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SPACE-TIME COORDINATE TRANSFORMATIONS IN CONTINUUM MECHANICS INVARIABLE EQUATIONS OF THERMOMECHANICS

1. Introduction. It was shown in [6] that the uniform three-dimensional Drobot formalism [3] which allows to classify interactions in a material deformable continuum in the case of statical problems can be generalized to the case of a continuum embedded in a four-dimensional space-time $T \times E^3$, where T denotes the one-dimensional time space. This method allows us to obtain equations of motion of the classical and Cosserat continuum with appropriate inertial terms (momenta, spins) appearing in a natural way. Thus we need not use the so-called d'Alembert principle.

The procedure of embedding the examined material continuum in space-time requires to assume a space-time definition of the continuum motion as well as to define an operator $(A_\mu u^\mu)(x)$ which classifies the interactions in the material continuum in a certain region of space-time. Obviously, we have also to make use of the methods of tensor analysis in space-time. Tensor-valued functions which appear in the formalism were treated in [6] as four-tensors under the following coordinate transformation group G_1 :

$$\tilde{x}_0 = x_0, \quad \tilde{x}_i = f_i(x_1, x_2, x_3), \quad \left| \frac{\partial \tilde{x}_i}{\partial x_j} \right| \neq 0,$$

where x_0 denotes the time coordinate and x_i the space ones. A displacement four-vector u_μ was defined as $u_\mu = \{0, u_i\}$, so it is of an invariant form under G_1 . The motivation for adopting here such a form of the vector u_μ lies in the conviction of a complete description of the classical continuum by the vector $u_i(x_0, x_1, x_2, x_3)$ and of the Cosserat continuum by the vectors $u_i(x_0, x_1, x_2, x_3)$ and $\omega_i(x_0, x_1, x_2, x_3)$.

Now we extend the coordinate transformation group G_1 to the following group G :

$$\tilde{x}_0 = x_0, \quad \tilde{x}_i = f_i(x_0, x_1, x_2, x_3), \quad \left| \frac{\partial x_\mu}{\partial \tilde{x}_\nu} \right| \neq 0.$$

The transformations G preserve the so-called absolute time required by classical mechanics and allow also to introduce local coordinate systems moving in the space E^3 . This permits us to interpret the continuum motion as a transition to a new local system of coordinates in space-time.

Traditionally in continuum mechanics certain three-tensors are defined which are then physically interpreted. But if we consider any nontrivial space-time transformation of coordinates (e.g., Galileo's transformations which are allowed by Newtonian mechanics), then we notice that certain three-dimensional tensors, called *objective*, preserve their tensor capacity in contrast to nonobjective ones. The subjects of objectivity have been controversial in mechanics [4].

Making use of the specific properties of tensors under the transformations G we propose here to extend the traditional objective tensors of continuum mechanics to the four-tensors covariant under G and the nonobjective tensors to the four-tensors contravariant under G .

We also consider in general the problem how to embed the traditional three-dimensional tensors in any four-dimensional tensors under G . With the aid of two natural four-vectors, the contravariant four-vector of velocity $v^a = (1, v^i)$ and the covariant time normal $t_a = (-1, 0, 0, 0)$, we define, after Precht [7], a projector $S_a{}^b$ on a three-dimensional subspace. This allows us to identify certain tensors under G as traditional three-dimensional tensors of continuum mechanics.

It is necessary to omit the assumption $u_0 = 0$ as a consequence of the admission of transformations G in place of G_1 . Making use of the methods of dimensional analysis we show that u_0 is connected in a certain way with a temperature field.

The set of four equations which is obtained from the generalized Drobot formalism is interpreted as stress equations of continuum thermomechanics.

The advantage of the presented formalism lies in that it shows evidently the deformations corresponding to the above-mentioned generalized stresses and it leads in a natural way to a precise definition of the continuum deformations.

The presented method is similar to the Lagrangian variational principle, it does not require however the assumption that a potential exists and it separates the problems of obtaining the equations of equilibrium, which are valid for all continua, from the problems of formulating the so-called constitutive equations. If we admit constitutive equations such as in thermoelasticity, we see certain analogies with the variational principles presented by Biot [1].

