DIFFERENTIAL GEOMETRY
BANACH CENTER PUBLICATIONS, VOLUME 12
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1984

ON QUASI-SASAKIAN MANIFOLDS

SHÔJI KANEMAKI

Department of Mathematics, Science University of Tokyo, Tokyo, Japan

Introduction

An almost contact metric structure on a differentiable manifold is called a *quasi-Sasakian structure* if the structure is normal and its fundamental 2-form is closed. Then a normal contact metric structure, which is also called a *Sasakian structure*, and a cosymplectic structure are quasi-Sasakian structures. A manifold with a quasi-Sasakian structure is called a *quasi-Sasakian manifold*. The notion of quasi-Sasakian manifolds was first introduced by D. E. Blair [1] in 1967 and some results were obtained also by S. Tanno [25] in 1971.

The present note is mainly devoted to exhibit a characterization of a quasi-Sasakian structure in terms of the covariant derivative of the fundamental linear transformation field of the structure in Section 3 ([10]). This characterization is applied to the study of quasi-Sasakian manifolds each of which is locally a product manifold of a Sasakian manifold and a Kähler manifold, and to the study of submanifolds. This note consists of the following sections.

(0) Notations.

(1) Almost contact metric structures.

(2) A relation between an almost contact metric structure σ on M and an almost Hermitian structure (J_0, G) on $M \times \boldsymbol{R}$.

(3) Quasi-Sasakian structures.

(4) Product manifolds.

(5) A parallelizable Riemannian manifold M^3 of dimension 3.

(6) $M(\sigma)$ of constant f-sectional curvature.

(7) A hypersurface N of a Kähler manifold $P(J, G)$.

(8) A submanifold M of codimension 2 of a quasi-Sasakian manifold.

The text is based on lectures given by the author at Stefan Banach International Mathematical Center during the Differential Geometry Semester in 1979.

The author expresses his gratitude to the members of the international organizing committee of this semester. He thanks sincerely also Professor J. Waniurski

(Department of Mathematics, Marii Curie-Skłodowska University (Lublin)), Professor J. Ławrynowicz (Institute of Mathematics, Polish Academy of Sciences, Łódź Branch (Łódź)), Professor W. Roter (Mathematical Institute, Wrocław University (Wrocław)) and others who kindly gave him the hearty encouragement during his stay in Poland, and Professor T. Adati (Department of Mathematics, Science University of Tokyo (Tokyo)) for his kind advice.

(0) Notations

We shall fix some notations and try to sketch terminology briefly for our discussion. Let M be an m-dimensional differentiable manifold of class C^∞. The manifold M is covered by coordinate neighborhoods $\{(U_\alpha, (x^1, \ldots, x^m))\}$, and the coordinate transformation: $y^i = y^i(x^1, \ldots, x^m)$ $(i = 1, 2, \ldots, m)$ or $x^i = x^i(y^1, \ldots, y^m)$ $(i = 1, 2, \ldots, m)$ arises on the non-empty intersection $U_\alpha \cap U_\beta$ of two coordinate neighborhoods $(U_\alpha, (x^i))$ and $(U_\beta, (y^i))$. Let $S(M)$ denote the set of all (C^∞-) scalar fields on M and let $\mathfrak{X}(M)$ denote the set of all vector fields on M. Any vector field $X \in \mathfrak{X}(M)$ at any point of U_α of M can be written as a linear combination $X = \sum_{i=1}^{m} X^i(\partial/\partial x^i)$ of the natural frame $\{\partial/\partial x^i\}$, where X^i ($\in S(M)$) are the components of X with respect to (x^i). The vector field X has two expressions

$$X = \sum_{i=1}^{m} X^i(\partial/\partial x^i) = \sum_{i=1}^{m} X'^i(\partial/\partial y^i)$$

at any point of the non-empty intersection $U_\alpha \cap U_\beta$. Since two natural frames $\{\partial/\partial x^i\}$ and $\{\partial/\partial y^i\}$ are combined by the element $(\partial y^j/\partial x^i)$ of the real general linear group $\mathrm{GL}(m; \boldsymbol{R})$ of degree m such as

$$\partial/\partial x^i = \sum_{j=1}^{m} (\partial y^j/\partial x^i)\,\partial/\partial y^j, \qquad i = 1, 2, \ldots, m,$$

at each point of $U_\alpha \cap U_\beta$ by means of $\sum_{j=1}^{m} (\partial y^i/\partial x^j)(\partial x^j/\partial y^k) = \delta_{ik}$, then we obtain the coordinate transformation law of the components:

$$X'^i = \sum_{j=1}^{m} (\partial y^i/\partial x^j) X^j, \qquad i = 1, 2, \ldots, m,$$

for X. $\mathfrak{X}(M)$ is a Lie algebra over the real number field $\boldsymbol{R}$ in consideration of the Lie bracket product $[X, Y]$ of X and Y, defined by $[X, Y]\lambda = X(Y\lambda) - Y(X\lambda)$ for any $\lambda \in S(M)$, which can be expressed locally

$$[X, Y] = \sum_{i,j} \left(X^j(\partial Y^i/\partial x^j) - Y^j(\partial X^i/\partial x^j)\right)\partial/\partial x^i$$

for $X = \sum X^i\partial/\partial x^i$ and $Y = \sum Y^i\partial/\partial x^i$.

We denote by $T^r_s(M)$ the set of all (r, s)-type tensor fields, which are of contravariant degree r and covariant degree s. Any element T of $T^r_s(M)$ is expressible locally in the form

$$T = \sum_{\substack{i_1,\dots,i_s\\ j_1,\dots,j_r=1}}^{m} T^{j_1\dots j_r}_{i_1\dots i_s} dx^{i_1}\otimes\dots\otimes dx^{i_s}\otimes\frac{\partial}{\partial x^{j_1}}\otimes\dots\otimes\frac{\partial}{\partial x^{j_r}}$$

with respect to (x^i). Then we obtain the coordinate transformation law of components for T where any two coordinate neighborhoods intersect. The direct sum $\mathcal{T}(M) = \sum_{r,s\geqslant 0} T^r_s(M)$ of $S(M)$-modules $T^r_s(M)$ forms the tensor algebra with the tensor multiplication $\otimes$. The tensor product $T\otimes S\in T^{r+r'}_{s+s'}(M)$ of $T\in T^r_s(M)$ and $S\in T^{r'}_{s'}(M)$ is given by $(T\otimes S)_x = T_x\otimes S_x$ for all $x\in M$, in this case, the components of $T\otimes S$ are represented as

$$(T\otimes S)^{j_1\dots j_{r+r'}}_{i_1\dots i_{s+s'}}(x) = T^{j_1\dots j_r}_{i_1\dots i_s}(x)\, S^{j_{r+1}\dots j_{r+r'}}_{i_{s+1}\dots i_{s+s'}}(x)$$

with respect to (x^i). A contraction is a mapping $C^\mu_\nu\colon T^r_s(M)\to T^{r-1}_{s-1}(M)$ which sends a mixed type tensor field $T\in T^r_s(M)$ to the tensor field $C^\mu_\nu T\in T^{r-1}_{s-1}(M)$ such that

$$(C^\mu_\nu T)^{j_1\dots j_{r-1}}_{i_1\dots i_{s-1}} = \sum_{k=1}^{m} T^{j_1\dots \overset{\mu\text{th}}{k}\dots j_r}_{i_1\dots \underset{\nu\text{th}}{k}\dots i_s}$$

for the components $T^{j_1\dots j_r}_{i_1\dots i_s}$ of T, with respect to a coordinate system (x^i), where the superscript k appears at the μth position and the subscript k appears at the νth position, $1\leqslant\mu\leqslant r$, $1\leqslant\nu\leqslant s$. Then, in particular, the following notations are available in terms of contractions in mind. For any 1-form η $\big(=\sum_{i=1}^{m}\eta_i dx^i$ on $(U_\alpha,(x^i))\big)$ $\in T^0_1(M)$, any vector fields $E\,(=\sum_i E^i\partial/\partial x^i$ on $U_\alpha)$, $X, X_1,\dots,X_p\in\mathfrak{X}(M)$, a linear transformation field $f\,(=\sum f^h_i dx^i\otimes\partial/\partial x^h$ on $U_\alpha)\in T^1_1(M)$ and a $(0,p)$-type tensor field $\omega\,(=\sum\omega_{i_1\dots i_p}dx^{i_1}\otimes\dots\otimes dx^{i_p}$ on $U_\alpha)\in T^0_p(M)$ the relations hold:

$C^1_1(\eta\otimes X) = \eta(X)$, which equals $\sum\eta_i X^i$ on U_α,

$C^1_1(f\otimes X) = (\operatorname{trace} f)X$, which equals $\big(\sum f^i_i\big)X$ on U_α,

$C^2_1(f\otimes X) = fX$, which equals $\sum_{i,j} f^j_i X^i(\partial/\partial x^j)$ on U_α,

$C^1_1(\eta\otimes E\otimes X) = \eta(E)X = (\eta\otimes X)E$, which equals $\big(\sum\eta_i E^i\big)X$ on U_α,

$C^2_1(\eta\otimes E\otimes X) = \eta(X)E = (\eta\otimes E)X$, which equals $\big(\sum\eta_i X^i\big)E$ on U_α,

$C^p_p\dots C^2_2 C^1_1(\omega\otimes X_1\otimes\dots\otimes X_p) = \omega(X_1,\dots,X_p)$, which equals

$$\sum\omega_{i_1 i_2\dots i_p}X_1^{i_1}X_2^{i_2}\dots X_p^{i_p}\quad\text{on}\quad U_\alpha.$$

