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ON CONTINUOUS MAPPINGS BETWEEN NON-NEGATIVELY AND NON-POSITIVELY CURVED MANIFOLDS

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Introduction

If M is a compact Riemannian manifold of positive Ricci curvature, then the first Betti number $b_1(M)$ vanishes [6]. On the other hand, the first Betti number $b_1(M)$ of a compact metric space M is positive if and only if there exists a continuous mapping $f\colon M \to S^1$ which is not homotopic to a constant [2]. Comparing these two results we get the following:

($*$) Every continuous mapping of a compact Riemannian manifold of positive Ricci curvature into S^1 is homotopic to a constant.

If N is a complete Riemannian manifold of non-positive sectional curvature, then, according to the Hadamard–Cartan Theorem, the universal covering $\tilde{N}$ of N is diffeomorphic to the Euclidean space. Therefore, all homotopy groups $\pi_k(\tilde{N})$ are trivial. On the other hand, $\pi_k(\tilde{N}) \approx \pi_k(N)$ for any $k \geqslant 2$ [4]. It follows that

($**$) the homotopy groups $\pi_k(N)$ of a complete Riemannian manifold N of non-positive sectional curvature are trivial for all $k \geqslant 2$.

In this note we generalize the above results ($*$) and ($**$). We prove the following:

THEOREM. *Let M be a compact Riemannian manifold of non-negative Ricci curvature and N a compact Riemannian manifold of non-positive sectional curvature such that $n = \dim N < \dim M = m$. If either the Ricci curvature of M is strictly positive at some point of M or $H^k(M, R) = 0$ for $k = 1, 2, \ldots, m-1$ and the fundamental group $\pi_1(M)$ of M does not contain any subgroup of index 2, then any continuous mapping of M into N is homotopic to a constant.*

1. Preliminaries

Let M and N be Riemannian manifolds and denote by ∇^M and ∇^N the Levi–Civita connections on M and N, respectively. If $f\colon M \to N$ is a differentiable mapping, then the differential df can be considered as a section of the bundle $T^*M \otimes f^*TN$.

The connections ∇^M and ∇^N induce a connection on this bundle. We denote it by ∇. The covariant derivative ∇df is a section of the bundle $T^*M \otimes T^*M \otimes f^*TN$. It is called the *second fundamental form of f*. The mapping f is said to be *totally geodesic* if and only if $\nabla df = 0$. The trace $\tau(f)$ of ∇df is a section of the bundle f^*TN called the *tension field of f*. The mapping f is said to be *harmonic* if and only if $\tau(f) = 0$.

LEMMA 1 [3]. *If M and N are compact Riemannian manifolds and the sectional curvature of N is non-positive, then any continuous mapping f: $M \to N$ is homotopic to a harmonic mapping.*

LEMMA 2 [3]. *If f: $M \to N$ is a harmonic mapping, M is compact, the Ricci curvature of M is non-negative and the sectional curvature of N is non-positive, then f is totally geodesic. If, moreover, the Ricci curvature of M is strictly positive at some points of M, then f is constant.*

If F is a k-dimensional oriented foliation of a Riemannian manifold M, then the characteristic form ω of F is defined by

$$\omega(v_1, \ldots, v_k) = \det[\langle v_i, w_j\rangle;\ i, j \leqslant k],$$

where $v_1, \ldots, v_k \in T_xM$ $(x \in M)$ and $w_1, \ldots, w_k$ is a positive oriented orthonormal frame of the space T_xL tangent at x to the passing through x leaf L of F. Clearly, ω is a k-form on M such that its restriction to any leaf L of F is a volume element on L.

LEMMA 3 [1], [8]. *The characteristic form ω of a foliation F is closed if and only if leaves of F are minimal submanifolds of M and the normal bundle of F is involutive.*

2. Proof of the Theorem

If the Ricci curvature of M is positive somewhere, then our statement follows immediately from Lemmas 1 and 2. (This case was considered in [3], § 11, Theorem B.)

