ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XVII, 4 (1983)

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ON THE BOUNDARY VALUE PROBLEM ARISING IN THE RADIALLY SYMMETRICAL FILTRATION OF FLUID

1. Introduction. Let S_T denote the half-stripe $(0, \infty) \times (0, T]$ in the (x, t)-plane for some fixed $T > 0$. In this paper we investigate the mixed boundary value problem

$$Lu \equiv -u_t + (u^m)_{xx} + (x+\xi)^{-1}(u^m)_x = 0 \quad \text{in } S_T, \tag{1}$$

$$u(x, 0) = u_0(x) \quad \text{for } x \in (0, \infty), \tag{2}$$

$$u(0, t) = u_1(t) \quad \text{for } t \in [0, T], \tag{3}$$

in which

I. $\xi > 0$ and $m > 1$ are fixed constants,

II. $u_0 = u_0(x)$ for $x \in (0, \infty)$ and $u_1 = u_1(t)$ for $t \in [0, T]$ are given nonnegative bounded functions.

This problem arises in the study of a radially symmetrical nonstationary filtration of fluids or gases in a porous, isotropic and homogeneous medium, surrounding the cylindrical reservoir. Under some physical simplifications, if $m = 2$, the solution $u = u(x, t)$ of equation (1) (the so-called Boussinesq equation) describes the boundary of the saturated region at time t (see [12], [13]). The parameter ξ denotes the radius of the reservoir, $u_0 = u_0(x)$ describes an initial level of fluid in the porous medium, and $u_1 = u_1(t)$ describes the height of fluid in the reservoir at time t. Especially: if $m = 2$ and

$$u_0(x) \equiv 0, \tag{2'}$$

$$u_1(t) \equiv c > 0, \tag{3'}$$

then (1) describes infiltration of fluid into the unsaturated porous medium; if $m = 2$ and

$$u_0(x) \equiv c_1 > 0, \tag{2''}$$

$$u_1(t) \equiv c_2 \geqslant 0, \tag{3''}$$

where $c_1 > c_2$, then (1) describes exfiltration from the porous medium.

Note that boundary data (2')-(3') and (2'')-(3'') do not satisfy the compatibility condition $u_0(0) = u_1(0)$.

For applications, the knowledge of the solution of problem (1)-(3) in a large time t is important. On the other hand, we may expect that after elapsing an arbitrarily small time $\tau > 0$ the solution of problem (1)-(3) becomes regular. Therefore, if we study the process of filtration since the time τ, then we may assume that at the initial moment the solution of problem (1)-(3) is regular. In this paper we assume that

III. $u_0(0) = u_1(0)$,

IV. u_0^{m-1}, u_1 are Lipschitz continuous and $u_1 \geqslant c > 0$ for some constant c.

Equation (1) is a degenerate parabolic equation: it is parabolic for $u > 0$, but it is not such if $u = 0$. In general, boundary value problems for equations of this type need not have classical solutions (see [8]), and so we shall interpret solutions of problem (1)-(3) as generalized solutions.

Let $0 \leqslant x < \bar{x} \leqslant \infty$, $0 \leqslant t < \bar{t} \leqslant T$, and $R = [x, \bar{x}] \times [t, \bar{t}]$. A real-valued function f, defined on R, is said to be *uniformly Hölder continuous with the exponent* $a \in (0, 1]$ on R if there exists a positive constant C such that

$$|f(x, t) - f(x', t')| \leqslant C(|x - x'|^2 + |t - t'|)^{a/2}$$

for (x, t), $(x', t') \in R$ (cf. [2]).

By $C^{l+a}(R)$, where $l = 0, 1, 2$ and $a \in (0, 1]$, we denote the Banach space of functions f, defined on R, bounded and uniformly Hölder continuous with the exponent a together with their derivatives of the form $(\partial/\partial t)^r(\partial/\partial x)^s f$, where $0 \leqslant 2r + s \leqslant l$, in R (cf. [2]).

By $C^{2,1}(R)$ we denote the set of continuous functions f, defined on R, with the derivatives f_t, f_x, f_{xx} continuous in R.

We use the following definition which is a modification of the definition introduced in [9].

Definition. A function u defined on $\bar{S}_T$ is called a *weak solution of problem* (1)-(3) if

(i) u is nonnegative and bounded on $\bar{S}_T$;

(ii) $u \in C^{0+a}(\bar{S}_T)$ for some $a \in (0, 1]$;

(iii) u satisfies the identity

(4)
$$\int_{x_1}^{x_2} \int_{t_1}^{t_2} \{f_t u + f_{xx} u^m - [(x + \xi)^{-1} f]_x u^m\} \, dx\, dt - \int_{x_1}^{x_2} f u \Big|_{t_1}^{t_2} dx - \int_{t_1}^{t_2} f_x u^m \Big|_{x_1}^{x_2} dt = 0$$

for all $0 \leqslant x_1 < x_2$, $0 \leqslant t_1 < t_2 \leqslant T$, and for all $f \in C^{2,1}([x_1, x_2] \times [t_1, t_2])$ such that $f|_{x=x_1} = f|_{x=x_2} = 0$;

(iv) u satisfies conditions (2)-(3).

