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G. SANKARANARAYANAN, K. USHA and R. RAMANARAYANAN
(Annamalainagar, India)

THE SINGLE SERVER QUEUE WITH ERLANG INPUT AND SEMI-MARKOV SERVICE TIMES

0. Assuming that the successive service times in a single server queue form an m-state semi-Markov process, new results are obtained when the input process is Erlang. Successive busy periods and queue length are investigated.

1. Introduction. We consider a single server queue in which customers arrive at instants $\tau_0, \tau_1, \ldots$, where $\{\tau_n - \tau_{n-1}\}$, $\tau_0 = 0$, $n = 1, 2, \ldots$, are independent and identically distributed Erlang random variables with shape and scale parameters k and λ, respectively. Assume that the customers are of one of m types and are served in order of arrival. Let the successive customer types form an m-state irreducible Markov chain. The successive service times are assumed to be conditionally independent given this chain and depend on the transition occurring in the chain. Such service times are called *semi-Markov service times* and were considered in [2]-[5]. Let J_n be the type of the $(n+1)$-st customer to enter the service and let X_n denote the service time of the n-th customer. Here we assume the double sequence $\{(J_n, X_n), n = 0, 1, \ldots\}$ has the following properties:

$$P[J_{n+1} = j, X_{n+1} \leqslant x \mid J_0, \ldots, J_n, X_1, \ldots, X_n] = P[J_{n+1} = j, X_{n+1} \leqslant x \mid J_n] = Q_{J_n, j}(x),$$

$$X_0 = 0 \text{ a.s.}, \quad P[J_0 = k] = p_k^0, \ k = 1, \ldots, m < \infty.$$

$Q_{ij}(x)$, $i, j = 1, \ldots, m$, are non-decreasing, right continuous, and satisfy

$$Q_{ij}(x) = 0 \text{ for } x \leqslant 0, \quad Q_{ij}(\infty) = P_{ij} \quad \text{and} \quad \sum_{j=1}^{m} P_{ij} = 1.$$

Denote the matrix $\{Q_{ij}(x)\}$ by $Q(x)$. Let J_t be the type of the last customer to join the queue before time t and $J_t = J_0$ during the arrival time of the first customer. Let $\psi(s)$ be the $(m\times m)$-matrix with entries

$$\psi_{i,j}(s) = \int_0^\infty e^{-sx}dQ_{ij}(x), \qquad \mathrm{Re}(s)\geqslant 0,$$

and let $\eta_\varrho(s)$, $\varrho = 1,\dots,m$, denote the m eigenvalues of $\psi(s)$ at the point s.

In what follows we obtain the busy period and the queue length for the above queueing system. Results for the Poisson arrival process are given in [4] and [5]. We prove the following lemma which is a generalization of [4] in the sense of [4] and [7].

2. Lemma. *The equation*

$$\det[z^kI - w\psi(s+\lambda-\lambda z)] = 0 \tag{1}$$

has exactly mk roots in the unit circle $|z|<1$ if either $\mathrm{Re}(s)\geqslant 0$, $|w|<1$ or $\mathrm{Re}(s)>0$, $|w|\leqslant 1$. The m eigenvalues $\eta_\varrho(s+\lambda-\lambda z)$ of the matrix $\psi(s+\lambda-\lambda z)$ can be defined as analytic functions of $s+\lambda-\lambda z$ if in the entire region $\mathrm{Re}(s)>0$ and $|z|\leqslant 1$ they are distinct or if some collection of eigenvalues is identical for all such values of s and z, while the remaining eigenvalues are distinct. In this case the equation

$$z^k - w\eta_\varrho(s+\lambda-\lambda z) = 0$$

has k roots $\bar{\nu}_{\varrho r}(s,w)$, $1\leqslant r\leqslant k$, in the open disc $|z|<1$ for either $\mathrm{Re}(s)\geqslant 0$, $|w|<1$ or $\mathrm{Re}(s)>0$, $|w|\leqslant 1$. The roots are

$$\bar{\nu}_{\varrho r}(s,w) = \sum_{j=1}^{\infty}\frac{(-\lambda)^{j-1}}{j!}(\varepsilon_r w^{1/k})^j\left(\frac{d^{j-1}}{ds^{j-1}}[\eta_\varrho(\lambda+s)]^{j/k}\right), \tag{2}$$