2. Coordinate transformations in space-time. In [6] an attempt to treat the time and space variables in a uniform way was made. Instead of the usual one-parameter family of functions f_i^t describing the continuum

motion, where $f_i^t: C_0 \rightarrow C_t$, $\tilde{x}_i = f_i^t(x_1, x_2, x_3)$, C_0 and C_t are the medium configurations at moments $t = 0$ and t , respectively, only one function $\psi_\mu: T \times C_0 \rightarrow T \times E^3$ was defined in [6]. The function ψ_μ assigns to each moment $x_0 = t$ and to each particle $x_i \in C_0$ the same moment t and the place $\tilde{x}_i \in C_t$ at which a particle is found at moment t . In this way the continuum motion described by ψ_μ can be identified as the following group G of coordinate transformations in space-time:

$$\begin{aligned}
 (1) \quad & \tilde{x}_\mu = \psi_\mu(x_0, x_1, x_2, x_3), \\
 & \tilde{x}_0 = x_0, \quad \tilde{x}_i = \psi_i(x_0, x_1, x_2, x_3), \quad \left| \frac{\partial x_\mu}{\partial x_\nu} \right| \neq 0,
 \end{aligned}$$

where Greek indices range from 0 to 3, and the Latin ones from 1 to 3.

The transformations G given by (1) are taken as the admitted coordinate transformations in the considered space-time $T \times E^3$. Now we deal with the four-tensors associated with such a group G .

Note that G preserves time as required by classical mechanics and that Galileo's group transformations of coordinates appropriate to Newtonian mechanics is a subgroup of G . Notice also that in this way we admit not only the inertial frame but also a linear acceleration of the coordinate system, and the Coriolis acceleration may be included to suitable affine connexion coefficients $\Gamma_{\mu\nu}^e$ associated with the transformations G . Kilmister [5] dealt with this subject generalizing the inertial frame of classical mechanics to the coordinate systems G .

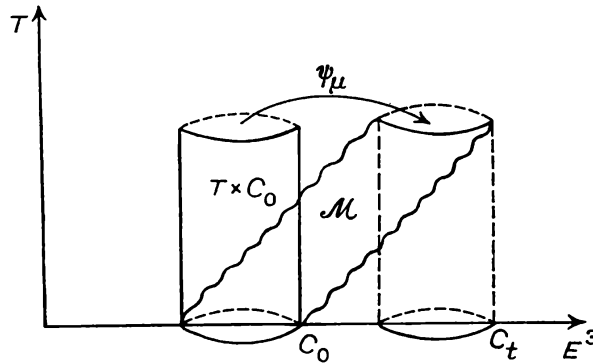


Fig. 1. Regions examined in space-time. Transformations $\psi_\mu: T \times C_0 \rightarrow \mathcal{M}$ describing the continuum motion are defined by formulae (1)

Denote the position vector of any point of a region $\mathcal{M} \subset T \times E^3$ (see [6]) by the notation such as in Fig. 1. Admitting in the region \mathcal{M} the transformations G and putting $R_{\mu,\nu} = \Gamma_{\mu\nu}^e R_e$ we obtain

$$\begin{aligned}
 (2) \quad & \Gamma_{\mu\nu}^e = \Gamma_{\nu\mu}^e, \quad \Gamma_{00}^j \frac{\partial \tilde{x}_k}{\partial x_j} = \frac{\partial^2 \tilde{x}_k}{\partial x^2}, \\
 & \Gamma_{\mu\nu}^0 = 0, \quad \Gamma_{0k}^j \frac{\partial \tilde{x}_i}{\partial x_j} = \frac{\partial}{\partial x_0} \frac{\partial \tilde{x}_i}{\partial x_k}.
 \end{aligned}$$

From equations (2) we get some relations between the coefficients Γ_{00}^j and the linear acceleration of the coordinate system, and we see that the coefficients Γ_{0k}^j are related with the acceleration of the rotating coordinate-system in the plane (x_i, x_k) with changeable velocity.

From formulae (2) we obtain also the following equations for the covariant derivatives $\bar{\nabla}_\mu \bar{u}^\nu$ of the four-vector \bar{u}^ν :

$$(3) \quad \begin{aligned} \bar{\nabla}_0 \bar{u}^0 &= \frac{\partial \bar{u}^0}{\partial x_0}, & \bar{\nabla}_i \bar{u}^0 &= \frac{\partial \bar{u}^0}{\partial x_i}, \\ \bar{\nabla}_i \bar{u}^j &= \Gamma_{i0}^j \bar{u}^0 + \nabla_i \bar{u}^j, & \bar{\nabla}_0 \bar{u}^i &= \frac{\partial}{\partial x_0} \bar{u}^i + \Gamma_{00}^i \bar{u}^0 + \Gamma_{0j}^i \bar{u}^j, \end{aligned}$$

where ∇_i denotes the traditional three-dimensional covariant derivative.