A large class of boundary value problems for degenerate parabolic equations for the one-dimensional nonstationary filtration was investigated in [11]. The problem (1)-(3) was studied in [4]-[6] for some classes of boundary conditions. In [4] the existence of a weak (in a suitably defined sense) solution of problem (1)-(2')-(3') was proved. In [6] some class of approximative solutions of problem (1)-(2')-(3') was examined.

In Section 2 we prove the existence of the weak solution of problem (1)-(3). We show that it is a classical solution of equation (1) in the neighbourhood of any point (x_0, t_0) where $u > 0$. Moreover, we show that the flux $(u^m)_x$ is a continuous function in S_T. In Section 3 we prove the uniqueness theorem. Some regularity properties of the solution of problem (1)-(3) are obtained in Section 4.

2. Existence of the weak solution. In this section we prove the following

THEOREM 1. *Let the assumptions* I-IV *of Section* 1 *be satisfied. Then there exists a weak solution of problem* (1)-(3).

Proof. In order to prove the existence of the weak solution we follow the construction method given in [11].

We use the construction given in [5]. It follows from the results of [5] that there exists a function u, defined on $\bar{S}_T$, which satisfies conditions (i), (iii), (iv) of the definition of a weak solution of problem (1)-(3) and such that for any $\delta \in (0,1)$ there is $u \in C^{0+\nu}([\delta, \infty) \times [0, T])$, where $\nu = \min\{1, (m-1)^{-1}\}$. It remains to show that $u \in C^{0+\alpha}(\bar{S}_T)$ for some $\alpha \in (0, 1]$. In order to prove this, it suffices to show that u is uniformly Hölder continuous in some neighbourhood of the segment $\{0\} \times [0, T]$.

In view of the construction given in [5] we have

$$u = \lim_k u_k,$$

where u_k for $k = 1, 2, \ldots$ are classical solutions of equation (1), defined on the rectangles $Q_k = [0, k] \times [0, T]$ for $k = 1, 2, \ldots$, respectively, and such that

$$u_k(x, t) \geqslant u_{k+1}(x, t) \quad \text{for } (x, t) \in Q_k \cap Q_{k+1}, \tag{5}$$

$$k^{-1} \leqslant u_k(x, t) \leqslant M \quad \text{for } (x, t) \in Q_k, \tag{6}$$

$$\left|\left(u_k^{m-1}(x, 0)\right)_x\right| \leqslant L_1 \quad \text{for } x \in [0, k], \tag{7}$$

$$\left|\left(u_k(0, t)\right)_t\right| \leqslant L_2 \quad \text{for } t \in [0, T], \tag{8}$$

(9) if $\delta \in (0, 1)$, then

$$|u_k^{m-1}(x, t) - u_k^{m-1}(x', t')| \leqslant K_1(|x-x'|^2 + |t-t'|)^{1/2}$$

for $(x, t), (x', t') \in [\delta, \infty) \times [0, T] \cap Q_k$,

where the constants K_1, L_1, L_2, M do not depend on k.

Note that by (9) we have

(10) if $\delta \in (0, 1)$, then

$$|u_k(x, t) - u_k(x', t')| \leqslant K_2(|x-x'|^2 + |t-t'|)^{\nu/2}$$

for $(x, t), (x', t') \in [\delta, \infty) \times [0, T] \cap Q_k$,

where $\nu = \min\{1, (m-1)^{-1}\}$ and the constant K_2 does not depend on k.

We shall prove that the functions u_k for $k = 1, 2, \ldots$ are bounded from below by a positive constant in some neighbourhood of the segment $\{0\} \times [0, T]$, uniformly with respect to k.

LEMMA 1. *Let the assumptions of Theorem 1 be satisfied and let u_k for $k = 1, 2, \ldots$ be the functions defined in the proof of Theorem 1. Then there exist positive constants μ, η independent of k and such that*

$$\mu \leqslant u_k(x, t) \tag{11}$$

for $(x, t) \in Q_\eta = [0, \eta] \times [0, T]$ and $k = 1, 2, \ldots$

Proof. Let $D = \{(x, t)\colon 0 < x < \varrho \log^{1/2}(t+\tau),\ 0 < t \leqslant T\}$, where $0 < \varrho < 1$, $\tau > 1$. For $(x, t) \in \bar{D}$ we consider the function

