Z}_+$, we put

$$(-\Delta_S)^{m/2}f(\omega) = \sum_{k=0}^{\infty}(k(k+n-2))^{m/2}S_k(f;\omega). \tag{2.3}$$

By Theorem 2.1, $(-\Delta_S)^{m/2}f(\omega)$ is a C^∞ function. Remark that the topology of the Fréchet space $C^\infty(S^{n-1})$ can be given by the system of seminorms

$$\|(-\Delta_S)^{m/2}f(\cdot)\|_{L_2}, \quad m \in \mathbf{Z}_+. \tag{2.4}$$

COROLLARY. *If $f \in C^\infty(S^{n-1})$, then the spherical harmonic development of f, $f(\omega) = \sum_{k=0}^{\infty} S_k(f;\omega)$, converges in the topology of $C^\infty(S^{n-1})$.*

Let us denote by $\mathscr{A}(S^{n-1})$ the space of the real-analytic functions on S^{n-1}. We define the complex sphere $\tilde{S}^{n-1}$ by

$$\tilde{S}^{n-1} = \{z \in C^n; z_1^2+z_2^2+\ldots+z_n^2 = 1\} \tag{2.5}$$

and put for $r > 1$

$$\tilde{S}^{n-1}(r) = \{z = x+iy \in \tilde{S}^{n-1}; \|y\| < \tfrac{1}{2}(r-1/r)\}. \tag{2.6}$$

Then $\tilde{S}^{n-1}$ is a complex manifold and its open sets $\tilde{S}^{n-1}(r)$, $r > 1$ form a fundamental system of complex neighborhoods of the real-analytic manifold S^{n-1}. Let us denote by $\mathscr{O}(\tilde{S}^{n-1}(r))$ the space of holomorphic functions on $\tilde{S}^{n-1}(r)$ equipped with the topology of the uniform convergence on every compact set of $\tilde{S}^{n-1}(r)$. The linear topology of $\mathscr{A}(S^{n-1})$ is, by definition, the locally convex inductive limit of the topology of $\mathscr{O}(\tilde{S}^{n-1}(r))$, $r > 1$:

$$\mathscr{A}(S^{n-1}) = \liminf_{r>1}\mathscr{O}(\tilde{S}^{n-1}(r)). \tag{2.7}$$

$\mathscr{A}(S^{n-1})$ is a DFS space. It is known (Lions–Magenes [7]) that we have also the following linear topological isomorphism:

$$\mathscr{A}(S^{n-1}) = \liminf_{h>0}\mathscr{A}_h(S^{n-1}), \tag{2.8}$$

where $\mathscr{A}_h(S^{n-1})$ is the Banach space defined as follows:

$$\mathscr{A}_h(S^{n-1}) = \Big\{f \in C^\infty(S^{n-1});\ \sup_{m\in Z_+}\frac{1}{m!\,h^m}\|(-\Delta_S)^{m/2}f\|_{L_2} < \infty\Big\}. \tag{2.9}$$

Using this isomorphism (2.8), Seeley [12] and later Hashizume–Minemura–Okamoto [3] proved the following theorem.

THEOREM 2.2. *If $f \in \mathscr{A}(S^{n-1})$, then we have*

$$\limsup_{k\to\infty}(\|S_k(f;\cdot)\|_{L_\infty})^{1/k} < 1. \tag{2.10}$$

On the contrary, if a sequence $\{S_k \in \mathscr{H}^k(S^{n-1}); k \in \mathbf{Z}_+\}$ satisfies the condition

$$\limsup_{k\to\infty}(\|S_k(\cdot)\|_{L_\infty})^{1/k} < 1, \tag{2.10'}$$

then $f(\omega) = \sum_{k=0}^{\infty} S_k(\omega)$ is a real-analytic function and we have (2.2).

COROLLARY. *If $f \in \mathscr{A}(S^{n-1})$, then the spherical harmonic development of f, $f(\omega) = \sum_{k=0}^{\infty} S_k(f;\omega)$ converges in the topology of $\mathscr{A}(S^{n-1})$.*

We will not reproduce here their proof, as we will give another proof relying on the definition formula (2.7). (Theorems 5.1 and 5.2 (ii), $r = 1$.)

Remark. The results of this section are valid even for a general real-analytic compact manifold. The references quoted above treat this general case.

3. Distributions and hyperfunctions on S^{n-1}

DEFINITION 3.1. A continuous linear functional on the space $C^\infty(S^{n-1})$ is called a *distribution* on S^{n-1}. We will denote by $\mathscr{D}'(S^{n-1})$ the space of the distributions on S^{n-1} and by $\langle\ ,\ \rangle$ the canonical inner product of duality. A continuous linear functional on the space $\mathscr{A}(S^{n-1})$ is called a *hyperfunction* on S^{n-1}. We will denote by $\mathscr{B}(S^{n-1})$ the space of the hyperfunctions on S^{n-1} and by $\langle\ ,\ \rangle$ the canonical inner product of duality.

As $\mathscr{A}(S^{n-1})$ is dense in $C^\infty(S^{n-1})$, $\mathscr{D}'(S^{n-1})$ can be identified with a subspace of $\mathscr{B}(S^{n-1})$. A function $f\in L^2(S^{n-1})$ defines a distribution T_f by the following formula:

$$\langle T_f, g\rangle = (f,\bar g)_{L_2} = \int_{S^{n-1}} f(\omega)g(\omega)d\Omega_n(\omega) \tag{3.1}$$

for $g\in C^\infty(S^{n-1})$. It is classical that $f\mapsto T_f$ is a continuous injection of $L^2(S^{n-1})$ into $\mathscr{D}'(S^{n-1})$. In the sequel, we will consider $L^2(S^{n-1})$ as a subspace of $\mathscr{D}'(S^{n-1})$ by this injection. We will denote the distribution T_f also by f.

Now we have the following sequence of spaces of functions or functionals on S^{n-1}:

$$\mathscr{A}(S^{n-1}) \subset C^\infty(S^{n-1}) \subset L^2(S^{n-1}) \subset \mathscr{D}'(S^{n-1}) \subset \mathscr{B}(S^{n-1}). \tag{3.2}$$

By Corollary to Theorem 2.1, we have, for $T\in\mathscr{D}'(S^{n-1})$ and $f\in C^\infty(S^{n-1})$,

$$\langle T,f\rangle = \Big\langle T, \sum_{k=0}^{\infty} S_k(f;\cdot)\Big\rangle = \sum_{k=0}^{\infty}\langle T, S_k(f;\cdot)\rangle.$$

By Definition 1.1, we have

$$\begin{aligned}\langle T, S_k(f;\cdot)\rangle &= \Big\langle T, \frac{N(n;k)}{\Omega_n}\int_{S^{n-1}} P_k(n;\langle\cdot,\tau\rangle)f(\tau)d\Omega_n(\tau)\Big\rangle \\ &= \frac{N(n;k)}{\Omega_n}\int_{S^{n-1}}\big\langle T, P_k(n;\langle\cdot,\tau\rangle)\big\rangle f(\tau)d\Omega_n(\tau) \\ &= \int_{S^{n-1}} S_k(T;\tau)f(\tau)d\Omega_n(\tau),\end{aligned} \tag{3.3}$$

where we denoted

$$S_k(T;\tau) = \frac{N(n;k)}{\Omega_n}\big\langle T, P_k(n;\langle\cdot,\tau\rangle)\big\rangle. \tag{3.4}$$

In view of the following theorem, $S_k(T;\tau)$ is called the *k-th spherical harmonic component* of T. Finally we get

$$\langle T,f\rangle = \sum_{k=0}^{\infty}\int_{S^{n-1}} S_k(T;\tau)f(\tau)d\Omega_n(\tau). \tag{3.5}$$

THEOREM 3.1. *Let T be a distribution on S^{n-1}. Then $S_k(T;\cdot)\in\mathscr{H}^k(S^{n-1})$ and the sequence $\|S_k(T;\cdot)\|_{L_\infty}$ is slowly increasing; that is, there exists an integer $N\geq 0$ such that* $\sup\{k^{-N}\|S_k(T;\cdot)\|_{L_\infty};\ k\in Z_+\}<\infty$.

On the contrary, suppose a sequence $\{S_k\in\mathscr{H}^k(S^{n-1}); k\in Z_+\}$ is given. If the sequence $\|S_k(\cdot)\|_{L_\infty}$ is slowly increasing, we can define a distribution T by the formula

$$\langle T,f\rangle = \sum_{k=0}^{\infty}\int_{S^{n-1}} S_k(\omega)f(\omega)d\Omega_n(\omega) = \sum_{k=0}^{\infty}\int_{S^{n-1}} S_k(\omega)S_k(f;\omega)d\Omega_n(\omega) \tag{3.6}$$

and we have

$$S_k(T;\omega) = S_k(\omega)\quad \textit{for every } k\in Z_+. \tag{3.7}$$

Proof. By the continuity of T, there exist an integer $N\in Z_+$ and $C_j\geq 0$, $j=0,1,2,\ldots,N$ such that

$$|\langle T,f\rangle| \leq \sum_{j=0}^{N} C_j\|(-\Delta_S)^{j/2}f\|_{L_2}$$

for any $f\in C^\infty(S^{n-1})$. Therefore, we have, by Lemma 1.5,

$$\begin{aligned}|S_k(T;\omega)| &= \frac{N(n;k)}{\Omega_n}\big|\big\langle T, P_k(n;\langle\cdot,\omega\rangle)\big\rangle\big| \\ &\leq \frac{N(n;k)}{\Omega_n}\sum_{j=0}^{N} C_j(k(k+n-2))^{j/2}\|P_k(n;\langle\cdot,\omega\rangle)\|_{L_2} \\ &= \sqrt{\frac{N(n;k)}{\Omega_n}}\sum_{j=0}^{N} C_j(k(k+n-2))^{j/2}.\end{aligned}$$

Thanks to Lemma 1.2, we have proved that the sequence $\|S_k(T;\cdot)\|_{L_\infty}$ is slowly increasing.

Let us prove the second part. We have to prove that the mapping

$$f\mapsto \sum_{k=0}^{\infty}\int_{S^{n-1}} S_k(\omega)f(\omega)d\Omega_n(\omega)$$

is linear and continuous on $C^\infty(S^{n-1})$. But

$$\int_{S^{n-1}} S_0(\omega)f(\omega)d\Omega_n(\omega) = \mathrm{const}\cdot\int_{S^{n-1}} f(\omega)d\Omega_n(\omega)$$

is clearly a linear continuous functional on $C^\infty(S^{n-1})$.