Objectivity and objective tensors. The subjects of objectivity of tensors appearing in the theory of continuum medium have been controversial in mechanics. For that reason it is necessary to begin with a closer investigation of coordinate transformations in space-time. As known, certain three-dimensional tensors under study in continuum mechanics, called *objective tensors*, preserve their tensor capacity when one considers any nontrivial space-time transformations of coordinates in contrast to the nonobjective tensors.

If we deal with a generalized group of transformations of coordinates, from the beginning we are already able to see the problem of objectivity from a new point of view. We cannot omit entirely the space-time coordinate transformations because even the Galileo transformations have such a capacity.

Note that the tensors under transformations G have a specific property. Namely, the time components T^0, T^{00}, \dots of the contravariant tensors $T^\mu, T^{\mu\nu}, \dots$ are the invariants under G , and the space components T_i, T_{ij}, \dots of the covariant tensors $T_\mu, T_{\mu\nu}, \dots$ map as three-dimensional tensors. For example, if v^μ and F_μ are the contravariant and covariant vectors under G , respectively, we have the following rule:

$$(4) \quad \begin{aligned} \tilde{v}^0 &= v^0, & \tilde{F}_0 &= F_0 + \frac{\partial x_i}{\partial \tilde{x}_0} F_i, \\ \tilde{v}^i &= \frac{\partial \tilde{x}_i}{\partial x_j} v^j + \frac{\partial \tilde{x}_i}{\partial x_0} v^0, & F_j &= \frac{\partial x_i}{\partial \tilde{x}_j} F_i. \end{aligned}$$

Making use of these properties of the tensors under G we propose to generalize the traditional continuum mechanics tensors which are objective to four-dimensional covariant tensors under G and the nonobjective tensors to the four-tensors contravariant under G .

It is known that the three-dimensional vector of velocity v^i is nonobjective. It maps under the Galileo transformations

$$(5) \quad \tilde{x}_0 = x_0, \quad \tilde{x}_i = A_i^j x_j + u^i x_0, \quad A A^T = 1$$

as follows:

$$(6) \quad \tilde{v}^i = v^i + u^i \quad \text{if } A_i^j = \delta_i^j.$$

Hence, let us try to generalize the three-vector v^i to the contravariant four-dimensional vector v^μ under transformations (5) in such a way that for its space components u^i the Galileo principle (6) of adding the velocity is valid. As Bonder has already seen [2], in order that the vector v^i satisfies the Galileo principle (6) it is necessary and sufficient to generalize v^i to the contravariant vector v^μ of the following form:

$$(7) \quad v^\mu = \{1, v^i\}.$$

The above remarks hold if we consider the transformations G instead of (5). Note that, by (4), $v^0 = 1$ is an invariant condition under G .

There exists one more natural four-vector under G , namely the covariant space normal

$$(8) \quad t_\mu = \{-1, 0, 0, 0\}.$$

It follows from (4) that the form of t_μ is invariant under G .

Projector on a three-dimensional subspace. If we admit the transformations G in space-time $T \times E^3$, then the following problem arises: how can one find among the four-tensors under G the traditional three-dimensional tensors introduced and interpreted in continuum mechanics up to now? In other words, how can one build the four-tensors under G making use of appropriate combinations of three-dimensional tensors?

Using the vectors v^μ and t_μ given by (7) and (8) we can construct a projector as follows (see [7]):

$$S_a^\beta = \delta_a^\beta + v^\beta t_a,$$

where δ_a^β denotes the Kronecker delta. This operator satisfies the relation $S_a^\beta S_\beta^\gamma = S_a^\gamma$ required for the projective operator.

Note that the kernel of S_a^β is generated by the vectors v^β and t_a . It is easy to verify that the contravariant four-vector w^β belongs to the kernel of S_a^β if and only if w^β is of the form $w^\beta = b v^\beta$, where b is a real number. It is known that the covariant vector l_β belongs to the kernel of S_a^β if and only if l_β is of the form $l_\beta = b t_\beta$. Consequently, S_a^β is a projector on the three-dimensional subspace of the four-dimensional space-time $T \times E^3$.

We use the projector S_a^β to decompose the tensors under G into parts which map as three-dimensional tensors and, therefore, can be understood as traditional tensors of continuum mechanics.