$$z(x, t) = \{A(t+\tau)^{-1}[\varrho^2 - x^2 \log^{-1}(t+\tau)]\}^{1/(m-1)},$$

where $A > 0$. (This function was also used in [3].) We have

$$Lz = z(m-1)(t+\tau)^{-1}[1 - 2Am\log^{-1}(t+\tau) - 2Amx(x+\xi)^{-1}\log^{-1}(t+\tau)] + \\ + z^{2-m}A(m-1)^{-1}x^2(t+\tau)^{-2}\log^{-2}(t+\tau)[4Am(m-1)^{-1} - 1]$$

in D for $0 < \varrho < 1$, $\tau > 1$ and $A > 0$. From the assumptions on u_0 and u_1 we have $u_1(t) \geqslant c > 0$ for $t \in [0, T]$ and $u_0(x) \geqslant c/2$ for $x \in [0, \eta_0]$, where $\eta_0 > 0$. For a fixed A we can choose a constant $\tau_0 > 1$ such that for $\tau \geqslant \tau_0$ and $0 < \varrho < 1$ we have $z(0, t) < u(0, t)$ for $t \in [0, T]$ and $z(x, 0) < u(x, 0)$ for $x \in [0, \eta_0]$. Set $A = \frac{1}{4}m^{-1}(m-1)$. Then for a sufficiently large $\tau_1 \geqslant \tau_0$ we have $Lz > 0$ in D.

Choose a number $\varrho_0 \in (0, 1)$ such that $\varrho_0 \log^{1/2}\tau_1 \leqslant \eta_0$. We have $u \leqslant u_k$ for $k = 1, 2, \ldots$ and, therefore, $z \leqslant z_k$ on the parabolic boundary of $D \cap Q_k$ for $k \geqslant \eta_0$. Since $Lu_k = 0$ in $D \cap Q_k$ for $k = 1, 2, \ldots$, from the maximum principle (see [2]) we obtain $z \leqslant u_k$ in $\bar{D} \cap Q_k$ for $k \geqslant \eta_0$. Hence we can choose positive constants μ and η, $\eta \leqslant \min\{1, \eta_0\}$, independent of k and such that $\mu \leqslant u_k(x, t)$ for $(x, t) \in Q_\eta = [0, \eta] \times [0, T]$ and $k = 1, 2, \ldots$, which completes the proof of the lemma.

Let $g = g(s)$ be a $C^\infty([0, \infty))$-function such that $g(s) = ms^{m-1}$ for $s \in [\mu, M]$ and $g(s) \in [m\mu^{m-1}/2, 2mM^{m-1}]$ for $s \in [0, \infty)$, where the constant μ is the same as in (11). Then for $k = 1, 2, \ldots$ the functions

u_k are classical solutions of the equation

(12) $$u_t = \big(g(u)u_x\big)_x + (x+\xi)^{-1}g(u)u_x$$

in Q_η. Note that equation (12) is uniformly parabolic. Moreover, from (7), (8), and (10) we have

$$|u_k(0,t)-u_k(0,t')| \leqslant C|t-t'|^{\nu/2} \quad \text{for } t, t' \in [0,T],$$

$$|u_k(x,0)-u_k(x',0)| \leqslant C|x-x'|^{\nu} \quad \text{for } x, x' \in [0,\eta],$$

$$|u_k(\eta,t)-u_k(\eta,t')| \leqslant C|t-t'|^{\nu/2} \quad \text{for } t, t' \in [0,T]$$

for $k = 1, 2, \ldots$, where the positive constant C does not depend on k. Hence, by Theorem 1 of [10], p. 476, there exist positive constants $\bar{C}$ and $a \in (0,1)$ independent of k and such that

$$|u_k(x,t)-u_k(x',t')| \leqslant \bar{C}(|x-x'|^2+|t-t'|)^{a/2}$$

for (x,t), $(x',t') \in Q_\eta$ and $k = 1, 2, \ldots$ Therefore, the limit function u belongs to $C^{0+a}(Q_\eta)$. Since $u \in C^{0+\nu}\big([\eta,\infty)\times[0,T]\big)$, we have $u \in C^{0+a'}(\bar{S}_T)$, where $a' = \min\{a, \nu\}$.

THEOREM 2. *Let the assumptions* I-IV *of Section* 1 *be satisfied and let u be the weak solution of problem* (1)-(3) *constructed in Theorem* 1. *Then*:

(i) *u is a classical solution of equation* (1) *in a neighbourhood of any point $(x_0, t_0) \in S_T$ where $u(x_0,t_0) > 0$.*

(ii) *The derivative $(u^m)_x$ exists and is continuous in S_T and, in particular, if $u(x,t) = 0$, then $(u^m)_x(x,t) = 0$. Moreover, if $m < 2$, then u_x exists and is continuous in S_T and, in particular, if $u(x,t) = 0$, then $u_x(x,t) = 0$.*