By Proposition 1.2, we have, for any $f\in C^\infty(S^{n-1})$ and any $m\in Z_+$,

$$\sum_{k=1}^{\infty}\int_{S^{n-1}} S_k(\omega)f(\omega)d\Omega_n(\omega) = \sum_{k=1}^{\infty}(k(k+n-2))^{-m/2}\int_{S^{n-1}} S_k(\omega)(-\Delta_S)^{m/2}f(\omega)d\Omega_n(\omega).$$

The second term converges if $m \geqslant N+2$. We have obtained in this way the following inequality:

$$|\langle T,f\rangle| \leqslant C_0\|f\|_{L_2}+C_1\|(-\Delta_S)^{(N+2)/2}f\|_{L_2},$$

from which results the continuity of T on $C^\infty(S^{n-1})$. (3.7) is a consequence of Lemma 1.4. ■

COROLLARY. *If $T\in\mathscr{D}'(S^{n-1})$, then we have*

$$T=\sum_{k=0}^{\infty}S_k(T;\ \cdot) \tag{3.8}$$

in the weak topology of $\mathscr{D}'(S^{n-1})$. We have also the following formula:

$$\langle T,f\rangle=\sum_{k=0}^{\infty}\int_{S^{n-1}}S_k(T;\omega)S_k(f;\omega)d\Omega_n(\omega). \tag{3.9}$$

In fact, (3.8) is equivalent to (3.5), while (3.9) results from Lemma 1.4.

Now we are going to consider the hyperfunctions on S^{n-1}. Because of Corollary to Theorem 2.2, the formulas (3.3), (3.4) and (3.5) are also valid for $T\in\mathscr{B}(S^{n-1})$ and $f\in\mathscr{A}(S^{n-1})$.

THEOREM 3.2. *For $T\in\mathscr{B}(S^{n-1})$, we have*

$$\limsup_{k\to\infty}(\|S_k(T;\ \cdot)\|_{L_\infty})^{1/k}\leqslant 1. \tag{3.10}$$

On the contrary, if a sequence $\{S_k(\omega)\in\mathscr{H}^k(S^{n-1});\ k\in\boldsymbol{Z}_+\}$ satisfies the condition

$$\limsup_{k\to\infty}(\|S_k(\cdot)\|_{L_\infty})^{1/k}\leqslant 1, \tag{3.10'}$$

then we can define a hyperfunction T by the formula (3.6) and the formula (3.7) is valid.

This theorem was first proved by Hashizume–Minemura–Okamoto [3], where they proved indeed the theorem for a general real-analytic compact manifold. In § 6, we will give it a new proof (Theorems 6.1 and 6.2 (ii), $r=1$).

Many classical theorems in Müller [10] can be extended to $\mathscr{B}(S^{n-1})$. For example, we cite the following.

THEOREM 3.3. *Let $T\in\mathscr{B}(S^{n-1})$. Then*

$$u(x)=\sum_{k=0}^{\infty}r^kS_k(T;\omega),\qquad x=r\omega,\ r\geqslant 0,\ \|\omega\|=1 \tag{3.11}$$

is a harmonic function in $B_1=\{x\in\boldsymbol{R}^n;\ \|x\|<1\}$. $u(r\omega)$ converges to T_ω as $r\to 1-0$ in the weak topology of $\mathscr{B}(S^{n-1})$. The harmonic function $u(x)$ can be represented by the Poisson formula:

$$u(x)=\frac{1}{\Omega_n}\Big\langle T_\omega,\frac{1-\|x\|^2}{(1+\|x\|^2-2\langle x,\omega\rangle)^{n/2}}\Big\rangle. \tag{3.12}$$

4. Holomorphic functions and analytic functionals on C^n

Let N be a norm on the complex vector space $\boldsymbol{C}^n$. Denote by

$$\tilde{B}(r;N)=\{z\in\boldsymbol{C}^n;\ N(z)<r\}\quad\text{for}\quad 0<r\leqslant\infty, \tag{4.1}$$

$$\tilde{B}[r;N]=\{z\in\boldsymbol{C}^n;\ N(z)\leqslant r\}\quad\text{for}\quad 0\leqslant r<\infty \tag{4.2}$$

the open and closed N-balls of radius r. (Note $\tilde{B}(\infty;N)=\boldsymbol{C}^n$ and $\tilde{B}[0;N]=\{0\}$.) It is clear that $\tilde{B}(r;N)$ is an open convex balanced set of $\boldsymbol{C}^n$. In particular, it is a domain of holomorphy. Let us denote by $\mathcal{O}(\tilde{B}(r;N))$ the space of holomorphic functions on $\tilde{B}(r;N)$. It is an FS space with the system of norms $\|f\|_{\varrho;N}$, $0\leqslant\varrho<r$, where

$$\|f\|_{\varrho;N}=\sup\{|f(z)|;\ z\in\tilde{B}[\varrho;N]\}. \tag{4.3}$$

Let us define

$$\mathcal{O}(\tilde{B}[r;N])=\liminf_{r'>r}\mathcal{O}(\tilde{B}(r';N)).$$

It is the space of germs of holomorphic functions on the closed N-ball $\tilde{B}[r;N]$ and is a DFS space.

LEMMA 4.1. *Suppose $f\in\mathcal{O}(\tilde{B}(r;N))$ (resp. $f\in\mathcal{O}(\tilde{B}[r;N])$). Let*

$$f(z)=\sum_{k=0}^{\infty}f_k(z) \tag{4.4}$$

be the development of the holomorphic function f by the series of homogeneous polynomials f_k of degree k; that is, $f_k\in H^k$. Then we have

$$\|f_k\|_{1;N}\leqslant\varrho^{-k}\|f\|_{\varrho;N} \tag{4.5}$$

for every ϱ with $0<\varrho<r$ (resp. for some ϱ with $\varrho>r$). In particular,

$$\limsup_{k\to\infty}(\|f_k\|_{1;N})^{1/k}\leqslant 1/r \tag{4.6}$$

(resp.

$$\limsup_{k\to\infty}(\|f_k\|_{1;N})^{1/k}<1/r). \tag{4.7}$$

On the contrary, if we have a sequence $\{f_k\in H^K;\ k\in\boldsymbol{Z}_+\}$ which satisfies the condition (4.6) (resp. (4.7)), then the series (4.4) converges in the topology of $\mathcal{O}(\tilde{B}(r;N))$ and $f\in\mathcal{O}(\tilde{B}(r;N))$ (resp. in the topology of $\mathcal{O}(\tilde{B}[r;N])$ and $f\in\mathcal{O}(\tilde{B}[r;N])$).

In fact, for $f\in\mathcal{O}(\tilde{B}(r;N))$, we have

$$f_k(z)=\frac{1}{2\pi i}\oint_{|t|=\varrho}\frac{f(tz)}{t^{k+1}}dt \tag{4.8}$$

for $z\in\boldsymbol{C}^n$, $N(z)\leqslant 1$ and $0<\varrho<r$. The lemma is a consequence of this integral formula.

Let $A > 0$. For an entire function f, we put

$$\|f\|_{(A;N)} = \sup\{|f(z)|\exp(-AN(z));\ z \in C^n\} \tag{4.9}$$

and

$$X_{A;N} = \{f \in \mathcal{O}(C^n);\ \|f\|_{(A;N)} < \infty\}. \tag{4.10}$$

Then $X_{A;N}$ is a Banach space. Define

$$\operatorname{Exp}(C^n; (A;N)) = \operatorname*{lim\,proj}_{A'>A} X_{A';N} \quad \text{for} \quad 0 \leqslant A < \infty, \tag{4.11}$$

$$\operatorname{Exp}(C^n; [A;N]) = \operatorname*{lim\,ind}_{A'<A} X_{A';N} \quad \text{for} \quad 0 < A \leqslant \infty. \tag{4.12}$$

$\operatorname{Exp}(C^n; (A;N))$ is an FS space and $\operatorname{Exp}(C^n; [A;N])$ is a DFS space. Remark that

$$\operatorname{Exp}(C^n; (0)) = \operatorname{Exp}(C^n; (0;N)), \tag{4.13}$$

$$\operatorname{Exp}(C^n) = \operatorname{Exp}(C^n; [\infty;N]) \tag{4.14}$$

are independent of the choice of the norm N. A function in $\operatorname{Exp}(C^n)$ will be called an *entire function of exponential type* while a function in $\operatorname{Exp}(C^n; (0))$ will be called an *entire function of exponential type zero*.

Lemma 4.2. *Let $f \in \operatorname{Exp}(C^n)$ and (4.4) be the development of f by homogeneous polynomials f_k of degree k. If f belongs to $\operatorname{Exp}(C^n; (A; N))$ (resp. $f \in \operatorname{Exp}(C^n; [A; N])$), then we have*

$$\|f_k\|_{1;N} \leqslant \|f\|_{A';N}\sqrt{2\pi k}\,\frac{A'^k}{k!} \tag{4.15}$$

for every $A' > A$ (resp. for some A' with $0 \leqslant A' < A$). In particular, we have

$$\limsup_{k\to\infty}(k!\,\|f_k\|_{1;N})^{1/k} \leqslant A \tag{4.16}$$

(resp.

$$\limsup_{k\to\infty}(k!\,\|f_k\|_{1;N})^{1/k} < A). \tag{4.17}$$

On the contrary, if we have (4.16) (resp. (4.17)), then the series (4.4) converges in the topology of $\operatorname{Exp}(C^n; (A;N))$ and $f \in \operatorname{Exp}(C^n; (A;N))$ (resp. the series (4.4) converges in the topology of $\operatorname{Exp}(C^n; [A;N])$ and $f \in \operatorname{Exp}(C^n; [A;N])$).