where $\varepsilon_r = e^{2\pi i r/k}$, $1\leqslant r\leqslant k$, are k roots of unity. If $\bar{\nu}_{\varrho r}(s,w)$ is defined by (2) for $\mathrm{Re}(s)\geqslant 0$ and $|w|\leqslant 1$, then in this domain $\bar{\nu}_{\varrho r}(s,w)$ is a continuous function of s and w. Moreover, $|\bar{\nu}_{\varrho r}(s,w)|\leqslant 1$ and $z=\bar{\nu}_{\varrho r}(s,w)$ satisfies the equation $z=\varepsilon_r[w\eta_\varrho(s+\lambda-\lambda z)]^{1/k}$. The roots $\bar{\nu}_{\varrho r}(s,w)$, $1\leqslant r\leqslant k$, are distinct if $w\neq 0$.

Proof. Let $\eta_\varrho(s+\lambda-\lambda z)$ be distinct and defined analytically in the region $\mathrm{Re}(s)>0$, $|z|\leqslant 1$. Since the spectral radius of $\psi(s+\lambda-\lambda z)$ is less than 1 in $\mathrm{Re}(s)>0$, $|z|\leqslant 1$, we have $\eta_\varrho(s+\lambda-\lambda z)<1$, $1\leqslant\varrho\leqslant m$, in this region. Equation (1) may be written as

$$\prod_{\varrho=1}^{m}[z^k - w\eta_\varrho(s+\lambda-\lambda z)] = 0. \tag{3}$$

By Rouche's theorem we know that each of the factors has k roots in $|z|<1$. In the region $\mathrm{Re}(s)\geqslant 0$, $|z|\leqslant 1$, we have $\eta_\varrho(s+\lambda-\lambda z)\leqslant 1$, $1\leqslant\varrho\leqslant m$. Since $|w|<1$, Rouche's theorem again gives the result. The analyticity of the roots $\bar{\nu}_{\varrho r}(s, w)$, $1 \leqslant \varrho \leqslant m$, $1 \leqslant r \leqslant k$, follows from the analyticity of $\eta_\varrho(s+\lambda-\lambda z)$. The Lagrange expansion gives (2). If the matrix $\psi(s+\lambda-\lambda z)$ has multiple eigenvalues at all points of the region $\mathrm{Re}(s) \geqslant 0$, $|z| \leqslant 1$, while the remaining eigenvalues are distinct, then also the above arguments remain valid.

NON-SINGULARITY ASSUMPTION. We assume $\psi(s+\lambda-\lambda z)$ has m distinct eigenvalues $\eta_\varrho(s+\lambda-\lambda z)$, which are not identically zero, in the entire region where $\mathrm{Re}(s) > 0$ and $|z| \leqslant 1$, $1 \leqslant \varrho \leqslant m$.

3. Successive busy periods. Let Y_i, $i \geqslant 1$, denote the lengths of the successive busy periods. Let $I_0 = J_0$ and let I_n be the type of the first customer to be served during the $(n+1)$-st busy period $(n = 1, 2, \ldots)$. Let $Y_0 = 0$ a.s. We apply the imbedded semi-Markov process method used in [1], [4], and [6].

Let $G_{ij}(n, r, \eta; x)$ denote the probability that a busy period consists of at least n services, that the total service time of the first n customers is at most x, that at the end of the n-th service r customers are waiting, and that the next arriving customer at the end of the n-th service is in the phase η related to the Erlang shape parameter and he is of type j under the condition that the first customer of the busy period is of type i. We put

$$\Gamma_{ij}(n, r, \eta; s) = \int_0^\infty e^{-st} dG_{ij}(n, r, \eta; x), \qquad n \geqslant 1,\ \mathrm{Re}(s) \geqslant 0,$$

$$C_{ij}(n, z; s) = \sum_{r=0}^{\infty} \sum_{\eta=1}^{k} \Gamma_{ij}(n, r, \eta; s) z^{rk+\eta}, \qquad |z| \leqslant 1,$$

$$D_{ij}(w, z; s) = \sum_{n=1}^{\infty} C_{ij}(n, z; s) w^n, \qquad |w| \leqslant 1,$$

$$\Gamma_{ij}(n, 0, \eta; s) = \Gamma_{ij}(n, \eta; s),$$

$$E_{ij}(w, y; s) = \sum_{n=1}^{\infty} \sum_{\eta=1}^{k} \Gamma_{ij}(n, \eta; s) w^n y^\eta, \qquad |w| \leqslant 1,\ |y| \leqslant 1.$$

Then $E_{ij}(1, 1; s)$ is the Laplace-Stieltjes transform of the probability $G_{ij}(x)$ that a busy period is at most x and that the next arriving customer is of type j under the condition that the initial customer is of type i.