If \bar{u}^β is the contravariant four-dimensional vector under G , then

1° $u^\beta = S_\alpha^\beta \bar{u}^\alpha$ is a vector belonging to E^3 ($u^0 = 0$);

2° there is an invariant $u = -t_\beta \bar{u}^\beta$ which corresponds to every four-vector \bar{u}^β ;

3° any contravariant four-vector \bar{u}^β can be decomposed as

$$(9) \quad \bar{u}^\beta = S_\alpha^\beta \bar{u}^\alpha + v^\beta u = u^\beta + v^\beta u.$$

Later we interpret the vector u^β as a displacement vector, while the invariant u is related in a certain manner to a temperature field.

If \bar{F}_α is the covariant vector under the transformations G , then

1° the vector $F_\alpha = S_\alpha^\beta \bar{F}_\beta$ belongs to the three-dimensional subspace of space-time $T \times E^3$ and is of the form $F_\alpha = \{-F_i v^i, F_i\}$;

2° there is an invariant $F = -v^\alpha \bar{F}_\alpha$ which corresponds to every four-vector \bar{F}_α ;

3° any covariant four-vector \bar{F}_α can be decomposed as

$$(10) \quad \bar{F}_\alpha = S_\alpha^\beta \bar{F}_\beta + t_\alpha F = F_\alpha + t_\alpha F.$$

Later we interpret the space components F_i of the vector F_α as the force vector, and the invariant F as the power of the source.

Applying the decomposition (10) to the vector of covariant derivative calculated with the aid of affine connexion coefficients which correspond to the transformations G , we have

$$(11) \quad \bar{\nabla}_\mu \dots = \nabla_\mu \dots - t_\mu D \dots,$$

where $\nabla_i \dots$ denotes the three-dimensional covariant derivative operator if it acts on projected four-quantities, and $D \dots$ is the so-called material derivative with respect to time.

Note that formulae (11) applied to tensors of any order replace the difficult formulae (3).

The tensors \bar{F}_α^β of mixed order under G have the following properties:

1° $F_\alpha^\beta = -S_\alpha^\gamma S_\delta^\beta \bar{F}_\delta^\gamma$ is of the form

$$F_\alpha^\beta = \left\{ \begin{array}{c|c} 0 & -v^i F_i^j \\ \hline 0 & F_i^j \end{array} \right\}.$$

2° There exist an invariant ε and vectors q^α, p_α which correspond to any tensor \bar{F}_α^β so that

$$\varepsilon = v^\alpha t_\beta \bar{F}_\alpha^\beta, \quad q^\alpha = -v^\beta S_\gamma^\alpha \bar{F}_\beta^\gamma, \quad p_\alpha = -t_\gamma S_\alpha^\beta \bar{F}_\beta^\gamma,$$

where $q^\alpha = \{0, q^i\}$ and $p_\alpha = \{-p_i v^i, p_i\}$.

3° Any tensor \bar{F}_α^β can be decomposed as follows:

$$(12) \quad \bar{F}_\alpha^\beta = \varepsilon t_\alpha v^\beta + t_\alpha q^\beta + p_\alpha v^\beta - F_\alpha^\beta.$$

Later we interpret the components F_i^j of the four-tensor F_a^β as the stress tensor and the vectors q^i and p_i as the three-vectors of heat flux and momentum, respectively. The invariant ε is interpreted as the internal energy of the considered continuum.

Decompositions (9) and (10) of the contravariant and covariant vectors under G imply analogous decompositions of tensors of any order. It will be shown in a separate paper that the decomposition of the mixed tensor $F_a^{\beta\gamma}$ of third order introduces new multipliers which can be interpreted physically on the ground of hydromechanics and rheology of solids.

3. Interactions in material deformable continua and their classification.

In this section we use the formalism presented by Drobot [3] and generalized in [6] to the case of continua embedded in space-time $T \times E^3$.

Interactions in the material deformable continuum are described and classified by the differential operator A_μ which assigns to each vector field $\bar{u}^\mu(a)$ defined in the region $\mathcal{M} \subset T \times E^3$ a scalar-valued function $\varphi(a) = A_\mu \bar{u}^\mu(a)$ defined on \mathcal{M} and belonging to a unitary space with a given inner product $\langle \cdot, \cdot \rangle$. The considered region \mathcal{M} is shown in Fig. 1. The adjoint operator A_μ^* is defined by the condition $\langle A_\mu \bar{u}^\mu, \psi \rangle = \langle \bar{u}_\mu, A_\mu^* \psi \rangle$.