Proof. By Lemma 4.1, if $f \in \operatorname{Exp}(C^n; (A;N))$, we have, for every $\varrho > 0$ and $A' > A$,

$$\|f_k\|_{1;N} \leqslant \varrho^{-k}\|f\|_{(A';N)}\exp(A'\varrho). \tag{4.18}$$

But we have

$$\inf\{\varrho^{-k}\exp(A'\varrho);\ \varrho > 0\} = \frac{A'^k}{k^k}e^k$$

and Stirling's formula

$$\exp(-1/12)k^k e^{-k}\sqrt{2\pi k} \leqslant k! \leqslant k^k e^{-k}\sqrt{2\pi k}. \tag{4.19}$$

Hence we get (4.15). The rest of the proof is a routine argument. ∎

Remark the following inclusion relations:

$$\begin{aligned}\operatorname{Exp}(C^n; (0)) &\subset \operatorname{Exp}(C^n; [A;N]) \subset \operatorname{Exp}(C^n; (A;N)) \subset \operatorname{Exp}(C^n)\\ &\subset \mathcal{O}(C^n) \subset \mathcal{O}(\tilde{B}[r;N]) \subset \mathcal{O}(\tilde{B}(r;N)) \subset \mathcal{O}(\{0\}).\end{aligned} \tag{4.20}$$

By Lemmas 4.1 and 4.2, the polynomials are dense in each of the spaces in (4.20). Let us denote by $\mathcal{O}'(\tilde{B}(r;N))$, $\mathcal{O}'(\tilde{B}[r;N])$, $\operatorname{Exp}'(C^n; (A;N))$ and $\operatorname{Exp}'(C^n; [A;N])$ etc. the dual spaces of $\mathcal{O}(\tilde{B}(r;N))$, $\mathcal{O}(\tilde{B}[r;N])$, $\operatorname{Exp}(C^n; (A;N))$ and $\operatorname{Exp}(C^n; [A;N])$, etc. Then the following inclusions can be defined transposing (4.20):

$$\begin{aligned}\mathcal{O}'(\{0\}) &\subset \mathcal{O}'(\tilde{B}(r;N)) \subset \mathcal{O}'(\tilde{B}[r;N]) \subset \mathcal{O}'(C^n)\\ &\subset \operatorname{Exp}'(C^n) \subset \operatorname{Exp}'(C^n; (A;N)) \subset \operatorname{Exp}'(C^n; [A;N]) \subset \operatorname{Exp}'(C^n; (0)).\end{aligned} \tag{4.21}$$

Now let us define the norm N^* dual to the norm N by

$$N^*(\zeta) = \sup\{\langle\zeta, z\rangle;\ N(z) \leqslant 1\}. \tag{4.22}$$

Theorem 4.1 (Martineau [8]). *Suppose $\lambda \in C$, $\lambda \neq 0$. The Fourier–Borel transformation*

$$\mathcal{P}_\lambda:\ T_z \mapsto \langle T_z, \exp(i\lambda\langle z, \zeta\rangle)\rangle \tag{4.23}$$

establishes the following linear topological isomorphisms:

$$\mathcal{P}_\lambda:\ \mathcal{O}'(\{0\}) \xrightarrow{\sim} \operatorname{Exp}(C^n; (0)), \tag{4.24}$$

$$\mathcal{P}_\lambda:\ \mathcal{O}'(\tilde{B}(r;N)) \xrightarrow{\sim} \operatorname{Exp}(C^n; [|\lambda|r; N^*]), \tag{4.25}$$

$$\mathcal{P}_\lambda:\ \mathcal{O}'(B[r;N]) \xrightarrow{\sim} \operatorname{Exp}(C^n; (|\lambda|r; N^*)), \tag{4.26}$$

$$\mathcal{P}_\lambda:\ \mathcal{O}'(C^n) \xrightarrow{\sim} \operatorname{Exp}(C^n), \tag{4.27}$$

$$\mathcal{P}_\lambda:\ \operatorname{Exp}'(C^n) \xrightarrow{\sim} \mathcal{O}(C^n), \tag{4.28}$$

$$\mathcal{P}_\lambda:\ \operatorname{Exp}'(C^n; (A;N)) \xrightarrow{\sim} \mathcal{O}(B[|\lambda|^{-1}A; N^*]), \tag{4.29}$$

$$\mathcal{P}_\lambda:\ \operatorname{Exp}'(C^n; [A;N]) \xrightarrow{\sim} \mathcal{O}(B(|\lambda|^{-1}A; N^*)), \tag{4.30}$$

$$\mathcal{P}_\lambda:\ \operatorname{Exp}'(C^n; (0)) \xrightarrow{\sim} \mathcal{O}(\{0\}). \tag{4.31}$$

5. Holomorphic functions on the complex sphere $\tilde{S}^{n-1}$

Recall that the complex sphere $\tilde{S}^{n-1}$ is defined by

$$\begin{aligned}\tilde{S}^{n-1} &= \{z \in C^n;\ z^2 = z_1^2+z_2^2+\ldots+z_n^2 = 1\}\\ &= \{z = x+iy \in C^n;\ \|x\|^2-\|y\|^2 = 1, \langle x, y\rangle = 0\}.\end{aligned} \tag{5.1}$$

We have

$$S^{n-1} = \tilde{S}^{n-1} \cap R^n. \tag{5.2}$$

Let us denote by

$$B(r) = \{x \in R^n;\ \|x\| < r\}, \quad B[r] = \{x \in R^n;\ \|x\| \leqslant r\} \tag{5.3}$$

the open and closed Euclidean balls of radius r. (We put $B(\infty) = R^n$.) Now we define

$$(5.4)\qquad L(z) = \Big\{\sum_{j=1}^{n}|z_j|^2+\Big\{\Big(\sum_{j=1}^{n}|z_j|^2\Big)^2-\Big|\sum_{j=1}^{n}z_j^2\Big|^2\Big\}^{1/2}\Big\}^{1/2} = \big\{||x||^2+||y||^2+2\{||x||^2||y||^2-\langle x, y\rangle^2\}^{1/2}\big\}^{1/2}.$$

LEMMA 5.1 (Drużkowski [1]). *$L(z)$ is the cross norm of the Euclidean norm $||x||$; that is,*

$$(5.5)\qquad L(z) = \inf\Big\{\sum_{j=1}^{m}|\lambda_j|\,||x_j||;\ z = \sum_{j=1}^{m}\lambda_j x_j,\ \lambda_j\in C,\ x_j\in R^n,\ m\in Z_+\Big\}.$$

We will call $L(z)$ the *Lie norm* of z. We define the *Lie ball* of radius r:

$$(5.6)\qquad \tilde{B}(r) = \tilde{B}(r; L) = \{z\in C^n; L(z) < r\}\quad \text{for}\quad 0 < r\leqslant\infty,$$

$$(5.7)\qquad \tilde{B}[r] = \tilde{B}[r; L] = \{z\in C^n; L(z) \leqslant r\}\quad \text{for}\quad 0 \leqslant r<\infty.$$

$\tilde{B}(r)$ is E. Cartan's classical domain of type 4. (See Hua [5] and Siciak [13].) We put also

$$(5.8)\qquad \tilde{S}^{n-1}(r) = \tilde{B}(r)\cap\tilde{S}^{n-1}\quad \text{for}\quad 1<r\leqslant\infty,$$

$$(5.9)\qquad \tilde{S}^{n-1}[r] = \tilde{B}[r]\cap\tilde{S}^{n-1}\quad \text{for}\quad 1\leqslant r<\infty.$$

Remark that the two definitions (2.6) and (5.8) are identical, for $L(x+iy) = ||x||+ +||y||$ if $\langle x, y\rangle = 0$.

Let us denote by $\mathcal{O}(\tilde{S}^{n-1}(r))$ the space of the holomorphic functions on the open set $\tilde{S}^{n-1}(r)$ of $\tilde{S}^{n-1}$ equipped with the topology of uniform convergence on every compact set of $\tilde{S}^{n-1}(r)$. $\mathcal{O}(\tilde{S}^{n-1}(r))$ is an FS space.

LEMMA 5.2. *The following sequence is exact:*

$$(5.10)\qquad 0\to\mathcal{I}(\tilde{B}(r))\xrightarrow{\iota}\mathcal{O}(\tilde{B}(r))\xrightarrow{\beta}\mathcal{O}(\tilde{S}^{n-1}(r))\to 0,$$

where $\mathcal{I}(\tilde{B}(r)) = \{f\in\mathcal{O}(\tilde{B}(r)); f(z) = 0$ for every $z\in\tilde{S}^{n-1}(r)\}$ is a closed subspace of $\mathcal{O}(\tilde{B}(r))$, ι is the canonical injection and β is the restriction mapping.

In fact, by Lemma 5.1, $\tilde{B}(r)$ is a domain of holomorphy. Therefore the lemma results from Theorem B of Oka–Cartan.

LEMMA 5.3. *We have the following linear topological isomorphism:*

$$(5.11)\qquad \mathcal{O}(\tilde{S}^{n-1}(r)) = \mathcal{O}(\tilde{B}(r))/\mathcal{I}(\tilde{B}(r))\quad \textit{for}\quad 1<r\leqslant\infty.$$

In fact, all spaces being FS space, a linear continuous isomorphism is topological by the closed graph theorem.

Now we put, for $1\leqslant r<\infty$,

$$(5.12)\qquad \mathcal{O}(\tilde{S}^{n-1}[r]) = \liminf_{r'>r}\mathcal{O}(\tilde{S}^{n-1}(r')).$$

$\mathcal{O}(\tilde{S}^{n-1}[r])$ is a DFS space. As we remarked in § 2, we have

$$(5.13)\qquad \mathcal{A}(S^{n-1}) = \mathcal{O}(\tilde{S}^{n-1}[1]).$$

As the exactness is stable under the inductive limit, taking the inductive limit in (5.10) we get

COROLLARY. *The following sequence is exact:*

$$(5.10')\qquad 0\to\mathcal{I}(\tilde{B}[r])\xrightarrow{\iota}\mathcal{O}(\tilde{B}[r])\xrightarrow{\beta}\mathcal{O}(\tilde{S}^{n-1}[r])\to 0,$$

where $\mathcal{I}(\tilde{B}[r]) = \liminf_{r'>r}\mathcal{I}(\tilde{B}(r'))$. We have also a linear topological isomorphism:

$$(5.11')\qquad \mathcal{O}(\tilde{S}^{n-1}[r]) = \tilde{\mathcal{O}}(B[r])/\mathcal{I}(\tilde{B}[r])\quad \textit{for}\quad 1\leqslant r<\infty.$$

Define the space $\mathrm{Exp}(\tilde{S}^{n-1})$ to be the image of the space $\mathrm{Exp}(C^n)$ of the entire functions of exponential type under the restriction mapping β. The topology of $\mathrm{Exp}(\tilde{S}^{n-1})$ is defined to be the quotient topology of $\mathrm{Exp}(C^n)$ by its closed subspace $\mathcal{I}_{\exp}(C^n) = \mathcal{I}(C^n)\cap\mathrm{Exp}(C^n)$. By the definition

$$(5.10'')\qquad 0\to\mathcal{I}_{\exp}(C^n)\to\mathrm{Exp}(C^n)\to\mathrm{Exp}(\tilde{S}^{n-1})\to 0$$

is an exact sequence and

$$(5.11'')\qquad \mathrm{Exp}(\tilde{S}^{n-1}) = \mathrm{Exp}(C^n)/\mathcal{I}_{\exp}(C^n).$$

$\mathrm{Exp}(\tilde{S}^{n-1})$ is a DFS space, being a quotient space of a DFS space by its closed subspace.

Because S^{n-1} is the real part of $\tilde{S}^{n-1}$ and that $\tilde{S}^{n-1}(r)$ and $S^{n-1}[r]$ are connected, the restriction mappings $\mathcal{O}(\tilde{S}^{n-1}(r))\xrightarrow{\gamma}C^\infty(S^{n-1})$ and $\mathcal{O}(\tilde{S}^{n-1}[r])\xrightarrow{\gamma}C^\infty(S^{n-1})$ are injective. In the sequel, we will consider the spaces $\mathcal{O}(\tilde{S}^{n-1}(r))$ and $\mathcal{O}(\tilde{S}^{n-1}[r])$ as subspaces of $C^\infty(S^{n-1})$ by these restriction mappings γ.