Every contraction defined above does not depend on any choice of the coordinate systems. We denote by L_X the Lie differentiation with respect to X. It is a type-preserving derivation $L_X\colon\mathcal{T}(M)\to\mathcal{T}(M)$, namely, $L_X T^r_s(M)$ is in $T^r_s(M)$, and it satisfies the well-known conditions: (i) $L_X \lambda = X\lambda$ for any $\lambda \in S(M)$, $X \in \mathfrak{X}(M)$, (ii) $L_X Y = [X, Y]$ for any $Y \in \mathfrak{X}(M)$, (iii) $L_X(S \otimes T) = (L_X S) \otimes T + S \otimes (L_X T)$ for any $S, T \in \mathscr{T}(M)$, (iv) L_X commutes with every contraction, (v) $L_X L_Y - L_Y L_X = L_{[X,Y]}$. A differential p-form α on M is a $(0, p)$-type tensor field which is skew-symmetric

$$\alpha(X_{\pi(1)}, \dots, X_{\pi(p)}) = \varepsilon(\pi)\alpha(X_1, \dots, X_p)$$

for an arbitrary permutation π of $(1, 2, \dots, p)$, where $\varepsilon(\pi)$ denotes its sign. Let $A^p(M)$ be the set of all differential p-forms on M and put $A(M) = \sum_{p \geqslant 0} A^p(M)$. Then we mean $A^0(M) = S(M)$ and $A^1(M) = T_1^0(M)$. $A(M)$ is an associative division graded algebra with the exterior multiplication $\wedge$, which is given by

$$(\alpha \wedge \beta)(X_1, \dots, X_{p+q}) = (1/p!q!) \sum_{\pi \in S_{p+q}} \varepsilon(\pi)\alpha(X_{\pi(1)}, \dots, X_{\pi(p)})\beta(X_{\pi(p+1)}, \dots, X_{\pi(p+q)})$$

for any p-form α and q-form β; in this case, the summation means the sum of terms obtained by all permutations $\pi \in S_{p+q}$, the permutation group of degree $p+q$. It follows that $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$. The exterior differential operator

$$d\colon A^p(M) \to A^{p+1}(M)$$

is given by

$$d\alpha(X_1, \dots, X_{p+1}) = \sum_{a=1}^{p+1} (-1)^{a-1} X_a\big(\alpha(X_1, \dots, \hat{X}_a, \dots, X_{p+1})\big) + \\ + \sum_{1 \leqslant a < b \leqslant p+1} (-1)^{a+b} \alpha([X_a, X_b], \dots, \hat{X}_a, \dots, \hat{X}_b, \dots, X_{p+1})$$

for any $\alpha \in A^p(M)$ $(p \geqslant 1)$ and by $d\alpha(X) = X\alpha$ for any 0-form α. The operator d satisfies the well-known relations $d^2 = 0$ and $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ for $\alpha \in A^p(M)$.

We consider on a differentiable manifold M a positive definite Riemannian metric g and denote by ∇ the Riemannian connection of the metric g. The covariant differentiation $\nabla_X\colon \mathscr{T}(M) \to \mathscr{T}(M)$ by X with respect to the connection ∇ is a type-preserving derivation and satisfies the conditions: (i) $\nabla_X \lambda = X\lambda$ for any $\lambda \in S(M)$, (ii) $\nabla_{\lambda X} T = \lambda \nabla_X T$ for any $\lambda \in S(M)$ and $T \in \mathscr{T}(M)$, (iii) $\nabla_X(S+T) = \nabla_X S + \nabla_X T$, (iv) $\nabla_X(S \otimes T) = (\nabla_X S) \otimes T + S \otimes (\nabla_X T)$, (v) ∇_X commutes with every contraction, (vi) $\nabla_{X+Y} = \nabla_X + \nabla_Y$. The covariant derivative $\nabla_X Y$ of Y by X is written locally as

$$\nabla_X Y = \sum_{i,j} \left(X^j (\partial Y^i / \partial x^j) + \sum_k \left\{ {i \atop j\,k} \right\} X^j Y^k \right) \partial / \partial x^i,$$

where $\left\{ {i \atop j\,k} \right\}$ denote the coefficients of the Riemannian connection ∇. Namely,

$$\nabla_X Y = \sum_{i,j} (\nabla_j Y^i) X^j \partial / \partial x^i \quad \text{where} \quad \nabla_j Y^i = \partial Y^i / \partial x^j + \sum_k \left\{ {i \atop j\,k} \right\} Y^k.$$

In general, we have

$$\nabla_X T = \sum_{\substack{i_1,\dots,i_s \\ j_1,\dots,j_r,k}} X^k \nabla_k T^{j_1\cdots j_r}_{i_1\cdots i_s} dx^{i_1} \otimes \dots \otimes dx^{i_s} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_r}}$$

for $T \in T^r_s(M)$, where

$$\nabla_k T^{j_1\cdots j_r}_{i_1\cdots i_s} = \partial T^{j_1\cdots j_r}_{i_1\cdots i_s}/\partial x^k + \sum_{\mu=1}^{r}\sum_{a=1}^{m} \begin{Bmatrix} j_\mu \\ k\ a \end{Bmatrix} T^{j_1\cdots \overset{\mu\text{th}}{a} \cdots j_r}_{i_1\cdots i_s} - \sum_{\nu=1}^{s}\sum_{a=1}^{m} \begin{Bmatrix} a \\ k\ i_\nu \end{Bmatrix} T^{j_1\cdots j_r}_{i_1\cdots \underset{\nu\text{th}}{a} \cdots i_s},$$

in this case, the second or third term on the right-hand side of this means the vain if the contravariant degree r vanishes or the covariant degree s of T vanishes. Since ∇ is a symmetric connection, the Lie bracket product $[X, Y]$ is written in the form: $[X, Y] = \nabla_X Y - \nabla_Y X$, and, in addition, $\nabla g = 0$. By a simple calculation we obtain, in particular, that

$$d\alpha(X, Y) = (\nabla_X \alpha)(Y) - (\nabla_Y \alpha)(X) \qquad \text{for} \quad \alpha \in A^1(M),$$

$$d\alpha(X, Y, Z) = (\nabla_X \alpha)(Y, Z) + (\nabla_Y \alpha)(Z, X) + (\nabla_Z \alpha)(X, Y) \qquad \text{for} \quad \alpha \in A^2(M).$$

The Riemannian curvature tensor field R is a (1,3)-type tensor field on M defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for any vector fields X, Y and Z on M. $R(X, Y)Z$ has the form:

$$R(X, Y)Z = \sum_{k,j,i,h} R^h_{kji} X^k Y^j Z^i \partial/\partial x^h$$

on a coordinate neighborhood $(U_\alpha, (x^i))$, namely,

$$R = \sum R^h_{kji} dx^k \otimes dx^j \otimes dx^i \otimes \partial/\partial x^h,$$

where R^h_{kij} are represented as

$$R^h_{kji} = \frac{\partial}{\partial x^k}\begin{Bmatrix} h \\ j\ i \end{Bmatrix} - \frac{\partial}{\partial x^j}\begin{Bmatrix} h \\ k\ i \end{Bmatrix} + \sum_a \begin{Bmatrix} h \\ k\ a \end{Bmatrix}\begin{Bmatrix} a \\ j\ i \end{Bmatrix} - \sum_a \begin{Bmatrix} h \\ j\ a \end{Bmatrix}\begin{Bmatrix} a \\ k\ i \end{Bmatrix}.$$

In fact,

$$\nabla_X \nabla_Y Z = \sum_{k,j,h} \{X^k (\nabla_k Y^j) \nabla_j Z^h + X^k Y^j \nabla_k \nabla_j Z^h\} \partial/\partial x^h$$

and

$$R(X, Y)Z = \sum (\nabla_k \nabla_j Z^h - \nabla_j \nabla_k Z^h) X^k Y^j \partial/\partial x^h,$$

namely,

$$\nabla_k \nabla_j Z^h - \nabla_j \nabla_k Z^h = \sum_i R^h_{kji} Z^i.$$

The quantities

$$\nabla_k \nabla_j Z^h = \nabla_k \left(\frac{\partial Z^h}{\partial x^j} + \sum_a \begin{Bmatrix} h \\ j\ a \end{Bmatrix} Z^a\right)$$

$$= \frac{\partial^2 Z^h}{\partial x^k \partial x^j} + \sum_a \left(\left(\frac{\partial}{\partial x^k}\begin{Bmatrix} h \\ j\ a \end{Bmatrix}\right) Z^a + \begin{Bmatrix} h \\ j\ a \end{Bmatrix} \frac{\partial Z^a}{\partial x^k} + \begin{Bmatrix} h \\ k\ a \end{Bmatrix}\left(\frac{\partial Z^a}{\partial x^j} + \sum_b \begin{Bmatrix} a \\ j\ b \end{Bmatrix} Z^b\right)\right)$$

lead to the above desired expression of R^h_{kji}.

(1) Almost contact metric structures

An almost contact metric structure $\sigma = (f, E, \eta, g)$ on a manifold M is a set consisting of a linear transformation field f ($\in T_1^1(M)$), a global vector field E, a 1-form η and a positive definite Riemannian metric g such that $f^2 = -I+\eta\otimes E$, $\eta(E) = 1$, $\eta(X) = g(X, E)$, $g(fX, Y) = -g(X, fY)$ for any vector fields X and Y on M, where I denotes the identity linear transformation field on M. A manifold $M(\sigma)$ with an almost contact metric structure σ is called an *almost contact metric manifold.* A differential 2-form F defined by $F(X, Y) = g(fX, Y)$ for any X and Y is called the fundamental 2-form of σ. The definition mentioned above implies that $fE = 0$, $\eta(fX) = 0$, and $\operatorname{rank} f =$ even, hence, $\dim M(\sigma) =$ odd, say $2n+1$. In fact, substituting $X = E$ into $f^2X = -X+\eta(X)E$, we have $f^2E = 0$, from which $0 = f^3E = -fE+\eta(fE)E$. Therefore, $0 = f^2E = (\eta(fE))^2E$, hence, $\eta(fE) = 0$, namely, $fE = 0$. On the other hand, since $\eta(f^2X) = 0$, we get $0 = \eta(f^3X) = -\eta(fX)+ +\eta(X)\eta(fE)$. Consequently, we have $\eta(fX) = 0$, which can also be obtained directly from $fE = 0$ making use of the metric g. Put $p = -f^2$ and $q = \eta\otimes E$ on M. Then we see that p and q are the projectors of an almost product structure $\{D(p), D(q)\}$ consisting of a $2n$-dimensional distribution $D(p)$ and a 1-dimensional distribution $D(q)$ which are complementary and mutually orthogonal with respect to g, because $p^2 = p$, $pq = qp = 0$, $q^2 = q$, $p+q = I$, $g(pX, Y) = g(X, pY)$, $g(qX, Y) = g(X, qY)$ hold for any X and Y on M.