In [6] the following definition of equilibrium of a material deformable continuum was admitted: to say that the *deformable continuum represented by the operator A_μ is in equilibrium* (motion is understood as an equilibrium in space-time) means that

$$(13) \quad A_\mu^* \psi = 0 \quad \text{if } \psi = \text{const.}$$

Now we describe conditions (13) assuming a suitable form of the operator A_μ and a suitable form of the inner product $\langle \psi, \varphi \rangle$. Taking the transformations G as the admitted coordinate transformations in the region \mathcal{M} and building the four-dimensional tensor fields based on G we consider interactions in a material continuum described by the differential operator A_μ of first order.

Assume that the differential operator acting on four-vectors \bar{u}^μ defined in the region $\mathcal{M} \subset T \times E^3$ and on its boundary $\partial\mathcal{M}$ is given by the formula

$$(14) \quad A_\mu \bar{u}^\mu(a) = \begin{cases} \bar{F}_\mu(a) \bar{u}^\mu(a) + \bar{F}_\mu{}^\nu(a) \bar{\nabla}_\nu \bar{u}^\mu(a) & \text{for } a \in \mathcal{M}, \\ \bar{f}_\mu(a) \bar{u}^\mu(a) & \text{for } a \in \partial\mathcal{M}, \end{cases}$$

where \bar{F}_μ , $\bar{F}_\mu{}^\nu$, \bar{f}_μ are tensor-valued functions given in the region \mathcal{M} and on its boundary $\partial\mathcal{M}$, respectively, and the symbols $\bar{\nabla}_\mu$ denote the covariant derivative in the region \mathcal{M} .

The inner product is defined as follows:

$$\langle A_\mu \bar{u}^\mu, \psi \rangle = \int_{\mathcal{M}} dv A_\mu \bar{u}^\mu \psi + \int_{\partial\mathcal{M}} d\sigma A_\mu \bar{u}^\mu \psi.$$

The adjoint operator is of the form

$$(15) \quad \Lambda_\mu^* \psi(a) = \begin{cases} F_\mu^*(a) \psi(a) + F_\mu^{**}(a) \nabla_\mu \psi(a) & \text{for } a \in \mathcal{M}, \\ f_\mu^*(a) \psi(a) & \text{for } a \in \partial\mathcal{M}. \end{cases}$$

If we take into consideration the decompositions of four-tensors given by (9)-(12), then from (14) and (15) we obtain

$$(16) \quad (\Lambda_\mu u^\mu)(a) = \begin{cases} F_i u^i - F_i^j \nabla_j u^i + p_i D u^i + \mathcal{F} u - q^j \nabla_j u - \varepsilon D u & \text{for } a \in \mathcal{M}, \\ f_i u^i - f u & \text{for } a \in \partial\mathcal{M}, \end{cases}$$

where $\mathcal{F} = p_i D v^i - F_i^j \nabla_j v^i - F$, and for the adjoint operator

$$(\Lambda_\mu^* \psi)(a) = \begin{cases} F_\mu^* + t_\mu F + (t_\mu q^{*j} - F_\mu^{*j}) \nabla_j \psi + (p_\mu^* + \varepsilon^* t_\mu) D \psi & \text{for } a \in \mathcal{M}, \\ f_\mu^* + t_\mu f & \text{for } a \in \partial\mathcal{M}, \end{cases}$$

where $F_\mu^* = \{-F_i^* v^i, F_i^*\}$, $p_\mu^* = \{-p_i^* v^i, p_i^*\}$, $f_\mu^* = \{-f_i^* v^i, f_i^*\}$, and

$$F_\mu^{**} = \left\{ \begin{array}{c|c} 0 & -v^i F_i^{*j} \\ \hline 0 & F_i^{*j} \end{array} \right\}.$$

Making the four-fold integration by parts in curvilinear coordinates and admitting in the presented formalism the Lagrangian description ($D = \partial/\partial t$) we obtain the following formulae for the coefficients of the adjoint operator: for $a \in \mathcal{M}$

$$(17) \quad \begin{aligned} F_i^* &= F_i + \nabla_j F_i^j - D p_i, & F^* &= -\mathcal{F} - \nabla_j q^j - D \varepsilon, \\ F_i^{*j} &= -F_i^j, & p_i^* &= -p_i, & q^{*j} &= -q^j, & \varepsilon^* &= -\varepsilon, \end{aligned}$$

and for $a \in \partial\mathcal{M}$

$$(18) \quad f_i^* = f_i - n_j F_i^j + n_0 p_i, \quad f^* = f + n_j q^j + n_0 \varepsilon,$$

where n_μ denotes the four-vector normal to the boundary $\partial\mathcal{M}$ and directed outside the region \mathcal{M} .