Proof. Let $u = \lim\limits_k u_k$ be the weak solution of problem (1)-(3) constructed in Theorem 1. Suppose that $u(x_0,t_0) > 0$ for some $(x_0,t_0) \in S_T$. It follows from (5) and from the continuity of u that there exists a number $\bar{\delta} > 0$ such that for $(x,t) \in R_{\bar{\delta}} = [x_0-\bar{\delta}, x_0+\bar{\delta}]\times[t_0-\bar{\delta}, t_0+(T-t_0)\bar{\delta}]$ and for $k \geqslant x_0+\bar{\delta}$ we have

$$u_k(x,t) \geqslant \frac{1}{2}u(x_0,t_0).$$

Let $v = u^m$ and $v_k = u_k^m$ for $k \geqslant x_0+\bar{\delta}$. Then

$$v = \lim_k v_k$$

and the functions v_k for $k \geqslant x_0+\bar{\delta}$ satisfy the equation

$$(v_k)_t = A_k(x,t)(v_k)_{xx} + B_k(x,t)(v_k)_x$$

in $R_{\bar{\delta}}$, where $A_k(x,t) = mu_k^{m-1}(x,t)$ and $B_k(x,t) = m(x+\xi)^{-1}u_k^{m-1}(x,t)$

for $k \geqslant x_0 + \delta$. We have

$$v_k(x, t) \geqslant \frac{1}{2^m} u^m(x_0, t_0),$$

$$|A_k(x, t) - A_k(x', t')| \leqslant K(|x-x'|^2 + |t-t'|)^{1/2},$$

$$|B_k(x, t) - B_k(x', t')| \leqslant K(|x-x'|^2 + |t-t'|)^{1/2}$$

for (x, t), $(x', t') \in R_{\delta}$ and $k \geqslant x_0 + \delta$, where the positive constant K does not depend on k. It follows from the interior a priori estimates for linear parabolic equations given in [2] (Theorem 5, p. 64) that there exist numbers $\bar{K} > 0$ and $\bar{a} \in (0, 1)$ independent of k and such that

$$|v_k(x, t) - v_k(x', t')| \leqslant \bar{K}(|x-x'|^2 + |t-t'|)^{\bar{a}/2},$$

$$|(v_k)_t(x, t) - (v_k)_t(x', t')| \leqslant \bar{K}(|x-x'|^2 + |t-t'|)^{\bar{a}/2},$$

$$|(v_k)_x(x, t) - (v_k)_x(x', t')| \leqslant \bar{K}(|x-x'|^2 + |t-t'|)^{\bar{a}/2},$$

$$|(v_k)_{xx}(x, t) - (v_k)_{xx}(x', t')| \leqslant \bar{K}(|x-x'|^2 + |t-t'|)^{\bar{a}/2}$$

for (x, t), $(x', t') \in R_{\delta/2}$. Hence the limit function v has continuous derivatives v_t, v_x, v_{xx} in $R_{\delta/2}$, and therefore there exist continuous derivatives u_t, u_x, u_{xx} in $R_{\delta/2}$. Since the functions u_k for $k \geqslant x_0 + \delta$ satisfy equation (1) in R_{δ}, u is a classical solution of (1) in $R_{\delta/2}$.

In order to prove (ii) note that, by (i), if $u(x, t) > 0$ and $(x, t) \in S_T$, then u_x exists and is continuous in a neighbourhood of (x, t), and the same is true for $(u^m)_x$. Using (10) we can show that if $(x_0, t_0) \in S_T$ and $u(x_0, t_0) = 0$, then $(u^m)_x(x_0, t_0) = 0$ and $(u^m)_x$ is continuous at (x_0, t_0). Moreover, if $m < 2$, then $u_x(x_0, t_0) = 0$ and u_x is continuous at (x_0, t_0). The proof is exactly the same as in [1], p. 466, and we omit the details.

3. Uniqueness. We shall prove the following result:

THEOREM 3. *Let the assumptions* I-IV *of Section* 1 *be satisfied. Then the weak solution of problem* (1)-(3) *is unique.*

Proof. Let u be the weak solution of problem (1)-(3) given in Theorem 1. Recall that

$$u = \lim_k u_k,$$

where $u_k = u_k(x, t)$ for $k = 1, 2, \ldots$ are strictly positive functions which satisfy (1) in the rectangles $Q_k = [0, k] \times [0, T]$ for $k = 1, 2, \ldots$, respectively. Consequently, the functions $u_k = u_k(x, t)$ for $k = 1, 2, \ldots$ satisfy (4) if $0 \leqslant x_1 < x_2 \leqslant k$ and $0 \leqslant t_1 < t_2 \leqslant T$. Moreover, by the construction given in Theorem 1, $u_k \searrow u$.