We shall see three examples of almost contact metric structures on the real number space $\boldsymbol{R}^5$, for the sake of simplicity. These examples can also be given on $\boldsymbol{R}^{2n+1}$ according to the same intention, namely, they suggest a cosymplectic structure on $\boldsymbol{R}^{2n+1}$, a Sasakian structure on $\boldsymbol{R}^{2n+1}$ and a quasi-Sasakian structure on $\boldsymbol{R}^{2n+1}$ which is non-cosymplectic and non-Sasakian.

We take a coordinate system (x^1, x^2, y^1, y^2, z) (or $(x^1, x^2, x^3, x^4, x^5)$) of $\boldsymbol{R}^5$.

EXAMPLE 1. We define an almost contact metric structure $\sigma = (f, E, \eta, g)$ on $\boldsymbol{R}^5$ by

$$f = \sum_{i=1}^{2}\left(dx^i\otimes\frac{\partial}{\partial y^i} - dy^i\otimes\frac{\partial}{\partial x^i}\right) = \left[\begin{array}{c|c|c} 0 & \begin{matrix}-1 & \\ & -1\end{matrix} & 0 \\ \hline \begin{matrix}1 & \\ & 1\end{matrix} & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right],$$

$$E = \frac{\partial}{\partial z} = {}^t(0\ 0\ 0\ 0\ 1), \qquad \eta = dz = (0\ 0\ 0\ 0\ 1),$$

$$g = \sum_{i=1}^{5} dx^i\otimes dx^i = \text{the unit matrix of degree } 5.$$

Then the fundamental 2-form F of σ has the form: $F = \sum_{i=1}^{2} dx^i \wedge dy^i$ (hence, $dF = 0$) and the 1-form η is closed, $d\eta = 0$. The rank of η is equal to 1. (This σ is a cosymplectic structure on $\boldsymbol{R}^5$.)

EXAMPLE 2. We define an almost contact metric structure $\sigma = (f, E, \eta, g)$ on $\boldsymbol{R}^5$ by

$$f = \sum_{i,h=1}^{5} f_i^h dx^i \otimes \frac{\partial}{\partial x^h} = \left[\begin{array}{cc|cc|c} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & -2y^1 & -2y^2 & 0 \end{array}\right],$$

$$E = \frac{\partial}{\partial z} = {}^t(0\ 0\ 0\ 0\ 1), \quad \eta = dz - 2\sum_{i=1}^{2} y^i dx^i = (-2y^1\ -2y^2\ 0\ 0\ 1),$$

$$g = \sum_{i,j=1}^{5} g_{ij} dx^i \otimes dx^j = \left[\begin{array}{cc|cc|c} 1+4(y^1)^2 & 4y^1y^2 & 0 & 0 & -2y^1 \\ 4y^2y^1 & 1+4(y^2)^2 & 0 & 0 & -2y^2 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline -2y^1 & -2y^2 & 0 & 0 & 1 \end{array}\right].$$

Then F has the form $F = \sum_{i=1}^{2} dx^i \wedge dy^i$ and the exterior derivative $d\eta$ of η has the form $d\eta = 2(dx^1 \wedge dy^1 + dx^2 \wedge dy^2)$, from which $\eta \wedge (d\eta)^2 \neq 0$. (This σ is a Sasakian structure on $\boldsymbol{R}^5$. The verification of the structure to be normal is shown after the proof of Theorem 2.2 (see [21]).)

EXAMPLE 3. We define an almost contact metric structure $\sigma = (f, E, \eta, g)$ on $\boldsymbol{R}^5$ by

$$f = \sum_{i,h=1}^{5} f_i^h dx^i \otimes \frac{\partial}{\partial x^h} = \left[\begin{array}{cc|cc|c} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & -2y^1 & 0 & 0 \end{array}\right],$$

$$E = \frac{\partial}{\partial z} = {}^t(0\ 0\ 0\ 0\ 1), \qquad \eta = dz - 2y^1 dx^1 = (-2y^1\ 0\ 0\ 0\ 1),$$

$$g = \sum_{i,j=1}^{5} g_{ij} dx^i \otimes dx^j = \left[\begin{array}{cc|cc|c} 1+4(y^1)^2 & & & & -2y^1 \\ & & 0 & & \\ & 1 & & & 0 \\ \hline & & 1 & & \\ 0 & & & & 0 \\ & & & 1 & \\ \hline -2y^1 & 0 & 0 & & 1 \end{array}\right].$$

Then F has the form $F = \sum_{i=1}^{2} dx^i \wedge dy^i$ and $d\eta$ has the form $d\eta = 2dx^1 \wedge dy^1$, which shows that $\eta \wedge d\eta \neq 0$ and $(d\eta)^2 = 0$. The rank of η is equal to 3. (This σ is a quasi-Sasakian structure on $\boldsymbol{R}^5$ which is non-cosymplectic and non-Sasakian.)

(2) A relation between an almost contact metric structure σ on M and an almost Hermitian structure (J_0, G) on $M \times \boldsymbol{R}$

This section is devoted to introduce a notion of normality for an almost contact metric structure σ on M ([22]). We shall prove the following assertion first.

THEOREM 2.1. *In order that a manifold M admit an almost contact metric structure σ, it is necessary and sufficient that the product manifold $M \times \boldsymbol{R}$ of M and a real line $\boldsymbol{R}$ admits an almost Hermitian structure (J_0, G) such that the vector $J_0(d/dt)$ is always tangent to the hypersurface M for the unit vector d/dt on $\boldsymbol{R}$.*

Proof. Suppose that M admits an almost contact metric structure $\sigma = (f, E, \eta, g)$. Then an almost Hermitian structure (J_0, G) can be defined by

$$J_0(X, \lambda d/dt) = (fX - \lambda E, \eta(X) d/dt),$$
$$G = g + dt \otimes dt,$$

which means

$$G((X, \lambda d/dt), (Y, \mu d/dt)) = g(X, Y) + \lambda\mu$$

for any vector fields $(X, \lambda d/dt)$ and $(Y, \mu d/dt)$ on $M \times \boldsymbol{R}$. In fact, the set (J_0, G) consists of an almost complex structure J_0 and a positive definite Riemannian metric G such that

$$G(J_0(X, \lambda d/dt), (Y, \mu d/dt)) = -G((X, \lambda d/dt), J_0(Y, \mu d/dt)),$$

and, in addition, $J_0(0, \lambda d/dt) = -(\lambda E, 0)$. Conversely, suppose that $M \times \boldsymbol{R}$ admits an almost Hermitian structure (J_0, G) such that $J_0(d/dt) \in T_{(x,t)}(M \times \{t\})$ in $T_{(x,t)}(M \times \boldsymbol{R})$ at any point (x, t) of $M \times \boldsymbol{R}$. Denoting by i_t the natural imbedding map $i_t\colon M \to M \times \boldsymbol{R}$ through a point $(x, t) \in M \times \boldsymbol{R}$ and by i_* its differential, we can write the decomposition law for $J_0 i_* X$ and $J_0(d/dt)$ as

$$J_0 i_* X = i_* fX + \eta(X)(d/dt), \qquad J_0(d/dt) = -i_* E,$$

for any vector field X on M, where f is a linear transformation field, η is a 1-form, E is a global vector field on M. Further, we obtain the induced metric g given by

$$g(X, Y) = G(i_*X, i_*Y)$$

for any $X, Y \in \mathscr{X}(M)$. The assumption implies that the set (f, E, η, g) of the induced tensor fields defines an almost contact metric structure on M. ∎

An almost contact metric structure $\sigma = (f, E, \eta, g)$ is called *normal* if the almost complex structure J_0 on $M \times \boldsymbol{R}$ mentioned above is integrable ([22]). By the following assertion we can express the normality for σ in terms of the tensor field $[f,f]+d\eta \otimes E$ on M, which is called the *torsion tensor field* of σ.

THEOREM 2.2. *A necessary and sufficient condition for an almost contact metric structure $\sigma = (f, E, \eta, g)$ to be normal is that the torsion tensor field $[f,f]+d\eta\otimes E$ of σ vanishes identically, where $[f,f]$ denotes the Nijenhuis tensor field of f, given by*

$$[f,f](X, Y) = [fX,fY]-f[fX, Y]-f[X,fY]+f^2[X, Y]$$

for any vector fields X and Y on M.