Let $\bar{a}_\varrho(s+\lambda-\lambda z) = [a_{1\varrho}, \ldots, a_{m\varrho}]$ denote the right eigenvectors of the matrix $\psi(s+\lambda-\lambda z)$ corresponding to the m eigenvalues $\eta_\varrho(s+\lambda-\lambda z)$, $1 \leqslant \varrho \leqslant m$. Let the matrix $T(w, s)$ with columns $\bar{a}_\varrho[s+\lambda-\lambda\bar{\nu}_{\varrho r}(s, w)]$, $1 \leqslant \varrho \leqslant m$, be non-singular for all s and w in the region $\mathrm{Re}(s) > 0$, $|w| \leqslant 1$.

We have the following recurrence relations for G_{ij}:

$$G_{ij}(1, r, \eta; x) = \int_0^x e^{-\lambda y} \frac{(\lambda y)^{rk+\eta-1}}{(rk+\eta-1)!} dQ_{ij}(y).$$

For $n > 1$ we obtain

$$G_{ij}(n, r, \eta; x) = \sum_{\nu=1}^{m} \sum_{\xi=1}^{k} \sum_{q=1}^{r} \int_0^x G_{i\nu}(n-1, q, \xi; x-u) \times$$
$$\times e^{-\lambda u} \frac{(\lambda u)^{[(r-q)k+\eta-\xi+k]}}{[(r-q)k+\eta-\xi+k]!} dQ_{\nu j}(u) +$$
$$+ \sum_{\nu=1}^{m} \sum_{\xi=1}^{\eta} \int_0^x G_{i\nu}(n-1, r+1, \xi; x-u) e^{-\lambda u} \frac{(\lambda u)^{\eta-\xi}}{(\eta-\xi)!} dQ_{\nu j}(u).$$

Taking the Laplace-Stieltjes transforms we get

$$\Gamma_{ij}(1, r, \eta; s) = \int_0^\infty e^{-(\lambda+s)y} \frac{(\lambda y)^{rk+\eta-1}}{(rk+\eta-1)!} dQ_{ij}(y), \tag{4}$$

$$\Gamma_{ij}(n, r, \eta; s) = \sum_{\nu=1}^{m} \sum_{\xi=1}^{k} \sum_{q=1}^{r} \Gamma_{i\nu}(n-1, q, \xi; s) \times \tag{5}$$
$$\times \int_0^\infty e^{-(\lambda+s)y} \frac{(\lambda y)^{[(r-q)k+\eta-\xi+k]}}{[(r-q)k+\eta-\xi+k]!} dQ_{\nu j}(y) +$$
$$+ \sum_{\nu=1}^{m} \sum_{\xi=1}^{\eta} \Gamma_{i\nu}(n-1, r+1, \xi; s) \int_0^\infty e^{-(s+\lambda)y} \frac{(\lambda y)^{\eta-\xi}}{(\eta-\xi)!} dQ_{\nu j}(y).$$

From (4) and (5) we obtain

$$C_{ij}(1, z; s) = z\psi_{ij}(s+\lambda-\lambda z), \tag{6}$$

$$z^k C_{ij}(n, z; s) = \sum_{\nu=1}^{m} C_{i\nu}(n-1, z; s)\psi_{\nu j}(s+\lambda-\lambda z) - \tag{7}$$
$$- \sum_{\nu=1}^{m} \sum_{\xi=1}^{k} \Gamma_{i\nu}(n-1, \xi; s) z^\xi \psi_{\nu j}(s+\lambda-\lambda z) \quad \text{for } n > 1,$$

whence

$$z^k D_{ij}(w, z; s) = wz^{k+1}\psi_{ij}(s+\lambda-\lambda z) + \tag{8}$$
$$+ w\sum_{\nu=1}^{m} D_{i\nu}(w, z; s)\psi_{\nu j}(s+\lambda-\lambda z) - w\sum_{\nu=1}^{m} \sum_{\xi=1}^{k} E_{i\nu}(w, \xi; s) z^\xi \psi_{\nu j}(s+\lambda-\lambda z).$$