Let u^* be an arbitrary weak solution of problem (1)-(3) and let $g \in C_0^\infty(S_T)$. We show that

$$\int_0^\infty \int_0^T (u-u^*) g\,dx\,dt = 0.$$

In (4) we set $x_1 = 0$ and $x_2 = r$, where $r > 0$, $t_1 = 0$, $t_2 = T$. Let $k \geqslant r$. Then for each $f \in C^{2,1}([0, r] \times [0, T])$ such that $f|_{x=0} = f|_{x=r} = 0$ we have

$$(13)\qquad \int_0^r \int_0^T (u_k - u^*)\{f_t + a_k(x,t)[f_{xx} - ((x+\xi)^{-1}f)_x]\}\,dx\,dt - \\ - \int_0^r f(u_k - u^*)\Big|_{t=0}^{t=T} dx - \int_0^T f_x(u_k^m - u^{*m})\Big|_{x=0}^{x=r} dt = 0,$$

where

$$(14)\qquad a_k(x,t) = \begin{cases} \dfrac{u_k^m(x,t) - u^{*m}(x,t)}{u_k(x,t) - u^*(x,t)} & \text{if } u_k(x,t) \neq u^*(x,t), \\ m u_k^{m-1}(x,t) & \text{if } u_k(x,t) = u^*(x,t) \end{cases}$$

for $(x, t) \in Q_k$.

Note that $a_k \in C^{0+\gamma}(Q_k)$ for some $\gamma \in (0, 1)$ and, by (6),

$$(15)\qquad 0 < c_k = \min_{(x,t)\in Q_k} m u_k^{m-1}(x,t) \leqslant a_k(x,t) \leqslant \bar{M}$$

for $(x, t) \in Q_k$, where the constants γ and $\bar{M}$ do not depend on k. Moreover, since $u_1(t) \geqslant c > 0$ and $u_k(x, t) \geqslant u(x, t)$ for $(x, t) \in Q_k$ and $k = 1, 2, \ldots$, there exist positive constants $\bar{\mu}$ and $\bar{\eta}$ independent of k and such that

$$(16)\qquad \bar{\mu} \leqslant a_k(x,t)$$

for $(x, t) \in Q_{\bar{\eta}} = [0, \bar{\eta}] \times [0, T]$.

Choose a number r_0 such that $g = 0$ for $x \geqslant r_0 - 1$ and consider, for given $r \geqslant r_0$ and $k \geqslant r$, the following problem:

$$(17)\qquad Mf \equiv f_t + a_k(x,t)[f_{xx} - ((x+\xi)^{-1}f)_x] = g \quad \text{in } (0, r) \times [0, T),$$

$$(18)\qquad f|_{x=0} = f|_{x=r} = f|_{t=T} = 0.$$

It follows from Theorem 7 of [2], p. 65, that for each $r \geqslant r_0$ and $k \geqslant r$ there exists a unique solution $f^{k,r}$ of problem (17)-(18) such that $f^{k,r} \in C^{2+\beta}([0, r] \times [0, T])$ for some $\beta \in (0, 1)$. We need some estimates of $f^{k,r}$.

Lemma 2. *Let $r \geqslant r_0$ and $k \geqslant r$ and let $f^{k,r}$ be the solution of problem (17)-(18). Then there exist positive constants C_1, C_2, C_3 independent of k and r and such that*

(19) $|f^{k,r}(x,t)| \leqslant C_1 e^{-x}$ *for* $0 \leqslant x \leqslant r,\ 0 \leqslant t \leqslant T$,

(20) $|f_x^{k,r}(0,t)| \leqslant C_2$ *for* $0 \leqslant t \leqslant T$,

(21) $|f_x^{k,r}(r,t)| \leqslant 2C_3(r+\xi)e^{-r-1} + C_1(r+\xi)^{-1}e^{-r}$ *for* $0 \leqslant t \leqslant T$.

Proof. Let $Q_r = [0,r]\times[0,T]$ and let

$$\Gamma_r = \{0\}\times[0,T]\cup[0,r]\times\{T\}\cup\{r\}\times[0,T].$$

Set $x_g = \sup\{x\colon g(x,t) \neq 0,\ 0 \leqslant t \leqslant T\}$. Put $f^{k,r} = [\exp(-\bar{M}\sigma t)]\bar{f}^{k,r}$, where the constant $\bar{M}$ occurs in (15) and $\sigma = \max\{1, \xi^{-2}\}$. Then we obtain

$$(22)\qquad \bar{M}\bar{f}^{k,r} \equiv \bar{f}_t^{k,r} + a_k(x,t)\bar{f}_{xx}^{k,r} - a_k(x,t)(x+\xi)^{-1}\bar{f}_x^{k,r} + \\ + [a_k(x,t)(x+\xi)^{-2} - \bar{M}\sigma]\bar{f}^{k,r} = g\exp(\bar{M}\sigma t)$$

in Q_r and $\bar{f}^{k,r}|_{x=0} = \bar{f}^{k,r}|_{x=r} = \bar{f}^{k,r}|_{t=T} = 0$. For $(x,t)\in Q_r$ we consider the functions