The assumed postulate defining the continuum equilibrium leads us to the following equations for the continua described by the operator Λ_μ of first order:

$$(19) \quad \begin{aligned} F_\mu^* + t_\mu F &= 0 & \text{for } a \in \mathcal{M}, \\ f_\mu^* + t_\mu f &= 0 & \text{for } a \in \partial\mathcal{M}. \end{aligned}$$

Now we shall interpret equations (19) as invariable stress equations under the transformations G of continuum thermomechanics.

4. Physical interpretation. Note that the form of equations (19) results from the assumed form of the operator Λ_μ . In order to obtain

mathematical models of a material continuum we must first interpret physically the tensor coefficients appearing in the operator Λ_μ . As in [6] we require that the expression

$$W = \int_{\mathcal{M}} dv \Lambda_\mu \bar{u}^\mu + \int_{\partial\mathcal{M}} d\sigma \Lambda_\mu \bar{u}^\mu$$

has the physical dimension of an action, i.e., $[\hat{W}] = [\text{work}] [\text{time}]$. This allows us to interpret physically the tensor-valued functions appearing in (16) if we make use of the fundamental assumption admitted in methods of dimensional analysis that the addition operation of physical quantities is defined only in the class of the same physical dimension. The following quantities are assigned to natural physical dimensions:

$$\hat{V}_i = V_i [\text{length}]^{-1}, \quad \hat{D} = D [\text{time}]^{-1}, \quad \hat{u}^i = u^i [\text{length}],$$

where V_i, D, u^i denote the dimensionless quantities corresponding to $\hat{V}_i, \hat{D}, \hat{u}^i$, respectively.

In this way the tensors in (16) obtain the following interpretation of physical quantities having physical dimensions:

$$\begin{aligned} \hat{F}_i &= F_i [\text{force}] [\text{volume}]^{-1}, & \hat{F}_i^j &= F_i^j [\text{force}] [\text{surface}]^{-1}, \\ \hat{p}_i &= p_i [\text{momentum}] [\text{volume}]^{-1}, & \hat{F} &= F [\text{energy}] [\text{volume}]^{-1} [\text{time}]^{-1}, \\ \hat{q}^j &= q^j [\text{energy}] [\text{surface}]^{-1} [\text{time}]^{-1}, & \hat{\varepsilon} &= \varepsilon [\text{energy}] [\text{volume}]^{-1}, \\ & & \hat{u} &= u [\text{time}]. \end{aligned}$$

As mentioned in [6] the boundary $\partial\mathcal{M}$ is not “dimensionally uniform”. It is formed by hypersurfaces S_0, S_1, S , where the notation is such as

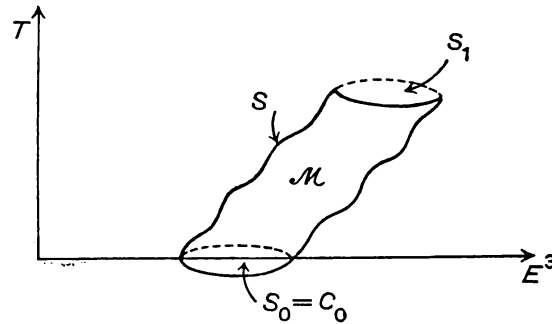


Fig. 2

in Fig. 2. Tensors on S_0 and S_1 have different physical dimensions from those on S . For example, on S we have

$$\hat{f}_k = f_k [\text{force}] [\text{surface}]^{-1}, \quad \hat{f} = f [\text{power}] [\text{surface}]^{-1},$$

and on S_0 and S_1 we obtain

$$\hat{f}_k = f_k[\text{momentum}][\text{volume}]^{-1}, \quad \hat{f} = f[\text{energy}][\text{volume}]^{-1}.$$

For that reason the notation is assumed as follows:

$$\tilde{p}_k = f_k \text{ for } a \in S \quad \text{and} \quad \tilde{\varepsilon} = f \text{ for } a \in S_0 \cup S_1.$$

Applying this notation to formula (16) we obtain the following form of the operator on the boundary $\partial\mathcal{M}$:

$$A_\mu u^\mu = \begin{cases} f_i u^i - f u & \text{for } a \in S, \\ p_i u^i - \tilde{\varepsilon} u & \text{for } a \in S_0 \cup S_1. \end{cases}$$