Proof. By Theorem 2.1 an almost contact metric structure σ determines the almost Hermitian structure (J_0, G) on the product manifold $M\times \boldsymbol{R}$, on which the identity equations

$$[J_0, J_0](i_*X, i_*Y) = i_*([f,f]+d\eta\otimes E)(X, Y)+\{d\eta(fX, Y)+d\eta(X,fY)\}d/dt,$$
$$[J_0, J_0](i_*X, d/dt) = i_*(L_E f)X+(L_E\eta)(X)d/dt$$

hold for any vector fields X and Y on M, in general. We state its verification first. For the sake of a simpler notation, we use the alternating summation $A_{X,Y}$ with respect to X and Y. The Nijenhuis tensor field $[J_0, J_0]$ of J_0 is given by

$$[J_0, J_0]\,(X, Y) = [J_0X, J_0Y]-A_{X,Y}J_0[J_0X, Y]-[X, Y]$$

for $X, Y \in \mathscr{X}(M\times \boldsymbol{R})$. For any vector fields i_*X and i_*Y tangent to the hypersurface M it follows that

$$\begin{aligned}[J_0, J_0](i_*X, i_*Y) &= [i_*fX+\eta(X)d/dt,\, i_*fY+\eta(Y)d/dt]-\\ &\quad -A_{X,Y}J_0[i_*fX+\eta(X)d/dt,\, i_*Y]-i_*[X, Y]\\ &= i_*[fX,fY]+A_{X,Y}[i_*fX,\, \eta(Y)d/dt]+[\eta(X)d/dt,\, \eta(Y)d/dt]-\\ &\quad -A_{X,Y}\{i_*f[fX, Y]+\eta([fX, Y])d/dt+J_0[\eta(X)d/dt,\, i_*Y]\}-\\ &\quad -i_*[X, Y].\end{aligned}$$

On the other hand, the vector fields $[i_*fX, \eta(Y)d/dt]$ and $J_0[\eta(X)d/dt, i_*Y]$ described in the above reduce to

$$[i_*fX, \eta(Y)d/dt] = \big(fX(\eta(Y))\big)d/dt+\eta(Y)\,[i_*fX, d/dt] = \{(\nabla_{fX}\eta)(Y)+\eta(\nabla_{fX}Y)\}d/dt,$$
$$J_0[\eta(X)d/dt, i_*Y] = \{(\nabla_Y\eta)(X)+\eta(\nabla_YX)\}i_*E,$$

respectively, and $[\eta(X)d/dt, \eta(Y)d/dt] = 0$ holds identically. Hence

$$\begin{aligned}[J_0, J_0](i_*X, i_*Y) &= i_*\{[fX, fY]-A_{X,Y}f[fX, Y]-[X, Y]\}+A_{X,Y}\{(\nabla_{fX}\eta)(Y)+\\ &\quad+\eta(\nabla_{fX}Y)\}d/dt-A_{X,Y}\{\eta(\nabla_{fX}Y)-\eta(\nabla_Y fX)\}d/dt-\\ &\quad-A_{X,Y}\{(\nabla_Y\eta)(X)+\eta(\nabla_Y X)\}i_*E\\ &= i_*\{[f,f](X, Y)+((\nabla_X\eta)(Y)-(\nabla_Y\eta)(X))E\}+\\ &\quad+A_{X,Y}\{(\nabla_{fX}\eta)(Y)-(\nabla_Y\eta)(fX)\}d/dt.\end{aligned}$$

Consequently, we obtain the first desired expression. Similarly, we can reduce $[J_0, J_0](i_*X, d/dt)$ to

$$\begin{aligned}[J_0, J_0](i_*X, d/dt) &= [i_*fX+\eta(X)d/dt, -i_*E]-\\ &\quad-J_0[i_*fX+\eta(X)d/dt, d/dt]+J_0[i_*X, i_*E]\\ &= i_*[E, fX]+[i_*E, \eta(X)d/dt]+i_*f[X, E]+\eta([X, E])d/dt\\ &= i_*(L_E f)X+(L_E\eta)(X)d/dt.\end{aligned}$$

Suppose now that σ is normal. Then we have the vanishing of the torsion tensor field, $[f,f]+d\eta\otimes E = 0$. Conversely, suppose that

$$[f,f](X, Y)+d\eta(X, Y)E = 0$$

holds for any vector fields X and Y on M. We first substitute $Y = E$ into this equation, and hence we get $d\eta(X, E) = 0$, from which $\nabla_E\eta = 0$ since $(\nabla_X\eta)(E) = g(\nabla_X E, E) = 0$. Therefore,

$$(L_E\eta)(X) = (\nabla_E\eta)(X)+\eta(\nabla_X E) = 0,$$

that is, $L_E\eta = 0$. Since $[f, f](X, E) = 0$, it follows that $f(L_E f)X = 0$ holds. Applying L_E to $f^2X = -X+\eta(X)E$, we attain to $(L_E f)fX = 0$ and then $(L_E f)f^2X = 0$. On the other hand, operating L_E to $fE = 0$, we have $(L_E f)E = 0$. Thus we obtain $L_E f = 0$. The condition

$$\eta([f,f](X, Y))+d\eta(X, Y) = 0$$

implies that

$$\eta([f^2X, fY])+d\eta(fX, Y) = 0.$$

By means of $L_E f = 0$ we come to the desired result:

$$d\eta(fX, Y)+d\eta(X, fY) = 0.$$

Thus we can conclude that J_0 is integrable. ■

We shall deal with the verification of three structures described in Examples 1, 2 and 3 to be normal. Actually we use the following expression of the torsion tensor field of σ:

$$[f,f]+d\eta\otimes E = \sum_{i,j,k}([f,f]^h_{ij}+(d\eta)_{ij}E^h)\,dx^i\otimes dx^j\otimes\partial_h,$$

where

$$[f,f]^h_{ij} = \sum_t\{f^t_i\partial_t f^h_j-f^t_j\partial_t f^h_i+f^h_t\partial_j f^t_i-f^h_t\partial_i f^t_j\},$$

$$(d\eta)_{ij}E^h = (\partial_i\eta_j-\partial_j\eta_i)E^h;$$

in this case, we have put $\partial_h = \partial/\partial x^h$ and have used (f_i^h), (η_i), (E^h) as the components of f, η, E, respectively, relative to a coordinate neighborhood $(U_\alpha, (x^i))$. In fact, $[fX, fY]$ reduces to

$$[fX, fY] = \sum_{i,j,t,s} \{f_t^j X^t \partial_j(f_s^i Y^s) - f_s^j Y^s \partial_j(f_t^i X^t)\} \partial_i$$

$$= \sum \{f_t^j(\partial_j f_s^i) X^t Y^s + f_t^j f_s^i X^t \partial_j Y^s - f_s^j(\partial_j f_t^i) X^t Y^s - f_s^j f_t^i Y^s \partial_j X^t\} \partial_i .$$

Accordingly, we attain to the first desired expression $[f, f]_{tj}^h$ by the definition of Lie bracket multiplication. And

$$d\eta(X, Y) = \sum_{i,j} \{X^i \partial_i(\eta_j Y^j) - Y^j \partial_j(\eta_i X^i) - \eta_i(X^j \partial_j Y^i - Y^j \partial_j X^i)\}$$

$$= \sum (\partial_i \eta_j - \partial_j \eta_i) X^i Y^j = \left(\sum (\partial_i \eta_j - \partial_j \eta_i) dx^i \otimes dx^j\right)(X, Y).$$

Thus we have checked that the components $[f, f]_{ij}{}^h + (d\eta)_{ij} E^h$ of $[f, f] + d\eta \otimes E$ can be written as in the above. Now we state the verification of normality for σ in Example 2. The facts that the two remainder examples are also normal are convinced by the similar way. The components (f_i^h) have been given by

$$f_3{}^1 = f_4{}^2 = -1, \quad f_1{}^3 = f_2{}^4 = 1, \quad f_3{}^5 = -2x^3, \quad f_4{}^5 = -2x^4,$$

the other f_i^h vanish, hence, $\partial_3 f_3{}^5 = \partial_4 f_4{}^5 = -2$ and the other $\partial_j f_i{}^h$ vanish. Furthermore, $\partial_3 \eta_1 = \partial_4 \eta_2 = -2$ and the other $\partial_i \eta_j$ vanish. Consequently, we come to the vanishing of the torsion tensor field of σ, which shows that σ in Example 2 is Sasakian.

(3) Quasi-Sasakian structures

Let us recall the following illustration of structures. We can easily see the correspondences among almost contact metric structures in the study of odd-dimensional manifolds and almost Hermitian structures in the study of even-dimensional manifolds.

Almost Hermitian structure on P	→	Hermitian structure	→	Kähler structure		
↑ ⋮ ↓		↑ ⋮ ↓		↑ ⋮ ↓		
Almost contact metric structure on M	→	Normal almost contact metric structure	→	Quasi-Sasakian structure	{	→ Sasakian structure → Cosymplectic structure

The arrows mean the specializations. Namely, an *almost Hermitian structure* (J, G) on a differentiable manifold P is a set consisting of an almost complex structure J and a positive definite Riemannian metric G such that $G(JX, Y) = -G(X, JY)$

for any vector fields X and Y on P. An *Hermitian structure* is an almost Hermitian structure whose almost complex structure is integrable. A *Kähler structure* is an Hermitian structure such that the fundamental 2-form Ω, given by $\Omega(X, Y) = G(JX, Y)$, is closed. By these definitions we may consider an almost contact metric structure, a normal almost contact metric structure or a quasi-Sasakian structure as analogous notion according as an almost Hermitian structure, an Hermitian structure or a Kähler structure.

A $(2n+1)$-dimensional manifold is said to have a *contact structure* if it carries a 1-form η with the property $\eta \wedge (d\eta)^n \neq 0$ ([4]). It is known that on a contact manifold there exists an almost contact metric structure such that $d\eta = 2F$ ([20], [22], [23]). A normal almost contact metric structure σ is called a *Sasakian structure* (resp. a *cosymplectic structure*) if $d\eta = 2F$ (resp. $dF = 0$ and $d\eta = 0$).

In general, on an almost Hermitian manifold $P(J, G)$ the Nijenhuis tensor field $[J, J]$ of J is written as

$$[J, J](X, Y) = (\nabla_{JX}J)Y-(\nabla_{JY}J)X-J(\nabla_X J)Y+J(\nabla_Y J)X$$

for any vector fields X and Y on P with respect to the Riemannian connection ∇ of the metric G. Since

$$(\nabla_X \Omega)(Y, Z) = G\big((\nabla_X J)Y, Z\big)$$

and

$$d\Omega(X, Y, Z) = (\nabla_X \Omega)(Y, Z)+(\nabla_Y \Omega)(Z, X)+(\nabla_Z \Omega)(X, Y),$$

then we have the identity equation

$$G\big([J, J](X, Y), Z\big) = d\Omega(JX, Y, Z)+d\Omega(X, JY, Z)-2(\nabla_Z \Omega)(JX, Y).$$

This expression is used to see the familiar result of a characterization of a Kähler structure. The assertion says that a necessary and sufficient condition for an almost Hermitian structure (J, G) on a manifold P to be Kählerian is that $\nabla J = 0$ holds identically with respect to the Riemannian connection ∇ of the metric G.