We have just defined a chain of spaces of functions as follows:

$$(5.14)\qquad \mathrm{Exp}(\tilde{S}^{n-1})\subset\mathcal{O}(\tilde{S}^{n-1})\subset\mathcal{O}(\tilde{S}^{n-1}[r])\subset\mathcal{O}(\tilde{S}^{n-1}(r)) \subset\mathcal{A}(S^{n-1})\subset C^\infty(S^{n-1})\subset L^2(S^{n-1}).$$

Theorems 5.1 and 5.2 below will characterize the subspaces $\mathcal{O}(\tilde{S}^{n-1}(r))$, $\mathcal{O}(\tilde{S}^{n-1}[r])$ and $\mathrm{Exp}(\tilde{S}^{n-1})$ of $C^\infty(S^{n-1})$ by the behavior of the spherical harmonic development.

THEOREM 5.1. *For $f\in C^\infty(S^{n-1})$, we will denote by $f(\omega) = \sum_{k=0}^{\infty}S_k(\omega)$ the spherical harmonic development of f; that is, $S_k(\omega)$ is the k-th spherical harmonic component of f: $S_k(\omega) = S_k(f;\omega)$.*

(i) *Suppose $1<r\leqslant\infty$. If $f\in\mathcal{O}(\tilde{S}^{n-1}(r))$, we have*

$$(5.15)\qquad \limsup_{k\to\infty}(||S_k(\cdot)||_{L\infty})^{1/k}\leqslant 1/r;$$

(ii) *Suppose $1\leqslant r<\infty$. If $f\in\mathcal{O}(\tilde{S}^{n-1}[r])$, we have*

$$(5.16)\qquad \limsup_{k\to\infty}(||S_k(\cdot)||_{L\infty})^{1/k}<1/r;$$

(iii) *If* $f \in \operatorname{Exp}(\tilde{S}^{n-1})$, *we have*

$$\limsup_{k\to\infty} (k!\, \|S_k(\cdot)\|_{L\infty})^{1/k} < \infty. \tag{5.17}$$

Proof. Let us prove (i). By Lemma 5.2, for any given function $f \in \mathcal{O}(\tilde{S}^{n-1}(r))$, there exists a holomorphic function F on the Lie ball $\tilde{B}(r)$ such that $f = F|_{\tilde{S}^{n-1}(r)}$. As $\tilde{B}(r)$ is balanced, we can develop F by homogeneous polynomials

$$F(z) = \sum_{k=0}^{\infty} f_k(z), \tag{5.18}$$

where $f_k \in H^k$ and the convergence is uniform on every compact set of $\tilde{B}(r)$. By Lemma 4.1, we have for every $1 < \varrho < r$,

$$\|f_k(\cdot)\|_{L\infty} \equiv \sup\{|f_k(\omega)|;\ \omega \in S^{n-1}\} \leqslant \varrho^{-k} M_\varrho, \tag{5.19}$$

where we put

$$M_\varrho = \max\{|F(z)|;\ L(z) \leqslant \varrho\}. \tag{5.20}$$

Now we have by Definition 1.1,

$$\begin{aligned} S_k(\omega) &= S_k(f;\omega) \\ &= \frac{N(n;k)}{\Omega_n} \int_{S^{n-1}} F(\tau) P_k(n; \langle\omega,\tau\rangle)\, d\Omega_n(\tau) \\ &= \frac{N(n;k)}{\Omega_n} \int_{S^{n-1}} \sum_{j=0}^{\infty} f_j(\tau) P_k(n; \langle\omega,\tau\rangle)\, d\Omega_n(\tau) \\ &= \frac{N(n;k)}{\Omega_n} \sum_{j=k}^{\infty} \int_{S^{n-1}} f_j(\tau) P_k(n; \langle\omega,\tau\rangle)\, d\Omega_n(\tau), \end{aligned} \tag{5.21}$$

where, in the last equality, we used Lemmas 1.3 and 1.4.

By (5.19), we get

$$\begin{aligned} |S_k(\omega)| &\leqslant N(n;k) \sum_{j=k}^{\infty} \varrho^{-j} M_\varrho \\ &= N(n;k) M_\varrho \varrho^{-k} (1-\varrho^{-1})^{-1} \quad \text{for } \omega \in S^{n-1}, \end{aligned} \tag{5.22}$$

provided $1 < \varrho < r$. Therefore we get

$$\limsup_{k\to\infty} (\|S_k(\cdot)\|_{L\infty})^{1/k} \leqslant 1/\varrho. \tag{5.23}$$

ϱ being arbitrary with $1 < \varrho < r$, we get (5.16).

(ii) is a corollary to (i). Let us prove (iii). By the definition of $\operatorname{Exp}(\tilde{S}^{n-1})$, there exists an entire function F of exponential type such that $F|_{\tilde{S}^{n-1}} = f$. Then there exists $A > 0$ such that $\|F\|_{(A;L)} < \infty$. Let (5.18) be the development of F by homogeneous polynomials. By Lemma 4.2, we get

$$\|f_k(\cdot)\|_{L\infty} \leqslant \sqrt{2\pi k}\, \|F\|_{(A;L)} \frac{A^k}{k!}. \tag{5.24}$$

Therefore using the formula (5.21), we get

$$\begin{aligned} |S_k(\omega)| &\leqslant N(n;k) \sum_{j=k}^{\infty} \sqrt{2\pi j}\, \|F\|_{(A;L)} \frac{A^j}{j!} \\ &\leqslant N(n;k) \sqrt{2\pi} \|F\|_{(\ ;L)} \frac{A^k}{(k-1)!} e^A. \end{aligned} \tag{5.25}$$

From Lemma 1.2 and (5.25), we can conclude

$$\limsup_{k\to\infty} (k!\, \|S_k(\cdot)\|_{L\infty})^{1/k} \leqslant A < \infty. \ \blacksquare$$

DEFINITION 5.1. Let us define

$$\mathcal{O}_\Delta(\tilde{B}(r)) = \{f \in \mathcal{O}(\tilde{B}(r));\ \Delta_z f(z) = 0\} \quad \text{for} \quad 1 < r \leqslant \infty, \tag{5.26}$$

$$\mathcal{O}_\Delta(\tilde{B}[r]) = \liminf_{r'>r} \mathcal{O}_\Delta(\tilde{B}(r')) \quad \text{for} \quad 1 \leqslant r < \infty, \tag{5.27}$$

$$\operatorname{Exp}_\Delta(\boldsymbol{C}^n) = \{f \in \operatorname{Exp}(\boldsymbol{C}^n);\ \Delta_z f(z) = 0\}, \tag{5.28}$$

where $\Delta_z = \dfrac{\partial^2}{\partial z_1^2} + \dfrac{\partial^2}{\partial z_2^2} + \ldots + \dfrac{\partial^2}{\partial z_n^2}$.

$\mathcal{O}_\Delta(\tilde{B}(r))$ being a closed subspace of the FS space $\mathcal{O}(\tilde{B}(r))$, it is an FS space. $\mathcal{O}_\Delta(\tilde{B}[r])$ and $\operatorname{Exp}(\boldsymbol{C}^n)$ being closed subspaces of the DFS spaces $\mathcal{O}(\tilde{B}[r])$ and $\operatorname{Exp}(\boldsymbol{C}^n)$, they are DFS spaces.

THEOREM 5.2. *Let* $\{S_k \in \mathscr{H}^k(S^{n-1});\ k \in \boldsymbol{Z}_+\}$ *be a sequence of spherical harmonics. We denote by* $\tilde{S}_k$ *the harmonic homogeneous polynomial of degree* k *corresponding to* S_k.

(i) *Suppose* $1 < r \leqslant \infty$. *If the sequence* $\{S_k\}$ *satisfies the condition* (5.15), *then the series*

$$F_0(z) = \sum_{k=0}^{\infty} \tilde{S}_k(z) \tag{5.29}$$

converges uniformly on every compact set of $\tilde{B}(r)$, $F_0 \in \mathcal{O}_\Delta(\tilde{B}(r))$, $f = F_0|_{\tilde{S}^{n-1}(r)} \in \mathcal{O}(\tilde{S}^{n-1}(r))$ *and we have* (2.2).

(ii) *Suppose* $1 \leqslant r < \infty$. *If the sequence* $\{S_k\}$ *satisfies the condition* (5.16), *then the series* (5.29) *converges in the topology of* $\mathcal{O}(\tilde{B}[r])$, $F_0 \in \mathcal{O}_\Delta(\tilde{B}[r])$, $f = F_0|_{\tilde{S}^{n-1}[r]} \in \mathcal{O}(\tilde{S}^{n-1}[r])$ *and we have* (2.2).

(iii) *If the sequence* $\{S_k\}$ *satisfies the condition* (5.17), *then the series* (5.29) *converges in the topology of* $\operatorname{Exp}(\boldsymbol{C}^n)$ *and* $F_0 \in \operatorname{Exp}_\Delta(\boldsymbol{C}^n)$, $f = F_0|_{S^{n-1}} \in \operatorname{Exp}(\tilde{S}^{n-1})$ *and we have* (2.2).

For the proof we need the following lemmas:

LEMMA 5.4. *The Bergman–Šilov boundary* $\Sigma(r)$ *of the Lie ball* $\tilde{B}(r)$ *is given by*

$$\Sigma(r) = \{e^{i\theta}x;\ x \in \boldsymbol{R}^n,\ \|x\| = r,\ \theta \in \boldsymbol{R}\}. \tag{5.30}$$

(See Hua [5].) The following fact was constantly used in Siciak [13]:

LEMMA 5.5. *Let f_k be a homogeneous polynomial of degree k. Then we have*

$$\begin{aligned}(5.31)\qquad \sup\{|f_k(\omega)|;\ \omega\in S^{n-1}\} &= \sup\{|f_k(x)|;\ x\in B[1]\}\\ &= \sup\{|f_k(z)|;\ z\in\Sigma(1)\} = \sup\{|f_k(z)|;\ z\in\tilde{B}[1]\}.\end{aligned}$$

Proof. The first equality results from the homogeneity of f_k, the second equality results from Lemma 5.4 and the last equality is a property of the Bergman–Šilov boundary. ∎

Proof of Theorem 5.2. Let us prove (i). By Lemma 5.5, we have

$$(5.32)\qquad |\tilde{S}_k(z)|\leqslant L(z)^k\|S_k(\cdot)\|_{L_\infty}\quad\text{for any } z\in \boldsymbol{C}^n.$$

Therefore by Lemma 4.1, the series (5.29) converges uniformly on every compact set of the Lie ball $\tilde{B}(r)$ and $F_0\in\mathcal{O}(\tilde{B}(r))$. $\tilde{S}_k|_{\boldsymbol{R}^n}$ being harmonic and the convergence being uniform on every compact set of the ball $B(r)$, the function $F_0|_{B(r)}$ is harmonic. By the uniqueness of analytic continuation, the function F_0 is in $\mathcal{O}_\Delta(\tilde{B}(r))$. Other statements are clear by Theorem 2.1.