Assume that $\pi_1(M)$ contains no subgroups of index 2 and $H^k(M, R) = 0$ for $k = 1, 2, \ldots, m-1$, and take an arbitrary continuous mapping φ: $M \to N$. Suppose that φ is not homotopic to a constant. According to Lemma 1, φ is homotopic to a harmonic mapping f: $M \to N$. From Lemma 2, it follows that f is totally geodesic. Therefore, f has constant rank k on M. Clearly, $1 \leqslant k \leqslant m-1$ because f is not constant and $m > n$.

For any point x of M, put

$$E_x = \ker df(x).$$

The union $E = \bigcup_{x \in M} E_x$ carries a natural structure of a subbundle of the tangent bundle TM of M. It is involutive. In fact, if X and Y are sections of E, then they are vector fields on M which are f-related to the zero section 0_N of TN. Therefore, $[X, Y]$ is also f-related to 0_N, i.e. $[X, Y]$ is a section of E. Consequently, E is tangent to an $(m-k)$-dimensional foliation F of M.

The foliation F is orientable because $\pi_1(M)$ contains no subgroups of index 2.

Leaves of F are totally geodesic submanifolds of M. In fact, if X and Y are sections of E, then

$$0 = (\nabla_X df)(Y) = \nabla^N_X(df \circ Y) - df(\nabla^M_X Y) = -df(\nabla^M_X Y)$$

because $df \circ Y = 0$. Therefore, $\nabla^M_X Y$ is a section of E.

The normal bundle $E^\perp$ of F is involutive. In fact, if X is a section of $E^\perp$ and Y is a section of E, then $\nabla^M_X Y$ is a section of E. Therefore, for any sections X_1, X_2 of $E^\perp$ and Y of E, we have

$$\begin{aligned}\langle [X_1, X_2], Y\rangle &= \langle \nabla^M_{X_1} X_2, Y\rangle - \langle \nabla^M_{X_2} X_1, Y\rangle \\ &= X_1\langle X_2, Y\rangle - X_2\langle X_1, Y\rangle - \langle X_2, \nabla^M_{X_1} Y\rangle + \langle X_1, \nabla^M_{X_2} Y\rangle = 0.\end{aligned}$$

This shows that $[X_1, X_2]$ is a section of $E^\perp$ if only X_1 and X_2 are.

From Lemma 3, it follows that the characteristic form ω of F is closed. The cohomology class $[\omega]$ is a non-trivial element of $H^{m-k}(M, R)$. In fact, if $\iota\colon L \to M$ is the inclusion of a leaf L of F into M, then

$$\iota^*[\omega] = [\iota^*\omega] \neq 0$$

because L is a compact orientable manifold and $\iota^*\omega$ is a volume element on L. This ends the proof.

3. Final remarks

(i) Simple examples (the canonical projection $T^m \to T^n$ and the Hopf fibering $S^3 \to S^2$) show that if either M has non-negative Ricci curvature which is not strictly positive at any point and $H^k(M, R) \neq 0$ for some $k \in \{1, 2, \ldots, m-1\}$ or M has strictly positive curvature, $H^k(M, R) = 0$ for any $k \in \{1, 2, \ldots, m-1\}$ but the curvature of N is not non-positive, then continuous mappings $f\colon M \to N$ which are not homotopic to a constant may occur.

(ii) A wide class of manifolds admitting Riemannian metrics of positive Ricci curvature is described in [7]. Meyer [5] showed that cohomology groups $H^k(M, R)$, $k = 1, 2, \ldots, m-1$, of a compact orientable m-dimensional Riemannian manifold M with positive curvature operator are trivial.

(iii) If the sectional curvature of N is negative everywhere, then a harmonic mapping of a compact manifold M of non-negative Ricci curvature is either constant or maps M onto a circle (i.e., onto a closed geodesic on N). From Lemma 1 and the mentioned in the Introduction Borsuk's result, it follows that any continuous mapping of a compact Riemannian manifold M of non-negative Ricci curvature into a manifold of negative sectional curvature is homotopic to a constant if only $H^1(M, R) = 0$.

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