Relation (8) gives the following matrix equation:

$$D(w, z;\ s)[z^k I - w\psi(s+\lambda-\lambda z)] = w\Big[z^{k+1}I - \sum_{\xi=1}^{k} E(w, \xi;\ s)z^{\xi}\Big]\psi(s+\lambda-\lambda z). \tag{9}$$

The inverse of the matrix $z^k I - w\psi(s+\lambda-\lambda z)$ exists for all z, w, and s in the region $|z| \leqslant 1$, $|w| \leqslant 1$, and $\mathrm{Re}(s) > 0$ except for the roots $z = \bar{\nu}_{\varrho r}(s, w)$, $1 \leqslant \varrho \leqslant m$, $1 \leqslant r \leqslant k$. Then $\psi(s+\lambda-\lambda z)$ takes the following form under the non-degeneracy assumption:

$$\psi(s+\lambda-\lambda z) = R(s+\lambda-\lambda z)H(s+\lambda-\lambda z)R^{-1}(s+\lambda-\lambda z), \tag{10}$$

where

$$H_{ij}(s+\lambda-\lambda z) = \delta_{ij}\eta_i(s+\lambda-\lambda z),$$

$$R_{ij}(s+\lambda-\lambda z) = a_{ij}(s+\lambda-\lambda z), \quad (R^{-1})_{ij}(s+\lambda-\lambda z) = \beta_{ij}(s+\lambda-\lambda z).$$

Using (9) and (10) we have

$$D_{ij}(w, z;\ s) = w\sum_{\nu=1}^{k}\Big[z^{k+1}\delta_{i\nu} - \sum_{\xi=1}^{k} E_{i\nu}(w, \xi;\ s)z^{\xi}\Big]\times \times \sum_{\varrho=1}^{m}\frac{a_{\nu\varrho}(s+\lambda-\lambda z)\eta_\varrho(s+\lambda-\lambda z)\beta_{\varrho j}(s+\lambda-\lambda z)}{z^k - w\eta_\varrho(s+\lambda-\lambda z)}$$

for all $z \neq \bar{\nu}_{\varrho r}(s, w)$, $1 \leqslant \varrho \leqslant m$, $1 \leqslant r \leqslant k$. Since the functions $D_{ij}(w, z;\ s)$ are analytic for all $|w| \leqslant 1$, $\mathrm{Re}(s) > 0$, $|z| \leqslant 1$, the zeros of the denominators are zeros of the numerators. For all ϱ and r we get

$$\sum_{\nu=1}^{m}\Big[\delta_{i\nu}\bar{\nu}_{\varrho r}^{k+1}(s, w) - \sum_{\xi=1}^{k} E_{i\nu}(w, \xi, s)\bar{\nu}_{\varrho r}^{\xi}(s, w)\Big]\times$$

$$\times a_{\nu\varrho}[s+\lambda-\lambda\bar{\nu}_{\varrho r}(s, w)]\eta_\varrho[s+\lambda-\lambda\bar{\nu}_{\varrho r}(s, w)]\beta_{\varrho j}[s+\lambda-\lambda\bar{\nu}_{\varrho r}(s, w)] = 0.$$

Since $\eta_\varrho[s+\lambda-\lambda\bar{\nu}_{\varrho r}(s, w)]$ does not vanish and $\beta_{\varrho j}[s+\lambda-\lambda\bar{\nu}_{\varrho r}(s, w)]$ is different from zero for at least one j, we get

$$\sum_{\nu=1}^{m}\sum_{\xi=1}^{k} E_{i\nu}(w, \xi;\ s)\bar{\nu}_{\varrho r}^{\xi}(s, w)a_{\nu\varrho}[s+\lambda-\lambda\bar{\nu}_{\varrho r}(s, w)] = \bar{\nu}_{\varrho r}^{k+1}(s, w)a_{i\varrho}[s+\lambda-\lambda\bar{\nu}_{\varrho r}(s, w)] \tag{11}$$

for all i, ϱ, and r. From (11) we obtain m^2k linear equations satisfied by m^2k unknowns $E_{ij}(w, \xi, s)$.