According to this discussion in the study of Kähler manifolds, we can obtain a characterization of a quasi-Sasakian structure on an odd-dimensional manifold as follows.

THEOREM 3.1 [10]. *A necessary and sufficient condition for an almost contact metric structure $\sigma = (f, E, \eta, g)$ on a manifold M to be quasi-Sasakian is that there exists a linear transformation field A on M such that*

$$(\nabla_X f)Y = \eta(Y)AX-g(AX, Y)E, \quad fAX = AfX, \quad g(AX, Y) = g(X, AY)$$

for any vector fields X and Y on M, where ∇ denotes the Riemannian connection of the metric g.

Proof. We first prove that the following relation holds on $M(\sigma)$ in general

$$g\big([f,f](X, Y)+d\eta(X, Y)E, Z\big) = dF(fX, Y, Z)+dF(X,fY, Z)-$$
$$-2(\nabla_Z F)(fX, Y)-\eta(Y)d\eta(X, Z)+\eta(X)(L_E g)(Y, Z).$$

In fact, the torsion tensor field of σ is written as

$$[f,f](X, Y)+d\eta(X, Y)E = A_{X,Y}\{(\nabla_{fX}f)Y-f(\nabla_X f)Y+(\nabla_X\eta)(Y)E\}.$$

Applying the covariant differential operator ∇_X to $f^2Y = -Y+\eta(Y)E$, we have

$$(\nabla_X f)fY+f(\nabla_X f)Y = (\nabla_X\eta)(Y)E+\eta(Y)\nabla_X E.$$

Since the relations $(\nabla_X\eta)(Y) = g(\nabla_X E, Y)$, $(\nabla_X F)(Y, Z) = g((\nabla_X f)Y, Z)$ and $(\nabla_X F)(Y, Z) = -(\nabla_X F)(Z, Y)$ are valid, then

$$\begin{aligned} g([f,f](X, Y)&+d\eta(X, Y)E, Z) \\ &= A_{X,Y}\{(\nabla_{fX}F)(Y, Z)+(\nabla_Y F)(Z, fX)+(\nabla_Z F)(fX, Y)\}- \\ &\quad -\eta(Y)(\nabla_X\eta)(Z)+\eta(X)(\nabla_Y\eta)(Z)-(\nabla_Z F)(fX, Y)+(\nabla_Z F)(fY, X). \end{aligned}$$

The last term $(\nabla_Z F)(fY, X)$ in the right-hand side of this equation reduces to

$$(\nabla_Z F)(fY, X) = -(\nabla_Z F)(fX, Y)+\eta(X)(\nabla_Z\eta)(Y)+\eta(Y)(\nabla_Z\eta)(X).$$

Using the identity $(L_E g)(X, Y) = g(\nabla_E X, Y)+g(\nabla_E Y, X)$, consequently, we obtain the desired expression.

Suppose now that σ is a quasi-Sasakian structure on M. Then by the above identity it follows that

$$2(\nabla_Z F)(fX, Y) = \eta(X)(L_E g)(Y, Z)+\eta(Y)d\eta(Z, X).$$

We make a substitution $X = E$ in this and then have $L_E g = 0$, which shows that E is a Killing vector field. Hence, $2(\nabla_Z F)(fX, Y) = \eta(Y)d\eta(Z, X)$, in which we replace X, Y and Z by $-fY$, Z and X, respectively. It follows that

$$2(\nabla_X F)(Y, Z) = 2\eta(Y)(\nabla_X F)(E, Z)+\eta(Z)d\eta(fY, X).$$

Since E is a Killing vector field, that is, $(\nabla_X\eta)(Y) = -(\nabla_Y\eta)(X)$, this leads to

$$(\nabla_X f)Y = \eta(Y)(\nabla_X f)E+g(f\nabla_X E, Y)E.$$

We now define a linear transformation field $\underline{A}$ on M by

$$\underline{A}X = -f\nabla_X E$$

for any vector field X. Hence, $(\nabla_X f)Y = \eta(Y)\underline{A}X-g(\underline{A}X, Y)E$ is obtained. Since the torsion tensor field of σ vanishes, we have $\nabla_E\eta = 0$, that is, $\nabla_E E = 0$, which shows that the integral curve of E is geodesic. Hence, $\underline{A}E = 0$. It follows that $\nabla_E f = 0$. Then the relation:

$$0 = (L_E f)X = [E, fX]-f[E, X] = (\nabla_E f)X-\nabla_{fX}E+f\nabla_X E$$

implies that $\nabla_{fX}E = f\nabla_X E$. Consequently, $\underline{A}$ commutes with f. Further, we have

$$g(\underline{A}X, Y) = g(\nabla_X E, fY) = -g(\nabla_{fY}E, X) = g(X, \underline{A}Y)$$

by using the fact that E is a Killing vector field.

Conversely, suppose that there exists such a linear transformation field A described in the present assertion on an almost contact metric manifold $M(\sigma)$. It is true that F is closed. The condition:

$$(\nabla_X f)fY+f(\nabla_X f)Y = (\nabla_X\eta)(Y)E+\eta(Y)\nabla_X E$$

implies that $\nabla_X E = fAX$ by substituting $Y = E$ into this equation. The torsion tensor field $[f,f]+d\eta\otimes E$ must necessarily vanish identically. Thus σ defines a quasi-Sasakian structure. ∎

We shall investigate the rank of the linear transformation field A described in Theorem 3.1. In general, A is written in the form $A = \underline{A}+\eta(AE)\eta\otimes E$, which follows from $(\nabla_X f)E = AX-\eta(AX)E$. Since $g(\underline{A}X, \eta(Y)E) = 0$ holds for any vector fields X and Y, the subspaces $\underline{A}(T_x(M))$ and $\eta(T_x(M))E$ are orthogonal in the tangent space $T_x(M)$ to M at each point x. This shows that the space $\underline{A}(T_x(M))$ is always contained in the value $-f^2(T_x(M))$ at x of the $2n$-dimensional distribution determined by the projector $-f^2$ on M of dimension $2n+1$. We can conclude that $\operatorname{rank} A$ satisfies

$$\text{even} = \operatorname{rank}\underline{A} \leqslant \operatorname{rank} A \leqslant \operatorname{rank}\underline{A}+1$$

at every point on M. In fact, we take a non-zero element X of the intersection of the kernel of $\underline{A}$ and the subspace $-f^2(T_x(M))$ if rank $\underline{A} \neq 0$. Then the vector fX is also non-zero and belongs to the intersection of both subspaces in $T_x(M)$ itself, because $\underline{A}fX = f\underline{A}X = 0$ and $fX = -f^2(fX)$. This process gives rise to rank $\underline{A}$ = even. Now we define a linear transformation field $\bar{A}$ on M by

$$\bar{A} = \underline{A}+\eta\otimes E,$$

which we shall call the *indicator tensor field of the quasi-Sasakian structure* σ. Then the rank of $\bar{A}$ has the form: $2p+1$ $(0 \leqslant p \leqslant n)$. In the study of quasi-Sasakian manifolds, a Sasakian structure and a cosymplectic structure are characterized as stated in the following two theorems.

THEOREM 3.2. *A necessary and sufficient condition for a quasi-Sasakian structure σ on a manifold M to be Sasakian is that the indicator tensor field $\bar{A}$ of σ coincides with the identity I, in this case,* rank $\bar{A} = 2n+1$.

Proof. Suppose that σ is Sasakian. The 2-form $d\eta$ reduces to

$$d\eta(X, Y) = g(\nabla_X E, Y)-g(\nabla_Y E, X) = 2g(fAX, Y) = 2g(f\bar{A}X, Y).$$

Then we have $\bar{A} = I$. The converse is also true. ∎

THEOREM 3.3. *A necessary and sufficient condition for a quasi-Sasakian structure σ on a manifold M to be cosymplectic is that the indicator tensor field $\bar{A}$ of σ coincides with $\eta\otimes E$, in this case,* rank $\bar{A} = 1$.

Proof. Similar to that of Theorem 3.2.

The characterization theorem (Theorem 3.1) is applied to prove the following result in the study of quasi-Sasakian manifolds.

THEOREM 3.4. *Let $\bar{A}$ be the indicator tensor field of a quasi-Sasakian structure σ on a manifold M of dimension $2n+1$. If $\bar{A}$ is parallel and has a constant rank $2p+1$ $(1 \leqslant p \leqslant n-1)$ on M, then the manifold M is locally a product manifold of a Sasakian manifold of dimension $2p+1$ and a Kähler manifold of dimension $2q$, where $p+q = n$.*

Proof. The covariant derivative $\nabla_X\bar{A}$ of $\bar{A}$ reduces to

$$(\nabla_X\bar{A})Y = g(\bar{A}X, f\bar{A}Y)E+(\nabla_X\eta)(Y)E-f\nabla_X\nabla_Y E+f\nabla_{\nabla_X Y}E+\eta(Y)\nabla_X E.$$