$$z_1(x,t) = f^{k,r} - M_1\exp\big(4\bar{M}\sigma(T-t)+(x_g-x)\big),$$
$$z_2(x,t) = f^{k,r} + M_1\exp\big(4\bar{M}\sigma(T-t)+(x_g-x)\big),$$

where

$$M_1 = 2\sup_{(x,t)\in\bar{S}_T} |g(x,t)\exp(\bar{M}\sigma t)|.$$

We have $z_1|_{\Gamma_r} < 0$ and

$$\bar{M}z_1 = g\exp(\bar{M}\sigma t) - M_1\big[\exp\big(4\bar{M}\sigma(T-t)+(x_g-x)\big)\big]\{-4\bar{M}\sigma + \\ + a_k(x,t)[1+(x+\xi)^{-1}+(x+\xi)^{-2}]\} > 0$$

in Q_r, $z_2|_{\Gamma_r} > 0$ and

$$\bar{M}z_2 = g\exp(\bar{M}\sigma t) + M_1\big[\exp\big(4\bar{M}\sigma(T-t)+(x_g-x)\big)\big]\{-4\bar{M}\sigma + \\ + a_k(x,t)[1+(x+\xi)^{-1}+(x+\xi)^{-2}]\} < 0$$

in Q_r. Hence, by the maximum principle (see [2]), $z_1 \leqslant 0$ and $z_2 \geqslant 0$ in Q_r. Therefore

$$|\bar{f}^{k,r}(x,t)| \leqslant M_1\exp\big(4\bar{M}\sigma(T-t)+(x_g-x)\big) \quad \text{for } (x,t)\in Q_r.$$

Hence $|f^{k,r}(x,t)| \leqslant C_1 e^{-x}$ for $(x,t)\in Q_r$, where $C_1 = M_1\exp(4\bar{M}\sigma T + x_g)$.

In order to prove (20) we use the method given in [7].

Let $f^{k,r} = [\exp(-\bar{M}\sigma t)]\bar{f}^{k,r}$ and let $Q_{\bar{\eta}} = [0,\bar{\eta}]\times[0,T]$ (see (16)). Then $\bar{f}^{k,r}$ satisfies (22) in $Q_{\bar{\eta}}$ and $\bar{f}^{k,r}|_{x=0} = \bar{f}^{k,r}|_{t=T} = 0$. Moreover, by (19), $|\bar{f}^{k,r}(\bar{\eta},t)| \leqslant C_1\exp(\bar{M}\sigma T)\exp(-\bar{\eta})$ for $t\in[0,T]$. For $(x,t)\in Q_{\bar{\eta}}$ we consider the function

$$w_1(x, t) = \bar{f}^{k,r} + N_1 e^{-Nx},$$

where the positive constants N and N_1 are as specified below. We have

$$\bar{M}w_1 = g\exp(\bar{M}\sigma t) + N_1 e^{-Nx}\{a_k(x, t)N^2 + a_k(x, t)(x+\xi)^{-1}N + \\ + [a_k(x, t)(x+\xi)^{-2} - \bar{M}\sigma]\}$$

in $Q_{\bar{\eta}}^-$. It follows from (15) and (16) that we can choose N sufficiently large, independent of k, and such that $\exp(-N\bar{\eta}) < 1/2$ and

$$a_k(x, t)N^2 + a_k(x, t)(x+\xi)^{-1}N + a_k(x, t)(x+\xi)^{-2} - \bar{M}\sigma > \delta_1$$

in $Q_{\bar{\eta}}^-$ for some positive constant δ_1 which does not depend on k. Choose $N_1 \geqslant \max\{M_1\delta_1^{-1}\exp(N\bar{\eta}), 2C_1\exp(\bar{M}\sigma T)\}$, where the positive constant C_1 appears in (19). Then $\bar{M}w_1 > 0$ in $Q_{\bar{\eta}}^-$ and, by the maximum principle, w_1 cannot have a positive maximum in $(0, \bar{\eta}) \times [0, T)$. The function w_1 attains the positive maximum on $Q_{\bar{\eta}}^- \setminus (0, \bar{\eta}) \times [0, T)$ if $x = 0$. Therefore $(w_1)_x(0, t) \leqslant 0$ for $t \in [0, T]$. Hence $\bar{f}_x^{k,r}(0, t) \leqslant NN_1$ for $t \in [0, T]$ and the same is true for $f_x^{k,r}(0, t)$.

Analogously, if we consider the function $w_2(x, t) = \bar{f}^{k,r} - N_1 e^{-Nx}$ for $(x, t) \in Q_{\bar{\eta}}^-$, then we obtain the inequality $-NN_1 \leqslant f_x^{k,r}(0, t)$ for $t \in [0, T]$. Therefore $|f_x^{k,r}(0, t)| \leqslant C_2$ for $t \in [0, T]$, where $C_2 = NN_1$.