It may be noticed that if $k = 1$, (11) reduces to the result of [4], and if $m = 1$ and $k = 1$, this gives the one of [7].

4. Queue length in continuous time. Let $\xi(t) = (\xi_1(t), \xi_2(t))$, where $\xi_1(t)$ is the number of customers in the queue, $\xi_2(t)$ is the phase of the arriving customer at time t, and let J_t^* be the type of the customer being served at time t. Let τ_n' be the time of departure of the n-th customer, $n = 0, 1, \ldots$, and let $\tau_0' = 0$. Let $\xi(0)$ be the initial value of $\xi(t)$. We have

$$(12) \qquad U_{ij}^n(s, a, \beta) = \int_0^\infty e^{-st} d\Pr[\tau_n' \leqslant t, \xi(\tau_n') = (a, \beta), J_{\tau_n'} = j \mid J_0 = i],$$

$$U_{ij}^n(s, z) = \sum_{a=0}^\infty \sum_{\beta=1}^k z^{ak+\beta} U_{ij}^n(s, a, \beta)$$

for $\mathrm{Re}(s) \geqslant 0$, $|z| \leqslant 1$ and $n = 0, 1, \ldots, 1 \leqslant i, j \leqslant m$, and

$$(13) \qquad \sum_{n=0}^\infty w^n U_{ij}^n(s, a, \beta) = V_{ij}(s, a, \beta, w), \quad |w| < 1,$$

$$(14) \qquad V_{ij}(s, z, w) = \sum_{n=0}^\infty U_{ij}^n(s, z) w^n, \ |w| < 1.$$

Using a similar argument as in the previous section we get

$$(15) \qquad z^k U_{ij}^{n+1}(s, z) = \sum_{\nu=1}^m U_{i\nu}^n(s, z)\psi_{\nu j}(s+\lambda-\lambda z) - \\ - \sum_{\nu=1}^m \sum_{\eta=1}^k U_{i\nu}^n(s, 0, \eta) z^\eta \psi_{\nu j}(s+\lambda-\lambda z) + \\ + z^{k+1} \sum_{\nu=1}^m \sum_{\eta=1}^k U_{i\nu}^n(s, 0, \eta)[\lambda/(\lambda+s)]^{k-\eta+1} \psi_{\nu j}(s+\lambda-\lambda z).$$

By definition we obtain $U_{ij}^0 = 0$ if $i \neq j$. Thus (15) gives

$$(16) \qquad z^k V_{ij}(s, z, w) = w \sum_{\nu=1}^m V_{i\nu}(s, z, w)\psi_{\nu j}(s+\lambda-\lambda z) - \\ -w \sum_{\nu=1}^m \sum_{\eta=1}^k V_{i\nu}(s, 0, \eta, w) z^\eta \psi_{\nu j}(s+\lambda-z\lambda) + z^k \delta_{ij} U_{ii}^0(s, z) + \\ + z^{k+1} w \sum_{\nu=1}^m \sum_{\eta=1}^k V_{i\nu}(s, 0, \eta, w)[\lambda/(\lambda+s)]^{k-\eta+1} \psi_{\nu j}(s+\lambda-z\lambda).$$

From (16) we get the following matrix equation:

$$(17) \qquad V(s, z, w)[z^k I - w\psi(s+\lambda-\lambda z)]$$
$$= z^k U^0(s, z) - w \sum_{\eta=1}^k V(s, 0, \eta, w)\left[z^\eta - z^{k+1}(\lambda/(\lambda+s))^{k-\eta+1}\right]\psi(s+\lambda-\lambda z).$$

The matrix $z^k I - \psi(s+\lambda-\lambda z)$ is non-singular for $\mathrm{Re}(s) \geqslant 0$, $|w| < 1$, $|z| \leqslant 1$ except for $z = \bar{v}_{\varrho r}(s, w)$, $1 \leqslant \varrho \leqslant m$, $1 \leqslant r \leqslant k$. Taking the inverse of this matrix we have

$$(18) \quad V_{ij}(s, z, w) = \sum_{\varrho=1}^{m} \frac{\beta_{\varrho j}(s+\lambda-\lambda z)}{z^k - w\eta_\varrho(s+\lambda-\lambda z)} \Big\{ z^k U_{ii}^0(s, z)\, a_{i\varrho}(s+\lambda-\lambda z) -$$

$$- w \sum_{\eta=1}^{k} V_{i\nu}(s, 0, \eta, w)\big[z^\eta - z^{k+1}(\lambda/(\lambda+s))^{k-\eta+1}\big] a_{\nu\varrho}(s+\lambda-\lambda z)\,\eta_\varrho(s+\lambda-\lambda z)\Big\}.$$