Hence, we have the identity $g((\nabla_X\bar{A})Y, E) = g(f(\bar{A}-\bar{A}^2)X, Y)$. Under the present assumption we obtain $\bar{A}^2 = \bar{A}$. When we put $\bar{B} = I-\bar{A}$, then we get that $\bar{A}$ and $\bar{B}$ are projectors of an almost product structure $D = \{D(\bar{A}), D(\bar{B})\}$ consisting of two complementary $(2p+1)$-dimensional distribution $D(\bar{A})$ and $2q$-dimensional distribution $D(\bar{B})$, since $\bar{B}^2 = \bar{B}$, $\bar{A}+\bar{B} = I$, $\bar{A}\bar{B} = \bar{B}\bar{A} = 0$, $g(\bar{A}X, Y) = g(X, \bar{A}Y)$, $g(\bar{B}X, Y) = g(X, \bar{B}Y)$ for any X and Y on $M(\sigma)$. Since these projectors are parallel, both distributions are completely integrable. Denoting the maximal integral manifolds passing through a point of M corresponding to $D(\bar{A})$ and $D(\bar{B})$ by N_1 and N_2, respectively, we can induce naturally a Sasakian structure on N_1 and a Kähler structure on N_2 from the given quasi-Sasakian structure. In fact, we let f_a $(a = 1, 2)$ and g_a represent the restrictions of f and g on N_a and we use the same symbol ∇ as the induced connections with respect to g_a on N_a. By Theorem 3.1 for any vector fields X and Y belonging to the distribution $D(\bar{A})$ the relations

$$(\nabla_X f_1)Y = \eta(Y)X-g_1(X, Y)E, \quad g_1(f_1X, Y) = -g_1(X, f_1Y), \quad E\in D(\bar{A})$$

hold on N_1, and for any vector fields X and Y belonging to $D(\bar{B})$ the relations

$$(\nabla_X f_2)Y = 0, \quad g_2(f_2X, Y) = -g_2(X, f_2Y)$$

hold on N_2. Consequently, the sets of the tensor fields (f_1, E, η, g_1) restricted to N_1 and (f_2, g_2) restricted to N_2 define the desired structures. ∎

(4) Product manifolds

Let M and N be two differentiable manifolds of dimensions m and n, respectively. The product manifold $M\times N$ is the Hausdorff space $M\times N$ provided with the differentiable atlas $\{(U_\alpha\times V_\lambda, \phi_\alpha\times\psi_\lambda)\}_{(\alpha,\lambda)\in A\times B}$ such that every $\phi_\alpha\times\psi_\lambda$ is a homeomorphism of $U_\alpha\times V_\lambda$ onto a product of an open set in $\boldsymbol{R}^m$ and an open set in $\boldsymbol{R}^n$, in $\boldsymbol{R}^m\times\boldsymbol{R}^n = \boldsymbol{R}^{m+n}$, formed with an atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ for M and an atlas $\{(V_\lambda, \psi_\lambda)\}_{\lambda\in B}$ for N. In this case, the coordinate transformation satisfies

$$\frac{\partial(x^i, z^a)}{\partial(y^i, w^a)} = \begin{bmatrix} \dfrac{\partial x^i}{\partial y^j} & 0 \\ 0 & \dfrac{\partial z^a}{\partial w^b} \end{bmatrix}$$

on any non-empty intersection $(U_\alpha\times V_\lambda)\cap(U_\beta\times V_\mu)$ for coordinate neighborhoods $(U_\alpha, (x^i))$, $(U_\beta, (y^i))$ of M and $(V_\lambda, (z^a))$, $(V_\mu, (w^a))$ of N. Take an arbitrary point (x, y) of $M\times N$ and fix it in mind. Then we have the natural imbedding maps:

$$i_{1y}\colon M\to M\times\{y\} \quad\text{in}\quad M\times N, \qquad i_{2x}\colon N\to \{x\}\times N \quad\text{in}\quad M\times N.$$

We shall write both merely i_1, i_2 instead of i_{1y}, i_{2x}, respectively. Let i_{a*} $(a = 1, 2)$ be the differentials of i_a and i_a^* their duals. Further, we consider the natural projections

$$\pi_1\colon M\times N\to M, \qquad \pi_2\colon M\times N\to N$$

and, in like manner, π_{a*} denote the differentials of π_a and π_a^* their duals. Therefore, we have the well-known Leibniz formula:

$$X = i_{1*}\pi_{1*}X + i_{2*}\pi_{2*}X$$

for any vector $X \in T_{(x,y)}(M \times N)$ at every point (x, y) of $M \times N$. A 1-form $\omega \in A^1(M \times \times N)$ can be written as

$$\omega(X) = \omega(i_{1*}\pi_{1*}X) + \omega(i_{2*}\pi_{2*}X) = (\pi_1^* i_1^* \omega + \pi_2^* i_2^* \omega)(X)$$

for any $X \in \mathfrak{X}(M \times N)$, in this case, $i_1^*\omega \in A^1(M)$ and $i_2^*\omega \in A^1(N)$ hold.

Define two multilinear mappings $\varrho_{as}^{\,r}$ $(a = 1, 2)$ of $T_s^r(M_a)$ into $T_s^r(M_1 \times M_2)$, where $M_1 = M$ and $M_2 = N$ by

$$\varrho_{as}^{\,r} = \underbrace{\pi_a^* \otimes \dots \otimes \pi_a^*}_{s \text{ factors}} \otimes \underbrace{i_{a*} \otimes \dots \otimes i_{a*}}_{r \text{ factors}} \quad \text{for} \quad (r, s) \neq (0, 0),$$

$$\varrho_{a0}^{\,0}(\lambda) = \lambda \circ \pi_a \quad \text{for any scalar field } \lambda \text{ on } M_a.$$

Then the set $\sum_{a=1}^{2} \varrho_{as}^{\,r}(T_s^r(M_a))$ forms the submodule of the module $T_s^r(M_1 \times M_2)$. The product manifold $M_1 \times M_2$ of any two Riemannian manifolds $M_a(g_a)$ is a Riemannian manifold. In fact, $M_1 \times M_2$ admits a positive definite Riemannian metric G defined by $G = \varrho_{12}^{\,0}(g_1) + \varrho_{22}^{\,0}(g_2)$, which shows also that $G(X, Y) = g_1(X_1, Y_1) + g_2(X_2, Y_2)$ for any $X, Y \in \mathfrak{X}(M_1 \times M_2)$ in consideration of the decomposition law for $X = i_{1*}X_1 + i_{2*}X_2$ and $Y = i_{1*}Y_1 + i_{2*}Y_2$, because $\pi_{a*}i_{a*}$ = (the identity) and $\pi_{a*}i_{b*} = 0$ $(a \neq b)$. Therefore, any vector fields $i_{1*}\pi_{1*}X$ and $i_{2*}\pi_{2*}Y$ are mutually orthogonal with respect to the metric G.

Let $M_a(J_a, g_a)$ $(a = 1, 2)$ denote any two almost Hermitian manifolds of dimensions m_1 and m_2. We may write any vector field X on $M_1 \times M_2$ as $X = \sum_{a=1}^{2} i_{a*}X_a$ for $X_a \in \mathfrak{X}(M_a)$ and since there is no fear of confusion, we abbreviate $\varrho_{as}^{\,r}$ to ϱ. Define a linear transformation field J and a (0, 2)-type tensor field g on $M_1 \times M_2$ by $J = \varrho(J_1) + \varrho(J_2)$ and $g = \varrho(g_1) + \varrho(g_2)$, respectively. Namely,

$$J = \left[\begin{array}{c|c} (J_1)_i^j & 0 \\ \hline 0 & (J_2)_a^b \end{array}\right], \qquad g = \left[\begin{array}{c|c} (g_1)_{ij} & 0 \\ \hline 0 & (g_2)_{ab} \end{array}\right],$$

at a point of $M_1 \times M_2$ with respect to a coordinate system (x^i, z^a). Then the set (J, g) of tensor fields defines an almost Hermitian structure on $M_1 \times M_2$. In fact, JX has the form: $JX = \sum_{a=1}^{2} i_{a*}J_aX_a$, hence $J^2 = -I$. In addition,

$$g(JX, Y) = \sum_{a=1}^{2} g_a(J_aX_a, Y_a) = -g(X, JY).$$

Consequently, *the product manifold of any two almost Hermitian manifolds is an almost Hermitian manifold.* Since the differentials i_{a*} of i_a are homomorphisms of Lie algebras: $[i_{a*}X_a, i_{a*}Y_a] = i_{a*}[X_a, Y_a]$ and $[i_{a*}X_a, i_{b*}Y_b] = 0$ $(a \neq b)$ are valid for any $X_a, Y_a \in \mathfrak{X}(M_a)$, then the Nijenhuis tensor field of J can be expressed as

$$[J, J](X, Y) = \sum_{a=1}^{2} i_{a*}[J_a, J_a](X_a, Y_a),$$

where $X_a = \pi_{a*}X$. Thus we obtain that *the product manifold of any two Hermitian manifold is an Hermitian manifold.* The exterior differential operator d commutes with every multilinear mapping

$$\varrho_{ap}^{0}(=\pi_a^*\otimes \ldots \otimes\pi_a^*)\colon A^p(M_a) \to A^p(M_1 \times M_2).$$

Hence, if the fundamental 2-forms Ω_a of (J_a, g_a) are both closed, then the exterior derivative $d\Omega$ of the fundamental 2-form Ω of (J, g) vanishes. Thus we can say that *the product manifold of any two Kähler manifolds is a Kähler manifold.*

We shall prove the following theorem.

THEOREM 4.1. (1) *The product manifold of an almost contact metric manifold and an almost Hermitian manifold is an almost contact metric manifold.*

(2) *The product manifold of a normal almost contact metric manifold and an Hermitian manifold is a normal almost contact metric manifold.*

(3) *The product manifold of a quasi-Sasakian manifold and a Kähler manifold is a quasi-Sasakian manifold.*

(4) *The product manifold of a cosymplectic manifold and a Kähler manifold is a cosymplectic manifold.*

Proof. Let $M_1(f_1, E_1, \eta^1, g_1)$ be an almost contact metric manifold and let $M_2(f_2, g_2)$ be an almost Hermitian manifold. We set $f = \varrho(f_1)+\varrho(f_2)$, $E = \varrho(E_1)$, $\eta = \varrho(\eta^1)$, $g = \varrho(g_1)+\varrho(g_2)$. Then (f, E, η, g) defines an almost contact metric structure on $M_1 \times M_2$. The torsion tensor field of the structure constructed here can be reduced to

$$([f,f]+d\eta\otimes E)(X, Y) = \sum_{a=1}^{2} i_{a*}[f_a,f_a](X_a, Y_a)+d\eta^1(X_1, Y_1)i_{1*}E_1$$

for any $X, Y \in \mathfrak{X}(M_1 \times M_2)$, where $X_a = \pi_{a*}X$. The exterior derivative dF of the fundamental 2-form F vanishes if $dF_a = 0$ $(a = 1, 2)$, where F_1, F_2 denote the fundamental 2-forms of (f_1, E_1, η^1, g_1), (f_2, g_2), respectively. ∎

THEOREM 4.2. (1) *The product manifold of any two almost contact metric manifolds is an almost Hermitian manifold.*

(2) *The product manifold of any two normal almost contact metric manifolds is a Hermitian manifold.*

(3) *The product manifold of any two cosymplectic manifolds is a Kähler manifold.*

Proof. Let $M_a(f_a, E_a, \eta^a, g_a)$ $(a = 1, 2)$ be two almost contact metric manifolds.