Now we prove (21). Let $D_r = [r-1, r] \times [0, T]$ and let

$$\Gamma_r = \{r-1\} \times [0, T] \cup [r-1, r] \times \{T\} \cup \{r\} \times [0, T].$$

Set $f^{k,r} = (x+\xi)h^{k,r}$. The function $h^{k,r}$ satisfies the equation

$$M_1 h^{k,r} \equiv h_t^{k,r} + a_k(x, t)[h_{xx}^{k,r} + (x+\xi)^{-1}h_x^{k,r}] = 0$$

in D_r and $h^{k,r}|_{x=r} = h^{k,r}|_{t=T} = 0$. Moreover, from (19) we get $|h^{k,r}(r-1, t)| \leqslant C_3 e^{-r-1}$ for $t \in [0, T]$, where $C_3 = C_1\max\{1, \xi^{-1}\}$. For $(x, t) \in D_r$ we consider the function

$$y_1(x, t) = h^{k,r} + 2C_3(x-r)e^{-r-1}.$$

Since $y_1|_{\Gamma_r} \leqslant 0$ and $M_1 y_1 > 0$, we have $y_1 \leqslant 0$ in D_r. Thus y_1 attains its maximum at $x = r$. Hence $-2C_3 e^{-r-1} \leqslant h_x^{k,r}(r, t)$ for $t \in [0, T]$. Since $f_x^{k,r} = (x+\xi)h_x^{k,r} + (x+\xi)^{-1}f^{k,r}$, we have

$$f_x^{k,r}(r, t) \geqslant -2C_3(r+\xi)e^{-r-1} - C_1(r+\xi)^{-1}e^{-r} \quad \text{for } t \in [0, T].$$

Analogously, if we consider the function

$$y_2(x, t) = h^{k,r} - 2C_3(x-r)e^{-r-1} \quad \text{for } (x, t) \in D_r,$$

then we obtain the inequality

$$f_x^{k,r}(r, t) \leqslant 2C_3(r+\xi)e^{-r-1} + C_1(r+\xi)^{-1}e^{-r} \quad \text{for } t \in [0, T].$$

Thus $|f_x^{k,r}(r,t)| \leqslant 2C_3(r+\xi)e^{-r-1} + C_1(r+\xi)^{-1}e^{-r}$ for $t \in [0, T]$, which completes the proof of the lemma.

If we put in (13) the functions $f^{k,r}$ instead of f, then we obtain

$$\int_0^r \int_0^T (u_k - u^*) g\,dx\,dt + \int_0^r f^{k,r}(u_k - u^*)\Big|_{t=0} dx - \int_0^T f_x^{k,r}(u_k^m - u^{*m})\Big|_{x=0}^{x=r} dt = 0$$

for $k \geqslant r \geqslant r_0$. Hence, using (19)-(21), we have

$$\Big|\int_0^r \int_0^T (u_k - u^*) g\,dx\,dt\Big| \leqslant C_1 \int_0^r e^{-x}|u_k(x,0) - u_0(x)|\,dx + \tag{23}$$
$$+[2C_3(r+\xi)e^{-r-1} + C_1(r+\xi)^{-1}e^{-r}] \int_0^T |u_k^m(r,t) - u^{*m}(r,t)|\,dt +$$
$$+ C_2 \int_0^T |u_k^m(0,t) - u_1^m(t)|\,dt$$

for $k \geqslant r \geqslant r_0$. Let $k \to \infty$ in (23). Then

$$\Big|\int_0^r \int_0^T (u - u^*) g\,dx\,dt\Big|$$
$$\leqslant [2C_3(r+\xi)e^{-r-1} + C_1(r+\xi)^{-1}e^{-r}] \int_0^T |u^m(r,t) - u^{*m}(r,t)|\,dt$$

for $r \geqslant r_0$. Hence, if $r \to \infty$, then

$$\int_0^\infty \int_0^T (u - u^*) g\,dx\,dt = 0,$$

which holds for each $g \in C_0^\infty(S_T)$. Therefore, in view of the continuity of u and u^*, we obtain $u \equiv u^*$.

4. Regularity properties of solutions. For the weak solutions of problem (1)-(3) the following regularity theorem holds.

THEOREM 2′. *Let the assumptions* I-IV *of Section* 1 *be satisfied. If u is the weak solution of problem* (1)-(3), *then* (i) *and* (ii) *of Theorem* 2 *hold.*

Theorem 2′ follows immediately from Theorem 2 and from the uniqueness Theorem 3 given in Section 3.

Remark. Physically, the continuity of $(u^m)_x$ means the continuity of the flux of fluid.