(ii) is a corollary to (i). Let us prove (iii). Suppose now (5.17). By Lemma 5.5, the sequence $\{S_k\}$ satisfies the condition (4.17) with $N = L$ and $A = \infty$. Therefore $F_0\in\operatorname{Exp}(\boldsymbol{C}^n)$ and the series (5.29) converges in the topology of $\operatorname{Exp}(\boldsymbol{C}^n)$. ∎

COROLLARY. *Let $f\in\mathcal{O}(\tilde{S}^{n-1}(r))$, $1<r\leqslant\infty$ (resp. $f\in\mathcal{O}(\tilde{S}^{n-1}[r])$, $1\leqslant r<\infty$, $\operatorname{Exp}(\tilde{S}^{n-1})$), $S_k(f;\omega)$ be the k-th spherical harmonic component of f and $\tilde{S}_k(f;z)$ be the corresponding harmonic homogeneous polynomial of degree k. Then the series*

$$(5.33)\qquad f(z)=\sum_{k=0}^{\infty}\tilde{S}_k(f;z),\qquad z\in\tilde{S}^{n-1}$$

converges in the topology of $\mathcal{O}(\tilde{S}^{n-1}(r))$ (resp. of $\mathcal{O}(\tilde{S}^{n-1}[r])$, of $\operatorname{Exp}(\tilde{S}^{n-1})$).

Remark that we have, for $z\in\tilde{S}^{n-1}$,

$$\tilde{S}_k(f;z)=\frac{N(n;k)}{\Omega_n}\int_{S^{n-1}} f(\tau)P_k(n;\langle z,\tau\rangle)d\Omega_n(\tau).$$

DEFINITION 5.2. Put

$$(5.34)\qquad C^\infty_\Delta(B(r))=\{f\in C^\infty(B(r));\ \Delta_x f(x)=0\}\quad\text{for}\quad 0<r\leqslant\infty,$$

$$(5.35)\qquad C^\infty_\Delta(B[r])=\liminf_{r'>r} C^\infty_\Delta(B(r'))\quad\text{for}\quad 0\leqslant r<\infty,$$

$$(5.36)\qquad C^\infty_\Delta(\boldsymbol{R}^n;\exp)=\{f\in C^\infty(\boldsymbol{R}^n);\ \text{there exists } A>0 \text{ such that } \sup\{|f(x)|\exp(-A\|x\|);\ x\in\boldsymbol{R}^n\}<\infty\},$$

where $\Delta_x=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\ldots+\frac{\partial^2}{\partial x_n^2}$ is the Laplacian.

THEOREM 5.3. *The following restriction mappings are topological linear isomorphisms:*

(i) α: $\mathcal{O}_\Delta(\tilde{B}(r))\xrightarrow{\sim} C^\infty_\Delta(B(r))$ *for* $0<r\leqslant\infty$,
β: $\mathcal{O}_\Delta(\tilde{B}(r))\xrightarrow{\sim}\mathcal{O}(\tilde{S}^{n-1}(r))$ *for* $1<r\leqslant\infty$;

(ii) α: $\mathcal{O}_\Delta(\tilde{B}[r])\xrightarrow{\sim} C^\infty_\Delta(B[r])$ *for* $0\leqslant r<\infty$,
β: $\mathcal{O}_\Delta(\tilde{B}[r])\xrightarrow{\sim}\mathcal{O}(\tilde{S}^{n-1}[r])$ *for* $1\leqslant r<\infty$;

(iii) α: $\operatorname{Exp}_\Delta(\boldsymbol{C}^n)\xrightarrow{\sim} C^\infty_\Delta(\boldsymbol{R}^n;\exp)$,
β: $\operatorname{Exp}_\Delta(\boldsymbol{C}^n)\xrightarrow{\sim}\operatorname{Exp}(\tilde{S}^{n-1})$.

Proof. Let us prove (i). It was proved by Kiselman [6] and Siciak [13] that the restriction α is a topological linear isomorphism. We reproduce here an outline of the proof. The continuity and injectivity of α are clear. So if we can prove the surjectivity, α is a topological linear isomorphism by the closed graph theorem. Suppose $f\in C^\infty_\Delta(B(r))$ is given. Then we have the expansion

$$f(\varrho\omega)=\sum_{k=0}^{\infty}\varrho^k S_k(\omega)=\sum_{k=0}^{\infty}\tilde{S}_k(\varrho\omega),\qquad 0\leqslant\varrho<r,\ \|\omega\|=1,$$

where $S_k\in\mathscr{H}^k(S^{n-1})$, $\tilde{S}_k$ is the corresponding harmonic homogeneous polynomial of degree k and the convergence is uniform on every compact set of $B(r)$. We have

$$\varrho^k S_k(\omega)=\frac{N(n;k)}{\Omega_n}\int_{S^{n-1}} f(\varrho\tau)P_k(n;\langle\tau,\omega\rangle)d\Omega_n(\tau)$$

by Theorem 1.1. Therefore we get

$$(5.37)\qquad |\varrho^k S_k(\omega)|\leqslant N(n;k)\sup\{|f(x)|;\ \|x\|\leqslant\varrho\}$$

for any $\varrho<r$, from which results

$$\limsup_{k\to\infty}(\|S_k(\cdot)\|_{L_\infty})^{1/k}\leqslant 1/r.$$

We can conclude by Theorem 5.2 (i) that $F_0(z)=\sum_{k=0}^{\infty}\tilde{S}_k(z)\in\mathcal{O}_\Delta(\tilde{B}(r))$ and $F_0|_{B(r)}=f$, which proves the surjectivity of α.

The surjectivity of β is proved by Theorem 5.2. We will show its injectivity. Suppose that $F\in\mathcal{O}_\Delta(\tilde{B}(r))$ satisfies $F|_{\tilde{S}^{n-1}(r)}=0$. Then the harmonic function $f=F|_{B(r)}$ vanishes on S^{n-1}. By the maximal modulus principle, we know $f(x)=0$ for all $x\in B[1]$. By the uniqueness of analytic continuation, F is identically zero. β being continuous, it is a topological linear isomorphism by the closed graph theorem.

(ii) is a corollary to (i). The proof of (iii) is similar. We have only to show the surjectivity of α. If $f\in C^\infty_\Delta(\boldsymbol{R}^n;\exp)$, we have, for some $A>0$,

$$\sup\{|f(x)|;\ \|x\|\leqslant\varrho\}=\sup\{|f(x)|;\ \|x\|=\varrho\}\leqslant\exp(A\varrho).$$

Therefore by (5.37)

$$|S_k(\omega)|\leqslant N(n;k)\varrho^{-k}\exp(A\varrho)\qquad\text{for any }\varrho>0;$$

By the same argument as in the proof of Lemma 4.2 and by Lemma 1.2, we can conclude the estimate (5.17). Finally by Theorem 5.2 (iii), we can conclude $F_0(z)\in\operatorname{Exp}_\Delta(\boldsymbol{C}^n)$.

6. Analytic functionals on $\tilde{S}^{n-1}$

We shall denote by $\mathcal{O}'(\tilde{S}^{n-1}(r))$, $1 < r \leqslant \infty$, $\mathcal{O}'(\tilde{S}^{n-1}[r])$, $1 \leqslant r < \infty$ and $\operatorname{Exp}'(\tilde{S}^{n-1})$ the dual spaces of $\mathcal{O}(\tilde{S}^{n-1}(r))$, $\mathcal{O}(\tilde{S}^{n-1}[r])$ and $\operatorname{Exp}(\tilde{S}^{n-1})$ respectively. By Corollary to Theorem 5.2, we know that the spherical harmonic functions form a dense subspace of $\mathcal{O}(\tilde{S}^{n-1}(r))$, $\mathcal{O}(\tilde{S}^{n-1}[r])$ and $\operatorname{Exp}(\tilde{S}^{n-1})$ respectively. By transposing the inclusion relations (5.14), we have the following inclusion relations:

$$\operatorname{Exp}'(\tilde{S}^{n-1}) \supset \mathcal{O}'(\tilde{S}^{n-1}) \supset \mathcal{O}'(\tilde{S}^{n-1}[r]) \supset \mathcal{O}'(\tilde{S}^{n-1}(r)) \supset \mathscr{B}(S^{n-1}) \supset \mathscr{D}'(S^{n-1}) \supset L^2(S^{n-1}). \tag{6.1}$$

LEMMA 6.1. *The following sequence are exact*:

$$0 \to \mathcal{O}'(\tilde{S}^{n-1}(r)) \xrightarrow{\beta^*} \mathcal{O}'(\tilde{B}(r)) \to \mathscr{I}'(\tilde{B}(r)) \to 0, \tag{6.2}$$

$$0 \to \mathcal{O}'(\tilde{S}^{n-1}[r]) \xrightarrow{\beta^*} \mathcal{O}'(\tilde{B}[r]) \to \mathscr{I}'(\tilde{B}[r]) \to 0, \tag{6.2'}$$

$$0 \to \operatorname{Exp}'(\tilde{S}^{n-1}) \to \operatorname{Exp}'(C^n) \to \mathscr{I}'_{\exp}(C^n) \to 0. \tag{6.2''}$$

In fact these sequences are obtained by transposing the sequences (5.10), (5.10′) and (5.10″). We may consider by the mapping β^* the spaces $\mathcal{O}'(\tilde{S}^{n-1}(r))$, $\mathcal{O}'(\tilde{S}^{n-1}[r])$ and $\operatorname{Exp}'(\tilde{S}^{n-1})$ as subspaces of $\mathcal{O}'(\tilde{B}(r))$, $\mathcal{O}'(\tilde{B}[r])$ and $\operatorname{Exp}'(C^n)$ respectively.