Since the left-hand side is regular in $|z| < 1$, $\mathrm{Re}(s) \geqslant 0$, $|w| < 1$, we get km^2 linear equations for the unknowns $V_{i\nu}(s, 0, \eta, w)$ for $1 \leqslant i$, $\nu \leqslant m$, $1 \leqslant \eta \leqslant k$. We have

$$(19) \quad U_{ii}^0[s, \bar{v}_{\varrho r}(s, w)]\, a_{i\varrho}[s+\lambda-\lambda\bar{v}_{\varrho r}(s, w)]$$

$$= \sum_{\nu=1}^{m} \sum_{\eta=1}^{k} V_{i\nu}(s, 0, \eta, w)\big[\bar{v}_{\varrho r}^{\eta}(s, w) - \bar{v}_{\varrho r}^{k+1}(s, w)(\lambda/(\lambda+s))^{k-\eta+1}\big] \times$$

$$\times\, a_{\nu\varrho}[s+\lambda-\lambda\bar{v}_{\varrho r}(s, w)],$$

which describes (17) completely.

Next we obtain the probabilities

$$(20) \quad P(i, h, \xi; j, a, \eta; t)$$

$$= \Pr[\xi(t) = (a, \eta),\ J_t^* = j \mid \xi(0) = (h, \xi),\ J_0^* = i].$$

Let

$$\pi(i, h, \xi; j, a, \eta; s) = \int_0^\infty e^{-st} P[i, h, \xi; j, a, \eta; t]\, dt,$$

$$(21) \quad \pi_{ij}(s, z) = \sum_{a=0}^{\infty} \sum_{\eta=1}^{k} z^{ak+\eta} \pi(i, h, \xi; j, a, \eta; s), \qquad \mathrm{Re}(s) > 0,\ |z| \leqslant 1,$$

and

$$N_{ij}^{a\beta}(t) = \sum_{n=1}^{\infty} \Pr[\tau_n' \leqslant t,\ J_n = j,\ \xi(\tau_n') = (a, \beta) \mid J_0 = i].$$

Then by (13), (14), (18), and (19) we have

$$\sum_{a=0}^{\infty} \sum_{\beta=1}^{k} z^{ak+\beta} \int_0^\infty e^{-st} dN_{ij}^{a\beta}(t) = V_{ij}(s, z, 1).$$

The expressions for (20) are

(22) $P(i, h, \xi; j, 0, \eta; t)$

$$= \delta^*_{\xi\eta}\delta^*_{h0}\delta_{ij}e^{-\lambda t}\frac{(\lambda t)^{\eta-\xi}}{(\eta-\xi)!}+\sum_{\beta=1}^{\eta}\int_0^t e^{-\lambda(t-u)}\frac{[\lambda(t-u)]^{\eta-\beta}}{(\eta-\beta)!}\,dN^{0\beta}_{ij}(u),$$

(23) $$P(i, h, \xi; j, a, \eta; t) = \delta_{ij}\delta^*_{hk+\xi,ak+\eta}[1-H_j(t)]\frac{e^{-\lambda t}(\lambda t)^{(a-h)k+\eta-\xi}}{[(a-h)k+\eta-\xi]!}+$$

$$+\sum_{\alpha=1}^{a-1}\sum_{\beta=1}^{k}\int_0^t[1-H_j(t-u)]e^{-\lambda(t-u)}\frac{[\lambda(t-u)]^{(a-\alpha)k+\eta-\beta}}{[(a-\alpha)k+\eta-\beta]!}\,dN^{\alpha\beta}_{ij}(u)+$$

$$+\sum_{\beta=1}^{\eta}\int_0^t[1-H_j(t-u)]e^{-\lambda(t-u)}\frac{[\lambda(t-u)]^{\eta-\beta}}{(\eta-\beta)!}\,dN^{a\beta}_{ij}(u)+$$

$$+\int_0^t[1-H_j(t-u)]e^{-\lambda(t-u)}\frac{[\lambda(t-u)]^{(a-1)k+\eta-1}}{[(a-1)k+\eta-1]!}\,dM^0_{ij}(u),$$

where

$$H_j(t) = \sum_{\nu=1}^{m}Q_{j\nu}(t),$$

δ^*_{00} and δ^*_{ha} are equal to 1 for $h = 1, 2, \ldots, a$ and to 0 otherwise.