THEOREM 4. *Let $\bar{u}_0 = \bar{u}_0(x)$, $u_0 = u_0(x)$ for $x \in [0, \infty)$ and $\bar{u}_1 = \bar{u}_1(t)$, $u_1 = u_1(t)$ for $t \in [0, T]$ satisfy the assumptions* II-IV *of Section* 1 *and, moreover, let*

$$\bar{u}_0(x) \leqslant u_0(x) \text{ for } x \in [0, \infty), \quad \bar{u}_1(t) \leqslant u_1(t) \text{ for } t \in [0, T].$$

Let $\bar{u}$ and u be the weak solutions of problem (1)-(3), *which satisfy the boundary conditions $\bar{u}_0, \bar{u}_1$ and u_0, u_1, respectively.*

Then $\bar{u}(x,t) \leqslant u(x,t)$ for $(x,t) \in \bar{S}_T$.

Proof. Suppose that $\bar{u}(x_0,t_0) > u(x_0,t_0)$ at some point $(x_0,t_0) \in S_T$. It follows from the continuity of $\bar{u}$ and u that there exists a positive number δ_0 such that $\bar{u}(x,t) > u(x,t)$ for $(x,t) \in R_{\delta_0} = [x_0-\delta_0, x_0+\delta_0] \times \times [t_0-\delta_0, t_0+(T-t_0)\delta_0]$. By Theorems 1 and 3, $u = \lim\limits_k u_k$, where $u_k = u_k(x,t)$, for $k = 1, 2, \ldots$ are strictly positive solutions of equation (1) in the rectangles $Q_k = [0,k]\times[0,T]$ for $k = 1, 2, \ldots$, respectively. Moreover, by (5) we have $u_k \searrow u$.

Let $g \in C_0^\infty(S_T)$ be a nonnegative function such that $g > 0$ in $R_{\delta_0/2}$ and $g = 0$ in $\bar{S}_T \setminus R_{\delta_0}$. Analogously as in the proof of Theorem 3, for $r \geqslant r_0$ and $k \geqslant r$ we obtain

$$\int\limits_0^r \int\limits_0^T (u_k-\bar{u})g\,dx\,dt + \int\limits_0^r f^{k,r}(u_k-\bar{u})\Big|_{t=0}dx + \int\limits_0^T f_x^{k,r}(u_k^m-\bar{u}^m)\Big|_{x=0}dt = \int\limits_0^T f_x^{k,r}(u_k^m-\bar{u}^m)\Big|_{x=r}dt, \tag{24}$$

where $f^{k,r}$ for $r \geqslant r_0$ and $k \geqslant r$ are solutions of problem (17)-(18) in which $a_k(x,t)$ is defined by (14), where we put $\bar{u}$ instead of u^*. Note that, by the maximum principle, $f^{k,r} \leqslant 0$ for $k \geqslant r \geqslant r_0$. Moreover, since $f^{k,r}(0,t) = 0$ for $t \in [0,T]$ and $k \geqslant r \geqslant r_0$, we have $f_x^{k,r}(0,t) \leqslant 0$ for $t \in [0,T]$ and $k \geqslant r \geqslant r_0$. Hence it follows from (19)-(21) and (24) that

$$\Big|\int\limits_0^r \int\limits_0^T (u_k-\bar{u})g\,dx\,dt\Big| \leqslant [2C_3(r+\xi)e^{-r-1}+C_1(r+\xi)^{-1}e^{-r}]\int\limits_0^T |u_k^m(r,t)-\bar{u}^m(r,t)|\,dt.$$

Let $k\to\infty$. Then

$$\Big|\int\limits_0^r \int\limits_0^T (u-\bar{u})g\,dx\,dt\Big| \leqslant [2C_3(r+\xi)e^{-r-1}+C_1(r+\xi)^{-1}e^{-r}]\int\limits_0^T |u^m(r,t)-\bar{u}^m(r,t)|\,dt. \tag{25}$$

If $r\to\infty$, then from (25) we obtain

$$\int\limits_0^\infty \int\limits_0^T (u-\bar{u})g\,dx\,dt = 0.$$

On the other hand, in view of the choice of g, we have

$$\int_0^{\infty}\int_0^{T}(u-\bar{u})g\,dx\,dt < 0.$$

This contradiction completes the proof of the theorem.

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Received on 6. 6. 1979

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O ZAGADNIENIU BRZEGOWYM POJAWIAJĄCYM SIĘ W RADIALNIE SYMETRYCZNEJ FILTRACJI CIECZY

STRESZCZENIE

W pracy badane jest zagadnienie mieszane dla nieliniowego równania różniczkowego cząstkowego Boussinesq'a. Zagadnienie to pojawia się przy opisie radialnie symetrycznego procesu filtracji cieczy w ośrodku porowatym, otaczającym zbiornik o kształcie walca. Przy założeniu pewnej regularności danych brzegowych udowodniono istnienie i jednoznaczność uogólnionego rozwiązania badanego zagadnienia. Udowodniono także pewne własności typu regularności tego rozwiązania. Otrzymane wyniki mogą mieć zastosowanie w przybliżonym opisie procesu nawilżania suchego gruntu przez ciecz wypełniającą zbiornik oraz procesu sczerpywania cieczy z nawilżonego gruntu.