Let $T \in \mathcal{O}'(\tilde{S}^{n-1}(r))$, $\mathcal{O}'(\tilde{S}^{n-1}[r])$ or $\operatorname{Exp}'(\tilde{S}^{n-1})$ and $f \in \mathcal{O}(\tilde{S}^{n-1}(r))$, $\mathcal{O}(\tilde{S}^{n-1}[r])$ or $\operatorname{Exp}(\tilde{S}^{n-1})$ respectively. Then by Corollary to Theorem 5.2, we have

$$\begin{aligned}\langle T,f\rangle &= \sum_{k=0}^{\infty}\langle T_z, \tilde{S}_k(f;z)\rangle \\ &= \sum_{k=0}^{\infty}\Big\langle T_z, \frac{N(n;k)}{\Omega_n}\int_{S^{n-1}} f(\tau)P_k(n;\langle z,\tau\rangle)d\Omega_n(\tau)\Big\rangle \\ &= \sum_{k=0}^{\infty}\int_{S^{n-1}}\frac{N(n;k)}{\Omega_n}\langle T_z, P_k(n;\langle z,\tau\rangle)\rangle f(\tau)d\Omega_n(\tau).\end{aligned}$$

Define the kth spherical harmonic component of $T \in \operatorname{Exp}'(\tilde{S}^{n-1})$ by

$$S_k(T;\tau) = \frac{N(n;k)}{\Omega_n}\langle T_z, P_k(n;\langle z,\tau\rangle)\rangle. \tag{6.3}$$

It is clear that $S_k(T;\tau)$ is a spherical harmonic function of degree k. Then we have

$$\begin{aligned}\langle T,f\rangle &= \sum_{k=0}^{\infty}\int_{S^{n-1}} S_k(T;\tau)f(\tau)d\Omega_n(\tau) \\ &= \sum_{k=0}^{\infty}\int_{S^{n-1}} S_k(T;\tau)S_k(f;\tau)d\Omega_n(\tau).\end{aligned} \tag{6.4}$$

THEOREM 6.1. *Suppose $T \in \operatorname{Exp}'(\tilde{S}^{n-1})$. Let us denote by $S_k(\tau)$ the k-th spherical harmonic component of T: $S_k(\tau) = S_k(T;\tau)$.*

(i) *If $T \in \mathcal{O}'(\tilde{S}^{n-1}(r))$, $1 < r \leqslant \infty$, we have*

$$\limsup_{k\to\infty}(\|S_k(\cdot)\|_{L\infty})^{1/k} < r; \tag{6.5}$$

(ii) *If $T \in \mathcal{O}'(\tilde{S}^{n-1}[r])$, $1 \leqslant r < \infty$, we have*

$$\limsup_{k\to\infty}(\|S_k(\cdot)\|_{L\infty})^{1/k} \leqslant r; \tag{6.6}$$

(iii) *If $T \in \operatorname{Exp}'(\tilde{S}^{n-1})$, we have*

$$\limsup_{k\to\infty}(\|S_k(\cdot)\|_{L\infty}/k!)^{1/k} = 0. \tag{6.7}$$

Proof. Let us prove (i). If $T \in \mathcal{O}'(\tilde{S}^{n-1}(r))$, by the continuity of T, there exist constants ϱ with $1 \leqslant \varrho < r$ and $C \geqslant 0$ such that

$$|\langle T,f\rangle| \leqslant C\sup\{|f(z)|;\, z \in \tilde{S}^{n-1},\, L(z) \leqslant \varrho\}.$$

Remark that

$$\tilde{P}_k(n;z;\tau) = (\sqrt{z^2})^k P_k(n;\langle z/\sqrt{z^2};\tau\rangle)$$

is the harmonic homogeneous polynomial of degree k. Consequently, by virtue of Lemmas 5.5 and 1.5, we can majorize $S_k(T;\tau)$ as follows:

$$\begin{aligned}|S_k(\tau)| = |S_k(T;\tau)| &\leqslant C\frac{N(n;k)}{\Omega_n}\sup\{|P_k(n;\langle z,\tau\rangle)|;\, z \in \tilde{S}^{n-1},\, L(z) \leqslant \varrho\} \\ &\leqslant C\frac{N(n;k)}{\Omega_n}\sup\{|\tilde{P}_k(n;z;\tau)|;\, z \in \tilde{S}^{n-1},\, L(z) \leqslant \varrho\} \\ &\leqslant C\frac{N(n;k)}{\Omega_n}\varrho^k\sup\{|\tilde{P}_k(n;x;\tau)|;\, x \in S^{n-1}\} \\ &= C\frac{N(n;k)}{\Omega_n}\varrho^k,\end{aligned}$$

from which results (6.5).

(ii) is a corollary to (i). Let us prove (iii). Suppose now $T \in \operatorname{Exp}'(\tilde{S}^{n-1})$. Then by the continuity of T, for every $A > 0$, there exists a constant $C_A \geqslant 0$ such that

$$|\langle T,f\rangle| \leqslant C_A\sup\{|f(z)|\exp(-AL(z));\, z \in C^n\} = C_A\|f\|_{(A;L)}$$

for every $f \in X_{A;L}$, where we are considering $T \in \operatorname{Exp}'(C^n)$ by Lemma 6.1. Therefore we have, by Lemma 5.5 and Lemma 1.5,

$$\begin{aligned}|S_k(\tau)| &= |S_k(T;\tau)| \\ &\leqslant C_A\frac{N(n;k)}{\Omega_n}\sup\{|\tilde{P}_k(n;z;\tau)|\exp(-AL(z));\, z \in C^n\} \\ &= C_A\frac{N(n;k)}{\Omega_n}\sup\{L(z)^k\exp(-AL(z));\, z \in C^n\} \\ &= C_A\frac{N(n;k)}{\Omega_n}\frac{k^k}{A^k}\exp(-k).\end{aligned}$$

By Stirling's formula (4.21), we get

$$\limsup_{k\to\infty} (\|S_k(\cdot)\|_{L_\infty}/k!)^{1/k} \leqslant \frac{1}{M} \lim_{k\to\infty} \frac{k^k e^{-k}}{k!} = 0. \ \blacksquare$$

THEOREM 6.2. *Suppose that a sequence* $\{S_k \in \mathscr{H}^k(S^{n-1});\ k \in \mathbf{Z}_+\}$ *is given.*

(i) *Suppose* $1 < r \leqslant \infty$. *If the sequence* $\{S_k\}$ *satisfies the condition* (6.5), *then the formula*

$$(6.8)\qquad \langle T, f\rangle = \sum_{k=0}^{\infty} \int_{S^{n-1}} S_k(\omega) f(\omega)\, d\Omega_n(\omega) = \sum_{k=0}^{\infty} \int_{S^{n-1}} S_k(\omega) S_k(f;\omega)\, d\Omega_n(\omega)$$

defines a linear continuous functional T *on* $\mathcal{O}(\tilde{S}^{n-1}(r))$ *such that*

$$(6.9)\qquad S_k(T;\tau) = S_k(\tau) \quad \text{for } k \in \mathbf{Z}_+.$$

(ii) *Suppose* $1 \leqslant r < \infty$. *If the sequence* $\{S_k\}$ *satisfies the condition* (6.6), *then the formula* (6.8) *defines* $T \in \mathcal{O}'(\tilde{S}^{n-1}[r])$ *such that* (6.9).

(iii) *If the sequence* $\{S_k\}$ *satisfies the condition* (6.7), *then the formula* (6.8) *defines* $T \in \operatorname{Exp}'(S^{n-1})$ *such that* (6.9).

Proof. Let us prove (i). By the condition (6.5), there exist constants $\varrho_0 < r$ and $C \geqslant 0$ such that

$$|S_k(\tau)| \leqslant C\varrho_0^k \quad \text{for all } k \in \mathbf{Z}_+.$$

Let $f \in \mathcal{O}(S^{n-1}(r))$ and $f(\omega) = \sum_{k=0}^{\infty} S_k(f;\omega)$ be the spherical harmonic development of f. Then by Theorem 5.1 (i), the series (6.8) converges and defines a linear functional T on $\mathcal{O}(\tilde{S}^{n-1}(r))$. On the other hand, by (5.22), for $\varrho_0 < \varrho < r$,

$$(6.10)\qquad |\langle T, f\rangle| \leqslant \sum_{k=0}^{\infty} \int_{S^{n-1}} |S_k(\omega)|\,|S_k(f;\omega)|\, d\Omega_n(\omega) \leqslant \Omega_n \sum_{k=0}^{\infty} C\varrho_0^k N(n;k) M_\varrho \varrho^{-k}(1-\varrho)^{-1} \leqslant \Omega_n C(1-\varrho)^{-1} \sum_{k=0}^{\infty} N(n;k)(\varrho_0/\varrho)^k M_\varrho,$$

where $M_\varrho = \sup\{|F(z)|;\ L(z) \leqslant \varrho\}$. As $\sum_{k=0}^{\infty} N(n;k)\,(\varrho_0/\varrho)^k < \infty$, (6.10) implies the continuity of T on $\mathcal{O}(\tilde{B}(r))$, hence on $\mathcal{O}(\tilde{S}^{n-1}(r))$. (6.9) is nothing but (1.9).

(ii) is a corollary to (i). Let us prove (iii). Suppose that the sequence $\{S_k\}$ satisfies the condition (6.7). For every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$|S_k(\omega)| \leqslant C_\varepsilon \varepsilon^k k! \quad \text{for any } k \in \mathbf{Z}_+,$$

by Theorem 5.1 (iii), the series (6.8) converges and defines a linear functional T on $\operatorname{Exp}(\tilde{S}^{n-1})$. If $F \in X_{A;L}$ satisfies $f = F|_{S^{n-1}}$, we showed in (5.25)

$$|S_k(f;\omega)| \leqslant N(n;k)\sqrt{2\pi}\,\|F\|_{(A;L)} \frac{A^k}{(k-1)!} e^A.$$

Therefore if $\varepsilon < A^{-1}$, we have

$$(6.11)\qquad |\langle T, f\rangle| \leqslant \sum_{k=0}^{\infty} \int_{S^{n-1}} |S_k(\omega)|\,|S_k(f;\omega)|\, d\Omega_n(\omega) \leqslant C\Omega_n \sum_{k=0}^{\infty} \varepsilon^k k!\, N(n;k)\sqrt{2\pi}\,\|F\|_{(A;L)} \frac{A^k}{(k-1)!} e^A \leqslant C\Omega_n \sqrt{2\pi} e^A \sum_{k=0}^{\infty} (\varepsilon A)^k N(n;k)\, k \|F\|_{(A;L)}.$$

As we have $\sum_{k=0}^{\infty} (\varepsilon A)^k N(n;k) k < \infty$, (6.11) implies the continuity of T with respect to the norm $\|F\|_{(A;L)}$, especially the continuity of T on $\operatorname{Exp}(\mathbf{C}^n)$. $\blacksquare$

7. The Fourier–Borel transformation of $\operatorname{Exp}'(\tilde{S}^{n-1})$

Following Hashizume–Kowata–Minemura–Okamoto [2] we define the transformation $\mathscr{P}_\lambda$ for a functional $T \in \operatorname{Exp}'(\tilde{S}^{n-1})$ by

$$(7.1)\qquad \mathscr{P}_\lambda T(\zeta) = \langle T_z, \exp(i\lambda\langle\zeta, z\rangle)\rangle,$$

where $\lambda \in \mathbf{C}$, $\lambda \neq 0$ is a fixed constant and $\langle\zeta, z\rangle = \zeta_1 z_1 + \zeta_2 z_2 + \ldots + \zeta_n z_n$. This transformation $\mathscr{P}_\lambda$ is nothing but the restriction of the Fourier–Borel transformation $\mathscr{P}_\lambda$ on $\operatorname{Exp}'(\mathbf{C}^n)$, defined in § 4, to the subspace $\operatorname{Exp}'(\tilde{S}^{n-1})$.