We set $f = \sum_a \varrho(f_a)$, $g = \sum_a \varrho(g_a)$, $E_a = \varrho(E_a)$, $\eta^a = \varrho(\eta^a)$. Then the set (J, g) consisting of a linear transformation field J given by

$$J = f + \eta^1 \otimes E_2 - \eta^2 \otimes E_1$$

and the Riemannian metric g defines an almost Hermitian structure on $M_1 \times M_2$. The Nijenhuis tensor field $[J, J]$ of J has the form:

$$\begin{aligned}[J, J](X, Y) = \sum_{a=1}^{2} i_{a*}(([f_a, f_a] + d\eta^a \otimes E_a)(X_a, Y_a)) - \{d\eta^2(f_2 X_2, Y_2) + \\ + d\eta^2(X_2, f_2 Y_2)\} E_1 + \{d\eta^1(f_1 X_1, Y_1) + d\eta^1(X_1, f_1 Y_1)\} E_2 + \\ + A_{X,Y}[\eta^1(X_1)\{(L_{E_2} f_2) Y_2 - (L_{E_2} \eta^2)(Y_2) E_1\} - \\ - \eta^2(X_2)\{(L_{E_1} f_1) Y_1 + (L_{E_1} \eta^1)(Y_1) E_2\}]\end{aligned}$$

for any vector fields X and Y on $M_1 \times M_2$, where $X_a = \pi_{a*} X$. The vanishing of $[J, J]$ follows from the fact that $[f_a, f_a] + d\eta^a \otimes E_a = 0$ $(a = 1, 2)$. The exterior derivative $d\Omega$ of Ω $(= F + \eta^1 \wedge \eta^2)$ of (J, g) is written as $d\Omega = dF + d\eta^1 \wedge \eta^2 - d\eta^2 \wedge \eta^1$. ∎

We note here that, in particular, the product manifold $S^{2p+1} \times S^{2q+1}$ of two any odd-dimensional spheres except for $(p, q) = (0, 0)$ is a well-known example of a Hermitian manifold which is not Kählerian. Since a $(2p+1)$-sphere S^{2p+1} $(p \geqslant 1)$ admits a Sasakian structure, applying our argument to the manifold $S^{2p+1} \times S^{2q+1}$, we can see that $S^{2p+1} \times S^{2q+1}$ is a Hermitian manifold on which the exterior derivative $d\Omega$ of the fundamental 2-form Ω has the form: $d\Omega = 2(F_1 \wedge \eta^2 - F_2 \wedge \eta^1)$ and, furthermore, $d\Omega$ reduces to $d\Omega = -2\eta^1 \wedge \Omega$ on $S^1 \times S^{2q+1}$.

(5) A parallelizable Riemannian manifold M^3 of dimension 3

We denote by M^3 a parallelizable Riemannian manifold of dimension 3 in this section. We take a global field of orthonormal 3-frames $\{E_1, E_2, E_3\}$ on M^3 and denote by $\{\eta^1, \eta^2, \eta^3\}$ its dual frame field with respect to a Riemannian metric g, $\eta^a(X) = g(X, E_a)$ for any $X \in \mathfrak{X}(M^3)$. Then we obtain a triple $\{(f_a, E_a, \eta^a, g)\}$ of almost contact metric structures on M^3 which are given by the relations

$$f_1 = \eta^2 \otimes E_3 - \eta^3 \otimes E_2, \quad f_2 = \eta^3 \otimes E_1 - \eta^1 \otimes E_3, \quad f_3 = \eta^1 \otimes E_2 - \eta^2 \otimes E_1.$$

THEOREM 5.1. *Suppose that any two, say (f_1, E_1, η^1, g) and (f_2, E_2, η^2, g), of the triple $\{(f_a, E_a, \eta^a, g)\}$ of almost contact metric structures on a parallelizable Riemannian manifold M^3 of dimension 3 are quasi-Sasakian. The remaining structure (f_3, E_3, η^3, g) is also quasi-Sasakian if and only if both indicator tensor fields $\bar{A}_i$ $(i = 1, 2)$ of (f_i, E_i, η^i, g) satisfy a condition $\bar{A}_1 E_3 = \bar{A}_2 E_3$.*

Proof. Suppose that $\bar{A}_1 E_3 = \bar{A}_2 E_3$. The covariant derivative of f_3 defined above satisfies

$$(\nabla_X f_3) Y = \eta^3(Y)\{\eta^1(\bar{A}_2 X) E_1 + \eta^2(\bar{A}_1 X) E_2\} - g(\eta^1(\bar{A}_2 X) E_1 + \eta^2(\bar{A}_1 X) E_2, Y) E_3$$

for any vector fields X and Y on M^3. Since $\bar{A}_1$ is symmetric, $\bar{A}_1$ can be written, in general, as

$$\bar{A}_1 = t_1\eta^1\otimes E_1+t_2(\eta^1\otimes E_2+\eta^2\otimes E_1)+t_3(\eta^1\otimes E_3+\eta^3\otimes E_1)+ \\ +t_4\eta^2\otimes E_2+t_5(\eta^2\otimes E_3+\eta^3\otimes E_2)+t_6\eta^3\otimes E_3$$

for some scalar fields $t_1, \dots, t_6$ on M^3. Using $\bar{A}_1E_1 = E_1$ and $f_1\bar{A}_1 = \bar{A}_1f_1$, $\bar{A}_1$ reduces to

$$\bar{A}_1 = \eta^1\otimes E_1+s_1(\eta^2\otimes E_2+\eta^3\otimes E_3)$$

since we can put $s_1 = t_4 = t_6$. Similarly,

$$\bar{A}_2 = \eta^2\otimes E_2+s_2(\eta^3\otimes E_3+\eta^1\otimes E_1)$$

for some scalar field s_2. Therefore, when we put

$$\bar{A}_3 = \eta^3\otimes E_3+s_1\eta^1\otimes E_1+s_2\eta^2\otimes E_2,$$

then $(\nabla_X f_3)Y$ can be arranged to the form:

$$(\nabla_X f_3)Y = \eta^3(Y)\bar{A}_3X-g(\bar{A}_3X, Y)E_3$$

and $\bar{A}_3$ satisfies

$$g(\bar{A}_3X, Y) = g(X, \bar{A}_3Y), \quad \bar{A}_3E_3 = E_3,$$
$$(\bar{A}_3f_3-f_3\bar{A}_3)X = (s_1-s_2)\big(\eta^1(X)E_1+\eta^2(X)E_2\big)$$

for any X. Since $\bar{A}_1E_3 = s_1E_3$, $\bar{A}_2E_3 = s_2E_3$, we can conclude by Theorem 3.1 that (f_3, E_3, η^3, g) defines a quasi-Sasakian structure. The converse is also true. ∎

COROLLARY 5.2. *If (f_i, E_i, η^i, g) $(i = 1, 2)$ of the triple $\{(f_a, E_a, \eta^a, g)\}$ of almost contact metric structures on a parallelizable Riemannian manifold M^3 of dimension 3 are Sasakian (resp. cosymplectic), then the remaining structure (f_3, E_3, η^3, g) is also Sasakian (resp. cosymplectic).*

Theorem 5.1 and Corollary 5.2 tell us the following. When $\{(f_a, E_a, \eta^a, g)\}$ is a triple of quasi-Sasakian structures, $\bar{A}_a$ have the expressions

$$\bar{A}_1 = \eta^1\otimes E_1+s(\eta^2\otimes E_2+\eta^3\otimes E_3),$$
$$\bar{A}_2 = \eta^2\otimes E_2+s(\eta^3\otimes E_3+\eta^1\otimes E_1),$$
$$\bar{A}_3 = \eta^3\otimes E_3+s(\eta^1\otimes E_1+\eta^2\otimes E_2)$$

for a scalar field s on M^3. Since $\nabla_X E_a = f_a\bar{A}_aX = sf_aX$, the Lie bracket products $[E_a, E_b]$ of E_a and E_b satisfy

$$[E_1, E_2] = -2sE_3, \quad [E_2, E_3] = -2sE_1, \quad [E_3, E_1] = -2sE_2.$$

Since $\eta^1\otimes E_1+\eta^2\otimes E_2+\eta^3\otimes E_3 = I$, a triple $\{(f_a, E_a, \eta^a, g)\}$ of quasi-Sasakian structures is a triple of Sasakian (resp. cosymplectic) structures if and only if $s = 1$ (resp. $s = 0$) holds identically on M^3.

(6) $M(\sigma)$ of constant f-sectional curvature

In the study of Riemannian manifolds, we well know that the sectional curvature $\varkappa$ with respect to a plane section $\{X, Y\}$ determined by two vectors X and Y at a point on a Riemannian manifold $N(g)$ is defined by

$$\varkappa = -\frac{R(X, Y, X, Y)}{|X|^2 \cdot |Y|^2 - (g(X, Y))^2}$$

and, furthermore, that if $\varkappa$ does not depend on any choice of sections at every point of N, then the manifold $N(g)$ is of constant curvature when $\dim N \geqslant 3$, and the Riemannian curvature tensor field R has the form:

$$R(X, Y, Z, W) = D_g(X, Y, Z, W), \quad X, Y, Z, W \in \mathfrak{X}(N),$$

where we have put $R(X, Y, Z, W) = g\big(R(X, Y)Z, W\big)$ and D_g denotes a $(0, 4)$-type tensor field defined by

$$D_g(X, Y, Z, W) = g(X, W)g(Y, Z) - g(X, Z)g(Y, W)$$

for any vector fields X, Y, Z and W on N.