Introducing the above notation to system (19) and taking into consideration (17) and (18), we obtain

$$(20) \quad \begin{aligned} F_i + \nabla_j F_i^j - D p_i &= 0 \\ p_i D v^i - F_i^j \nabla_j v^i - F + \nabla_j q^j + D \varepsilon &= 0 \end{aligned} \quad \text{for } a \in \mathcal{M},$$

$$(21) \quad f_i - n_j F_i^j + n_0 p_i = 0, \quad f + n_j q^j + n_0 \varepsilon = 0 \quad \text{for } a \in S,$$

$$(22) \quad \tilde{p}_i + p_i = 0, \quad \tilde{\varepsilon} + \varepsilon = 0 \quad \text{for } a \in S_0 \cup S_1.$$

Now, let us discuss the physical and geometrical interpretation of the tensor multipliers F_i^j , q^j , p_i , and ε . Note that equations (22) are initial and final conditions satisfied by the momentum p_i and the energy ε , where \tilde{p}_i and $\tilde{\varepsilon}$ are given functions having the physical dimensions of momentum and energy, respectively. Therefore, we have to interpret only the multipliers F_i^j and q^j appearing in equations (21) which are valid on the boundary C_t at any moment t .

For small velocity motions ($n_0 \approx 0$), equations (21) take the form

$$(23) \quad f_i - n_j F_i^j = 0, \quad f + n_j q^j = 0 \quad \text{for } a \in S.$$

From (23) we obtain immediately the interpretation of the tensor multipliers F_i^j and q^j .

Let us choose inside the region $C_t \subset E^3$ any region \mathcal{A} with boundary $\partial\mathcal{A}$ having a unit normal vector n_i directed outside \mathcal{A} . We adopt here the generalized stress principle of Cauchy which states that the material outside \mathcal{A} acts partly inside \mathcal{A} through the field of density forces (per unit volume element) σ_i and through the scalar field σ of density of nonmechanical (thermal) power on the oriented surface element ds of the boundary $\partial\mathcal{A}$. Obviously, the vector field σ_i , called the *stress vector*, and the scalar field σ of thermal power are from the physical and geometrical point of view of the same type as the fields f_i and f , respectively. In every region \mathcal{A} with boundary $\partial\mathcal{A} \subset C_t$, equations (23) hold with σ_i and σ in place of f_i and f , respectively, i.e.,

$$\bar{\sigma} = \sigma_j \bar{R}^j = n_i F_i^j \bar{R}^j,$$

where \bar{R}^j denote the base contravariant vectors in C_i . With regard to this interpretation the multipliers F_i^j are called a *stress tensor* in a material deformable continuum. However, for the scalar field σ of thermal power we have $\sigma = -n_j q^j$. The multipliers q^j obtain now the interpretation of the heat flux vector.

Now we discuss the geometrical interpretation of the components of the stress tensor F_i^j as well as of the components of the heat flux vector q^j . Denote by $\bar{\sigma}_{(i)}$ the stress vectors and by $\sigma_{(i)}$ the scalar fields of thermal power which act on the oriented surface elements $ds_{(i)}$ of the curvilinear system of coordinates $\xi_i = \text{const}$ in the region C_i . The vectors normal to $ds_{(i)}$ are denoted by $\bar{n}_{(i)}$. The following relations hold:

$$\bar{\sigma}_{(i)} = F_i^j \frac{1}{\sqrt{R^{ii}}} \bar{R}_j \quad \text{and} \quad \sigma_{(i)} = -\frac{1}{\sqrt{R^{ii}}} q^i.$$

Furthermore, the stress vector $\bar{\sigma}$ and the scalar field σ of thermal power acting on any oriented (by the normal n_i) surface element ds form the following linear combinations of vectors $\bar{\sigma}_{(i)}$ and scalars $\sigma_{(i)}$:

$$\bar{\sigma} = \sum_i \bar{\sigma}_{(i)} R^{ii} n_i \quad \text{and} \quad \sigma = \sum_i \sigma_{(i)} R^{ii} n_i.$$

In this way we obtain the interpretation of equations (20)-(22) as the equations of thermomechanics of the classical continuum in the Lagrangian description, invariable under the transformations G . Notice that the components of the velocity vector v^i are related with the displacement vector u^i as follows:

$$Du^i = v^i.$$

Now we consider the problem of formulating the constitutive relations. The decomposition given by (16) can be written here in the form

$$(24) \quad \varphi = A_\mu \bar{u}^\mu = F_i^j u^i - F_i^j \varepsilon_j^i + p_i Du^i + \mathcal{F}u - q^j \varepsilon_j - \varepsilon Du,$$