DEFINITION 7.1. Put

$$(7.2)\qquad \mathcal{O}_\lambda(\mathbf{C}^n) = \{F \in \mathcal{O}(\mathbf{C}^n);\ (\Delta_\zeta + \lambda^2)F(\zeta) = 0\},$$

$$(7.3)\qquad C_\lambda^\infty(\mathbf{R}^n) = \{F \in C^\infty(\mathbf{R}^n);\ (\Delta_\zeta + \lambda^2)F(\zeta) = 0\}.$$

The spaces $\mathcal{O}_\lambda(\mathbf{C}^n)$ and $C_\lambda^\infty(\mathbf{R}^n)$ are FS spaces, being closed subspaces of the FS spaces $\mathcal{O}(\mathbf{C}^n)$ and $C^\infty(\mathbf{R}^n)$ respectively.

LEMMA 7.1. *Let* $T \in \operatorname{Exp}'(\tilde{S}^{n-1})$. *Then* $F(\zeta) = \mathscr{P}_\lambda T(\zeta)$ *is an entire function of* $\zeta \in \mathbf{C}^n$ *and satisfies the partial differential equation*

$$(7.4)\qquad (\Delta_\zeta + \lambda^2)F(\zeta) = 0,$$

where $\Delta_\zeta = \dfrac{\partial^2}{\partial\zeta_1^2} + \dfrac{\partial^2}{\partial\zeta_2^2} + \ldots + \dfrac{\partial^2}{\partial\zeta_n^2}$; *that is,* $\mathscr{P}_\lambda$ *maps* $\operatorname{Exp}'(\tilde{S}^{n-1})$ *into* $\mathcal{O}_\lambda(\mathbf{C}^n)$.

In fact, the function $\zeta \mapsto \exp(i\lambda\langle\zeta, z\rangle)$ satisfies (7.4) and the differentiation with respect to ζ is continuous in the topology of $\operatorname{Exp}(\tilde{S}^{n-1})$.

LEMMA 7.2. *The restriction mapping $\alpha\colon F\mapsto F|_{R^n}$ establishes a linear topological isomorphism*:

$$\alpha\colon \mathcal{O}_\lambda(C^n) \xrightarrow{\sim} C^\infty_\lambda(R^n). \tag{7.5}$$

In fact, the lemma results from the fact that the Laplacian Δ_ζ is an elliptic operator with constant coefficients (see Kiselman [6] for the details of this phenomenon).

The Bessel function of order ν, $\nu \neq -1, -2, \ldots$ is defined as follows:

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{2k}. \tag{7.6}$$

LEMMA 7.3. *For fixed $\zeta\in C^n$ and $\lambda\in C$, consider the function $f(\omega) = \exp(i\lambda\langle\zeta,\omega\rangle)$ of $\omega\in S^{n-1}$. The k-th spherical harmonic component $S_k(f;\omega)$ of the function f is given as follows*:

$$\begin{aligned} S_k(f;\omega) &= N(n;k)i^k\Gamma\left(\frac{n}{2}\right)\left(\frac{\lambda}{2}\sqrt{\zeta^2}\right)^{(2-n)/2} J_{k+(n-2)/2}(\lambda\sqrt{\zeta^2})P_k\left(n;\left\langle\frac{\zeta}{\sqrt{\zeta^2}},\omega\right\rangle\right)\\ &= N(n;k)i^k\Gamma\left(\frac{n}{2}\right)\left(\frac{\lambda}{2}\sqrt{\zeta^2}\right)^{-k+(2-n)/2} J_{k+(n-2)/2}(\lambda\sqrt{\zeta^2})\times\\ &\qquad\times\left(\frac{\lambda}{2}\sqrt{\zeta^2}\right)^k P_k\left(n;\left\langle\frac{\zeta}{\sqrt{\zeta^2}},\omega\right\rangle\right). \end{aligned} \tag{7.7}$$

(Cf. Lemma 3 of [2], where we must read $a_n = \Gamma(n/2)$.)

Proof. Remark first that the functions

$$\left(\frac{\lambda}{2}\sqrt{\zeta^2}\right)^{-k+(2-n)/2} J_{k+(n-2)/2}(\lambda\sqrt{\zeta^2}) \quad\text{and}\quad \left(\frac{\lambda}{2}\sqrt{\zeta^2}\right)^k P_k\left(n;\left\langle\frac{\zeta}{\sqrt{\zeta^2}},\omega\right\rangle\right)$$

are entire functions of $(\lambda,\zeta)\in C\times C^n$. By Definition 1.1, we have

$$S_k(f;\omega) = \frac{N(n;k)}{\Omega_n}\int_{S^{n-1}} \exp(i\lambda\langle\zeta,\tau\rangle)P_k(n;\langle\omega,\tau\rangle)\,d\Omega_n(\tau). \tag{7.8}$$

As $S_k(f;\omega)$ is an entire function of $(\lambda,\zeta)\in C\times C^n$, we have only to prove (7.7) when $\zeta\in R^n$ and $\lambda\in R$. Suppose $\zeta\in R^n$, $\zeta\neq 0$ and put $r=\sqrt{\zeta^2}>0$ and $\tau_0 = \zeta/\sqrt{\zeta^2}$.

By the Funk–Hecke formula (see for example Müller [10]), we have

$$S_k(f;\omega) = \frac{N(n;k)}{\Omega_n}\Omega_{n-1}\Lambda P_k(n;\langle\tau_0,\omega\rangle), \tag{7.9}$$

where

$$\Lambda = \int_{-1}^{1} \exp(i\lambda rt)P_k(n;t)(1-t^2)^{(n-3)/2}dt. \tag{7.10}$$

By Gegenbauer's integral formula (see for example Vilenkin [15], p. 555), we have

$$\Lambda = i^k\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)\left(\frac{\lambda r}{2}\right)^{(2-n)/2} J_{k+(n-2)/2}(\lambda r).$$

Using the formula (1.2), we get (7.7). ■

LEMMA 7.4. *We have the following estimate*:

$$\left(\frac{1}{2}\right)^\nu\frac{1}{\Gamma(\nu+1)} > J_\nu(1) > \left(\frac{1}{2}\right)^\nu\frac{\cos 1}{\Gamma(\nu+1)} > 0 \quad \textit{for} \quad \nu\geqslant 0. \tag{7.11}$$

Proof. We have the following integral formula (see for example Stein–Weiss [14], p. 151):

$$\begin{aligned} \frac{\sqrt{\pi}\,\Gamma(\nu+1/2)}{(z/2)^\nu}J_\nu(z) &= \int_{-1}^{1}\exp(izt)(1-t^2)^{\nu-1/2}dt\\ &= \int_0^\pi \cos(z\cos\theta)\sin^{2\nu}\theta\,d\theta. \end{aligned}$$

Therefore we have

$$\frac{\sqrt{\pi}\,\Gamma(\nu+1/2)}{(1/2)^\nu}J_\nu(1) = \int_0^\pi \cos(\cos\theta)\sin^{2\nu}\theta\,d\theta.$$

But we have

$$\int_0^\pi \sin^{2\nu}\theta\,d\theta > \int_0^\pi\cos(\cos\theta)\sin^{2\nu}\theta\,d\theta > \cos 1\int_0^\pi\sin^{2\nu}\theta\,d\theta$$

and

$$\int_0^\pi \sin^{2\nu}\theta\,d\theta = \sqrt{\pi}\,\frac{\Gamma(\nu+1/2)}{\Gamma(\nu+1)},$$

from which we can conclude (7.11). ■

THEOREM 7.1. (i) *Suppose $T\in \mathrm{Exp}'(\tilde{S}^{n-1})$. Let us denote by $S_k(T;\omega)$ the k-th spherical harmonic component of T and by $\tilde{S}_k(T;\zeta)$ the corresponding harmonic homogeneous polynomial of degree k. Then we have*

$$\begin{aligned} \mathcal{P}_\lambda T(\zeta) &= 2\pi^{n/2}\left(\frac{\lambda}{2}\sqrt{\zeta^2}\right)^{(2-n)/2}\sum_{k=0}^\infty i^k J_{k+(n-2)/2}(\lambda\sqrt{\zeta^2})\,S_k\left(T;\frac{\zeta}{\sqrt{\zeta^2}}\right)\\ &= 2\pi^{n/2}\sum_{k=0}^\infty i^k\left(\frac{\lambda}{2}\right)^k\left(\frac{\lambda}{2}\sqrt{\zeta^2}\right)^{-k-(n-2)/2} J_{k+(n-2)/2}(\lambda\sqrt{\zeta^2})\,\tilde{S}_k(T;\zeta). \end{aligned} \tag{7.12}$$

(ii) *The transformation* $\mathscr{P}_\lambda$ *establishes linear topological isomorphism of* $\operatorname{Exp}'(\tilde{S}^{n-1})$ *onto* $\mathcal{O}_\lambda(C^n)$ *and* $\mathcal{O}'(\tilde{S}^{n-1})$ *onto* $\operatorname{Exp}_\lambda(C^n)$:

$$\mathscr{P}_\lambda\colon \operatorname{Exp}'(\tilde{S}^{n-1}) \xrightarrow{\sim} \mathcal{O}_\lambda(C^n), \tag{7.13}$$

$$\mathscr{P}_\lambda\colon \mathcal{O}'(\tilde{S}^{n-1}) \xrightarrow{\sim} \operatorname{Exp}_\lambda(C^n), \tag{7.14}$$

where we put

$$\operatorname{Exp}_\lambda(C^n) = \operatorname{Exp}(C^n)\cap\mathcal{O}_\lambda(C^n).$$

Proof. (i) Suppose a functional $T \in \operatorname{Exp}'(\tilde{S}^{n-1})$ is given. Then by the formulas (7.7) and (6.4), we have

$$\begin{aligned}
&\langle T_z, \exp(i\lambda\langle\zeta, z\rangle)\rangle \\
&= \sum_{k=0}^{\infty}\int_{S^{n-1}} S_k(T;\omega)N(n;k)i^k\Gamma\left(\frac{n}{2}\right)\left(\frac{\lambda}{2}\sqrt{\zeta^2}\right)^{-k+(2-n)/2}\times \\
&\qquad\times J_{k+(n-2)/2}(\lambda\sqrt{\zeta^2})\left(\frac{\lambda}{2}\sqrt{\zeta^2}\right)^k P_k\left(n;\left\langle\frac{\zeta}{\zeta^2},\omega\right\rangle\right)d\Omega_n(\omega) \\
&= \sum_{k=0}^{\infty} N(n;k)i^k\Gamma\left(\frac{n}{2}\right)\left(\frac{\lambda}{2}\sqrt{\zeta^2}\right)^{-k+(2-n)/2} J_{k+(n-2)/2}(\lambda\sqrt{\zeta^2})\times \\
&\qquad\times\left(\frac{\lambda}{2}\sqrt{\zeta^2}\right)^k\frac{\Omega_n}{N(n;k)}S_k\left(T;\frac{\zeta}{\sqrt{\zeta^2}}\right) \\
&= \Omega_n\Gamma\left(\frac{n}{2}\right)\left(\frac{\lambda}{2}\sqrt{\zeta^2}\right)^{(2-n)/2}\sum_{k=0}^{\infty} i^k J_{k+(n-2)/2}(\lambda\sqrt{\zeta^2})S_k\left(T;\frac{\zeta}{\sqrt{\zeta^2}}\right).
\end{aligned}$$