The Laplace-Stieltjes transforms $m^0_{ij}(s)$ of $M^0_{ij}(x)$ and $n^{0\eta}_{ij}(s)$ of $N^{0\eta}_{ij}(x)$ are related by the following equation:

(24) $$m^0_{ij}(s) = \sum_{\eta=1}^{k}\big(\lambda/(\lambda+s)\big)^{k-\eta+1}n^{0\eta}_{ij}(s)+\delta_{ij}\delta^*_{h0}\big(\lambda/(\lambda+s)\big)^{k-\xi+1}.$$

The fact that the event $\xi(t) = (0, \eta)$, $J^*_t = j$ occurs if there are no customers in the queue at time 0 and no new customers arrive, or the customer of type j arrives at some time in the interval $(0, t]$, $\xi(u) = (0, \beta)$, and no new customers arrive is used to set up equation (22). For (23) notice that the event $\xi(t) = (a, \eta)$, $J^*_t = j$ can occur if the service of the customer of type j started at time 0 is not over or the service of the customer of type j started at some time in the interval $(0, t]$, $\xi(u) = (\alpha, \beta)$, $\alpha > 0$, and the service is not over up to time t. The fourth term corresponds to the case $\alpha = 0$, the server is idle for some time before u, $\xi(u) = (1, 1)$, and new arrivals occur afterwards. The generating function

(21) can be written by using (22)-(24). After some calculations we obtain

$$\pi_{ij}(s,z) = [1-h_j(s+\lambda-\lambda z)](s+\lambda-\lambda z)^{-1}\Big\{V_{ij}(s,z,1)+$$
$$+\sum_{\eta=1}^{k}\big((\lambda/(\lambda+s))^{k-\eta+1}n_{ij}^{0\eta}(s)\Big\}+$$
$$+\Big(\sum_{\eta=1}^{k} z^{\eta} n_{ij}^{0\eta}(s)\Big)(s+\lambda-\lambda z)^{-1}\big[h_j(s+\lambda-\lambda z)-\big(\lambda z/(\lambda+s)\big)^{k}\big]+$$
$$+\begin{cases} z^{\xi}\delta_{ij}(s+\lambda-\lambda z)^{-1}\big[1-h_j(s+\lambda-\lambda z)\big(\lambda z/(\lambda+s)\big)^{k-\xi+1}\big], & h=0,\\ z^{hk+\xi}\delta_{ij}(s+\lambda-\lambda z)^{-1}[1-h_j(s+\lambda-\lambda z)], & h\neq 0.\end{cases}$$

The expected number of departures of customers of type j in the interval $(0, t]$ can also be written using the previous discussion. Assuming the i-th customer leaves at $\tau_0' = 0$ and setting

$$N_{ij}(t) = \sum_{n=1}^{\infty} P[\tau_n' \leqslant t,\ J_n = j \mid J_0 = i],$$

by (12) and (14) we have

$$\int_0^{\infty} e^{-st} dN_{ij}(t) = V_{ij}(s, 1-, 1-) - \delta_{ij},$$

where $V_{ij}(s, 1-, 1-)$ are given by (18) and (19).

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DEPT. OF MATHEMATICS
ANNAMALAI UNIVERSITY
ANNAMALAINAGAR, 608 101 INDIA

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G. SANKARANARAYANAN, K. USHA i R. RAMANARAYANAN (Annamalainagar, India)

JEDNOKANAŁOWY SYSTEM OBSŁUGI MASOWEJ Z ERLANGOWSKIM PROCESEM WEJŚCIA I SEMIMARKOWOWSKIMI CZASAMI OBSŁUGI

STRESZCZENIE

Rozpatruje się jednokanałowy system obsługi masowej, w którym czasy obsługi tworzą proces semimarkowowski o m stanach. Otrzymuje się nowe wyniki dla procesu wejścia typu erlangowskiego. Bada się kolejne okresy zajętości systemu oraz długość kolejki.