where $\varepsilon_i^j = \nabla_i u^j$ and $\varepsilon_j = \nabla_j u$. The scalar function φ defined by (24) is interpreted in our considerations as the density of work per volume element of the region C_i . Therefore we can assume that the generalized stresses F_i^j , p_i , q^j act on the derivatives of the displacement vector u^i such as ε_i^j , v^i , and on the derivatives ε_j , Du of the invariant u . The tensors ε_i^j , v^j , ε_i , Du are interpreted as generalized deformations corresponding to suitable generalized stresses. We obtain a physical meaning of the invariant u combining it with the thermal field in the following way:

$$Du = \frac{T - T_0}{T_0} = \frac{\theta}{T_0},$$

where $T = T(x_0, x_i)$ denotes the absolute temperature, and T_0 the temperature of undeformed and unstressed state. Note that in this way $\hat{u} = u[\text{time}]$, according to the above-mentioned requirements of dimensional analysis.

If we assume furthermore the so-called principle of equipresention, then we obtain the following relations between the stresses and deformations:

$$F_i^j = F_i^j(\varepsilon_i^j, \varepsilon_i, \theta), \quad q^j = q^j(\varepsilon_i^j, \varepsilon_i, \theta), \\ \varepsilon = \varepsilon(\varepsilon_i^j, \varepsilon_i, \theta).$$

In thermoelasticity, constitutive relations take the form

$$(25) \quad F_i^j = c_i^j k^l \varepsilon_k^l - \beta_i^j \theta, \quad \varepsilon = \frac{1}{2} c_i^j k^l \varepsilon_j^i \varepsilon_l^k + \beta_i^j \varepsilon_j^i T_0 + c\theta,$$

$$q^j = -\lambda^{ij} T_0 D\varepsilon_i = -\lambda^{ij} \nabla_j T,$$

where the material constants are related to the isothermal state, and c denotes the specific heat related to unit volume at constant deformation.

If we apply constitutive relations of the form (25) in (24), then we see that each of the generalized stresses "works" linearly on the generalized deformation corresponding to it, but the first two relations are of algebraic type and the last one is of differential type. The dependence of the generalized stress q^j on $D\varepsilon_j$ suggests that the term $q^j \varepsilon_j$ in (24) describes the dissipation.

If we take into consideration the constitutive relations (25) in decomposition (24), we can show certain analogies of the presented method with the variational principles developed by Biot [1]. Introducing after Biot the potential $R = \frac{1}{2} \lambda^{ij} T_0 D\varepsilon_i D\varepsilon_j$, we can write the analogue of the term $-q^i \varepsilon_i$ in decomposition (24) in the form

$$-q^i \delta \varepsilon_i = \delta R, \quad \text{where } \delta R = \frac{\partial R}{\partial D\varepsilon_i} \delta \varepsilon_i.$$

Biot introduces such a function R and defines such a variation δR interpreting R as a dissipation function. Instead of the vector ε_j introduced here, Biot defines the vector $H_i = -\lambda_i^j T_0 \varepsilon_j$, i.e., $DH_i = q_i$.

If constitutive relations on F_i^j and ε such as (25) in thermoelasticity hold for the case $\theta/T_0 \ll 1$, then the expression $-F_i^j \varepsilon_j^i - \varepsilon Du$ in (24) can be written in the form

$$V = -F_i^j \varepsilon_j^i - \varepsilon Du = -F_i^j \varepsilon_j^i - \varepsilon \frac{\theta}{T_0} = \frac{1}{2} c_i^j k^l \varepsilon_j^i \varepsilon_l^k + \frac{c\theta^2}{2T_0},$$

where V is the thermal potential introduced by Biot [1].

Note that variational methods efface partially the differences between problems with entirely different genesis such as equations of equilibrium,

which are valid for all classes of materials, and constitutive equations for different classes of materials. The presented methods separate distinctly considerations of geometrical nature from physical interpretations, creating at the same time the possibilities to generalize these methods on wider classes of materials not only on thermoelastic ones.

If we consider material continua defined by the operator A_μ of second order which are not Cosserat ones ([3], [6]), we obtain mathematical models of material deformable continua examined by hydromechanics and rheology of solids. In a separate paper, admitting constitutive relations on generalized stresses and generalized deformations of linear and algebraic type, we shall show that the Navier-Stokes equations can be obtained from the equations of equilibrium (13). One may also discuss in a general way the problem of constitutive equations for the class of continua defined by the operator A_μ of second order which has a potential. The potential can be built up by invariants of generalized deformations. But in that case it will be convenient to introduce a metric tensor in space-time $T \times E^3$. The constitutive equations of thermomechanics and hydromechanics can be obtained as special cases of the general algebraic constitutive equations imposed on generalized stresses and deformations. These topics will be discussed in a separate paper.

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