Therefore we get (7.12) if we use the formula (1.2). ∎

(ii) The Fourier–Borel transformation $\mathscr{P}_\lambda$ maps $\operatorname{Exp}'(\tilde{S}^{n-1})$ into $\mathcal{O}_\lambda(C^n)$ by Lemma 7.1. The injectivity of $\mathscr{P}_\lambda$ is clear by the formula (7.12). Suppose $F \in \mathcal{O}_\lambda(C^n)$ is given. Put for $\varrho \in C$, $F_\varrho(z) = F(\varrho z)$. Then by (1.6), we have $F(\varrho\omega) = \sum_{k=0}^{\infty} S_k(F_\varrho;\omega)$, $\omega \in S^{n-1}$, where

$$S_k(F_\varrho;\omega) = \frac{N(n;k)}{\Omega_n}\int_{S^{n-1}} F(\varrho\tau)P_k(n;\langle\tau,\omega\rangle)d\Omega_n(\tau). \tag{7.15}$$

Because F satisfies the differential equation (7.4), we have

$$\begin{aligned}
-\lambda^2 S_k(F_\varrho;\omega) &= \frac{N(n;k)}{\Omega_n}\int_{S^{n-1}} \Delta F(\varrho\tau)P_k(n;\langle\tau,\omega\rangle)d_n(\tau) \\
&= \frac{N(n;k)}{\Omega_n}\int_{S^{n-1}}\left(\frac{d^2}{d\varrho^2}+\frac{n-1}{\varrho}\frac{d}{d\varrho}+\frac{1}{\varrho^2}\Delta_S\right)F(\varrho\tau)\times \\
&\qquad\times P_k(n;\langle\tau,\omega\rangle)d\Omega_n(\tau).
\end{aligned}$$

From Proposition 1.2, we can conclude that the holomorphic function $G(\varrho) = S_k(F_\varrho;\omega)$ satisfies the Bessel differential equation:

$$\frac{d^2}{d\varrho}G+\frac{n-1}{\varrho}\frac{d}{d\varrho}G+\left(\lambda^2-\frac{k(k+n-2)}{\varrho^2}\right)G = 0, \tag{7.16}$$

from which we can conclude

$$S_k(F_\varrho;\omega) = C_k(\omega)\varrho^{(2-n)/2}J_{k+(n-2)/2}(\lambda\varrho), \tag{7.17}$$

where $C_k(\omega)$ is a function of $\omega \in S^{n-1}$. Putting $\varrho = 1/\lambda$ in (6.17), we get

$$C_k(\omega) = \frac{S_k(F_{1/\lambda};\omega)}{J_{k+(n-2)/2}(1)}\lambda^{(2-n)/2} \tag{7.18}$$

and

$$F(\varrho\omega) = \sum_{k=0}^{\infty}(\lambda\varrho)^{(2-n)/2}J_{k+(n-2)/2}(\lambda\varrho)\frac{S_k(F_{1/\lambda};\omega)}{J_{k+(n-2)/2}(1)} \tag{7.19}$$

for $\varrho \in C$, $\omega \in S^{n-1}$. Now define a spherical harmonic function $S_k(\omega)$ of degree k by

$$S_k(\omega) = (-i)^k\left(\frac{1}{2\pi}\right)^{n/2}\frac{S_k(F_{1/\lambda};\omega)}{J_{k+(n-2)/2}(1)}. \tag{7.20}$$

Now we have, by Lemma 7.4,

$$\begin{aligned}
&\limsup_{k\to\infty}(\|S_k(\cdot)\|_{L_\infty}/k!)^{1/k} \\
&\qquad= \limsup_{k\to\infty}\left(\frac{\|S_k(F_{1/\lambda};\cdot)\|_{L_\infty}}{J_{k+(n-2)/2}(1)k!}\right)^{1/k} \\
&\qquad= \limsup_{k\to\infty}\left(\|S_k(F_{1/\lambda};\cdot)\|_{L_\infty}2^{k+(n-1)/2}\frac{\Gamma(k+(n-2)/2+1)}{k!}\right)^{1/k} \\
&\qquad= 2\limsup_{k\to\infty}(\|S_k(F_{1/\lambda};\cdot)\|_{L_\infty})^{1/k} = 0
\end{aligned}$$

by Theorem 5.1 (i) $r = \infty$. Therefore by Theorem 6.2 (iii), the sequence $\{S_k(\omega) \in \mathscr{H}^k(S^{n-1})\}$ determines a functional $T \in \operatorname{Exp}'(\tilde{S}^{n-1})$ such that (6.9). By the formula (7.12), we can conclude that $F(\zeta) = \mathscr{P}_\lambda T(\zeta)$, which proves the surjectivity of (7.13). As it is clearly continuous, it is a linear topological isomorphism.

As is shown in Lemma 6.1, we can consider $\mathcal{O}'(\tilde{S}^{n-1}) \subset \mathcal{O}'(C^n)$. Therefore it is classical that the $\mathscr{P}_\lambda$-image of $\mathcal{O}'(\tilde{S}^{n-1})$ is contained in $\operatorname{Exp}(C^n)$ (Theorem 4.1 (4.27)). Now suppose $F \in \operatorname{Exp}_\lambda(C^n)$. Then by Theorem 5.1 (iii), the functions $S_k(\cdot)$ defined by (7.20) satisfy

$$\begin{aligned}
\limsup_{k\to\infty}(\|S_k(\cdot)\|_{L_\infty})^{1/k} &= \limsup_{k\to\infty}\left(\frac{\|S_k(F_{1/\lambda};\cdot)\|_{L_\infty}}{J_{k+(n-2)/2}(1)}\frac{k!}{k!}\right)^{1/k} \\
&= 2\limsup_{k\to\infty}(\|S_k(F_{1/\lambda};\cdot)\|_{L_\infty}k!)^{1/k} < \infty,
\end{aligned}$$

where we used Lemma 7.4. Therefore by Theorem 6.2(i) $r = \infty$, there exists a functional $T \in \mathcal{O}'(\tilde{S}^{n-1})$ such that (6.9) and that $F(\zeta) = \mathscr{P}_\lambda T(\zeta)$, which proves the surjectivity of (7.14). The rest of the proof is same as above. ∎

Remark. The linear topological isomorphism (7.13) and (7.14) are very special cases of the Ehrenpreis–Palamodov fundamental principle.

References

[1] L. Drużkowski, *Effective formula for the cross norm in the complexified unitary spaces*, Zeszyty Nauk. Uniw. Jagiell., Prace Mat. 16 (1974), 47–53.

[2] M. Hashizume, A. Kowata, K. Minemura and K. Okamoto, *An integral representation of the Laplacian on the Euclidean space*, Hiroshima Math. J. 2 (1972), 535–545.

[3] M. Hashizume, K. Minemura and K. Okamoto, *Harmonic functions on Hermitian hyperbolic spaces*, ibid. 3 (1973), 81–108.

[4] S. Helgason, *Eigenspaces of the Laplacian; Integral representations and irreducibility*, J. Functional Analysis 17 (1974), 328–353.

[5] L.K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in Classical Domains* (Russian), Moscow 1959. Translations of Math. Monographs, 6, AMS, 1963.

[6] C.O. Kiselman, *Prolongement des solutions d'une équation aux dérivées partielles à coefficients constants*, Bull. Soc. Math. France 97 (1969), 329–356.

[7] J.L. Lions and E. Magenes, *Problèmes aux limites non-homogènes VII*, Ann. Math. Pura Appl. 63 (1963), 201–224.

[8] A. Martineau, *Equations différentielles d'ordre infini*, Bull. Soc. Math. France 95 (1967), 109–154.

[9] M. Morimoto, *A generalization of the Fourier–Borel transformation for the analytic functions with non convex carrier*, Tokyo J. Math. 2 (1979), 301–322.

[10] C. Müller, *Spherical Harmonics*, Lecture Notes in Math. 17, Springer, 1966.

[11] R.T. Seeley, *Spherical harmonics*, AMS Monthly 73, part II, 4 (1966), 115–121.

[12] —, *Eigenfunction expansions of analytic functions*, Proc. Amer. Math. Soc. 21 (1969), 734–738.

[13] J. Siciak, *Holomorphic continuation of harmonic functions*, Ann. Pol. Math. 29 (1974), 67–73.

[14] E.M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton, New Jersey, 1971.

[15] N.Ja. Vilenkin, *Special Functions and the Theory of Group Representations*, Translation of Math. Monographs, 22, AMS, 1968.

Presented to the Semester
COMPLEX ANALYSIS
February 15–May 30, 1979

COMPLEX ANALYSIS
BANACH CENTER PUBLICATIONS, VOLUME 11
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1983

COMPLEX METHODS IN NON-LINEAR ANALYSIS

JOSEF NAAS

Institut für Mathem. d. Akad. d. Wiss. d. DDR
DDR-1086 Berlin, Mohrenstr. 39, DDR

WOLFGANG TUTSCHKE

Sektion Mathematik der Universität, Halle, DDR-401, Universitätsplatz 8/9, DDR

One of the most important questions of analysis is the investigation of functional dependences using the concept of the limit. With it, on the one hand, many conclusions are valid under rather weak assumptions (they hold, for instance, for mappings of topological spaces). This fact may give an impression that the possibilities of the complex function theory (which starts from the consideration of complex-valued functions of complex variables) are contained in an abstract mapping theory of topological (or some more general) spaces.

On the other hand it is important to take into account that more specific assumptions permit a richer theory. The functions regarded within the complex function theory lead to the concept of holomorphy. Holomorphic functions have various specific properties. Their local behaviour determines, for instance, their global behaviour. Such properties of holomorphic functions cause that the complex function theory is an autonomous theory describing the general concept of holomorphy.

From this, however, it is not yet possible to conclude that a boundless development of the concept of holomorphy gives the unique end of a general "complex analysis". In our opinion from this the possibility of too affected generalization started indeed (as again in the case of other mathematical theories). For some generalizations of the concept of holomorphy, for instance, the applicability seems to be not satisfactory at all.

There are, for sure, many immediate applications of complex analysis (for instance those connected with the approximation theory of one or several complex variables). Fundamental applications of complex analysis, however, are connected with the theory of partial differential equations. This is true not only in the case of Cauchy–Riemann systems and the Laplace equation (holomorphic functions are, as it is well-known, connected immediately with these partial differential equa-