On an almost Hermitian manifold $P(J, g)$ the holomorphic sectional curvature $\varkappa$ with respect to a plane section $\{X, JX\}$ determined by X and JX at a point x of P is defined by

$$\varkappa = -\frac{R(X, JX, X, JX)}{|X|^2 \cdot |JX|^2}.$$

The manifold $P(J, g)$ is called to *be of constant holomorphic sectional curvature* if the holomorphic sectional curvature is always constant with respect to any plane section at every point of the manifold. K. Yano and I. Mogi [32] showed that on a Kähler manifold of constant holomorphic sectional curvature the Riemannian curvature tensor field R has the form:

$$R(X, Y, Z, W) = \frac{\varkappa}{4}\{D_g(X, Y, Z, W) + K_\Omega(X, Y, Z, W)\}$$

and $\varkappa$ must necessarily be constant on the manifold, where K_Ω denotes a $(0, 4)$-type tensor field defined by

$$K_\Omega(X, Y, Z, W) = \Omega(X, W)\Omega(Y, Z) + \Omega(X, Z)\Omega(W, Y) - 2\Omega(X, Y)\Omega(Z, W)$$

for any vector fields X, Y, Z and W on $P(J, g)$.

On an almost contact metric manifold $M(\sigma)$ the f-sectional curvature $\varkappa$ with respect to a plane section $\{fX, f^2X\}$ determined by fX and f^2X at a point x of $M(\sigma)$ is defined by

$$\varkappa = -\frac{R(fX, f^2X, fX, f^2X)}{|fX|^2 \cdot |f^2X|^2}.$$

The manifold $M(\sigma)$ is called to *be of constant f-sectional curvature* if f-sectional curvature is always constant with respect to any plane section at every point of the manifold. It is shown that on a Sasakian (resp. cosymplectic) manifold of constant f-sectional curvature the Riemannian curvature tensor field R has the form:

$$R(X, Y, Z, W) = \frac{\varkappa+3\varepsilon}{4} D_g(X, Y, Z, W) + \frac{\varkappa-\varepsilon}{4} K_{F,\eta}(X, Y, Z, W),$$

where $\varepsilon = 1$ (resp. $\varepsilon = 0$) and $K_{F,\eta}$ denotes a (0, 4)-type tensor field defined by

$$\begin{aligned} K_{F,\eta}(X, Y, Z, W) = F(X, W)F(Y, Z)+F(X, Z)F(W, Y)- \\ -2F(X, Y)F(Z, W)-\eta(X)\eta(W)g(Y, Z)+\eta(X)\eta(Z)g(W, Y)- \\ -\eta(Y)\eta(Z)g(X, W)+\eta(W)\eta(Y)g(X, Z) \end{aligned}$$

for any vector fields X, Y, Z and W on $M(\sigma)$. In this case, $\varkappa$ must necessarily be constant under the condition $\dim M > 3$ ([13], [17]). In fact, we shall describe how to find the curvature tensor field R briefly in the case of a cosymplectic structure. The above definition of $\varkappa$ is written locally as

$$R_{kjih}(fX)^k(f^2X)^j(fX)^i(f^2X)^h + \varkappa(g_{kj}(fX)^k(fX)^j)(g_{ih}(f^2X)^i(f^2X)^h) = 0, \tag{6.1}$$

where Einstein's summation convention is used with respect to the system of indices k, j, i, h and $R_{kjih} = g_{ah}R^a_{kji}$. Putting $T_{kjih} = f^a_k f^b_i R_{ajbh}$ for the components f^a_i of f, we have

$$T_{ihkj} = T_{kjih}. \tag{6.2}$$

Applying $\nabla_j\eta_i = 0$ and $\nabla_j f^h_i = 0$ to the Ricci identities:

$$\nabla_k\nabla_j\eta_i - \nabla_j\nabla_k\eta_i = -R^a_{kji}\,\eta_a$$

and

$$\nabla_k\nabla_j f^h_i - \nabla_j\nabla_k f^h_i = f^a_i R^h_{kja} - f^h_a R^a_{kji},$$

we have

$$R^a_{kji}\eta_a = 0, \quad f^a_k f^b_j R_{abih} = R_{kjih}, \tag{6.3}$$

from which it follows that

$$T_{kjhi} = T_{kjih}. \tag{6.4}$$

Using (6.3) again, we obtain

$$T_{kijh} = -f^a_k f^b_j (R_{ibah} + R_{baih}) = T_{khji} + R_{kjih}. \tag{6.5}$$

Since (6.1) is also written as

$$\{T_{kjih} + \varkappa(g_{kj} - \eta_k\eta_j)(g_{ih} - \eta_i\eta_h)\} X^k X^j X^i X^h = 0$$

for any vector X^h, the relation

$$T_{(kjih)} + \varkappa\{g_{(kj}g_{ih)} - 2\eta_{(k}\eta_j g_{ih)} + \eta_{(k}\eta_j\eta_i\eta_{h)}\} = 0 \tag{6.6}$$

is valid, where $(kjih)$ denotes the symmetrization of indices k, j, i, h. In consideration of (6.2), (6.4) and the relation

$$T_{kjih} + T_{kihj} + T_{khji} = 3T_{kjih} - R_{khji} + R_{kihj}$$

obtained from (6.5), relation (6.6) reduces to the form:

$$3T_{kjih}-R_{khji}+R_{kihj}+\varkappa\{g_{kj}g_{ih}+g_{ki}g_{hj}+g_{kh}g_{ji}+3\eta_k\eta_j\eta_i\eta_h-$$
$$-(\eta_k\eta_j g_{ih}+\eta_k\eta_i g_{hj}+\eta_k\eta_h g_{ji}+\eta_i\eta_h g_{kj}+\eta_h\eta_j g_{ki}+\eta_j\eta_i g_{kh})\}=0.$$

Transvecting this with $f_a^k f_b^i$, we have

$$3R_{ajbh}-f_a^k f_b^i R_{khji}+R_{abhj}+\varkappa(f_{aj}f_{bh}+g_{ab}g_{hj}-\eta_a\eta_b g_{hj}+$$
$$+f_{ah}f_{bj}-\eta_h\eta_j g_{ab}+\eta_h\eta_j\eta_a\eta_b)=0,$$

where $f_{ij}=g_{ja}f_i^a$, that is,

$$T_{kjih}=3R_{khji}-R_{kijh}-\varkappa(f_{kh}f_{ij}+g_{ki}g_{hj}-\eta_k\eta_i g_{hj}+f_{kj}f_{ih}-\eta_h\eta_j g_{ki}+\eta_h\eta_j\eta_k\eta_i).$$

Taking the skew-symmetric part of T_{kjih} with respect to i and h, we finally have the curvature tensor field

$$R_{kjih}=\frac{\varkappa}{4}(g_{kh}g_{ji}+g_{ki}g_{jh})+\frac{\varkappa}{4}(f_{kh}f_{ji}+f_{ki}f_{hj}-2f_{kj}f_{ih}-$$
$$-\eta_k\eta_h g_{ji}+\eta_k\eta_i g_{hj}-\eta_j\eta_i g_{kh}+\eta_h\eta_j g_{ki}).$$

THEOREM 6.1. *Let M be a manifold admitting, at least, two cosymplectic (resp. Sasakian) structures $\sigma_a=(f_a, E_a, \eta^a, g)$ $(a=1,2)$ such that the fundamental vector fields E_1 and E_2 are mutually orthogonal, $g(E_1, E_2)=0$. If M is of constant f_a-sectional curvature with respect to both cosymplectic (resp. Sasakian) structures, then M is flat (resp. of constant curvature 1).*

Proof. Denote by $\varkappa_a$ $(a=1,2)$ the f_a-sectional curvatures corresponding to cosymplectic (resp. Sasakian) structures σ_a. Then the Riemannian curvature tensor field R has two expressions:

$$R(X,Y)Z=\frac{\varkappa_a+3\varepsilon}{4}\{g(Y,Z)X-g(X,Z)Y\}+\frac{\varkappa_a-\varepsilon}{4}\{F_a(Y,Z)f_aX-$$
$$-F_a(X,Z)f_aY-2F_a(X,Y)f_aZ-\eta^a(X)g(Y,Z)E_a+\eta^a(X)\eta^a(Z)Y+$$
$$+\eta^a(Y)g(X,Z)E_a-\eta^a(Y)\eta^a(Z)X\}$$

for $a=1$ and 2, where $\varepsilon=0$ (resp. $\varepsilon=1$). Apply the formulae: the trace of the map $X\to X$ equals $2n+1$, and $\mathrm{trace}[X\to f_aX]=0$, $\mathrm{trace}[X\to\eta^a(X)Y]=\eta^a(Y)$, to the Ricci tensor field R_0 defined by $R_0(Y,Z)=\mathrm{trace}[X\to R(X,Y)Z]$. Then R_0 has the two expressions from the above:

$$R_0(Y,Z)=\frac{\varkappa_a+3\varepsilon}{2}\{ng(Y,Z)\}+\frac{\varkappa_a-\varepsilon}{2}\{g(Y,Z)-(n+1)\eta^a(Y)\eta^a(Z)\}$$
$$=\frac{(n+1)(\varkappa_a-\varepsilon)}{2}g(f_aY,f_aZ)+2n\varepsilon g(Y,Z)$$

for $a=1$ and 2. Therefore, $(\varkappa_1-\varepsilon)g(f_1X,f_1Y)=(\varkappa_2-\varepsilon)g(f_2X,f_2Y)$ holds for any vector fields X and Y on M. Put $X=Y=E_2$. Then $\varkappa_1=\varepsilon$, hence, $\varkappa_2=\varepsilon$. Consequently, we obtain $R(X,Y,Z,W)=\varepsilon D_g(X,Y,Z,W)$. ■

Let us recall the theory of Riemannian direct product manifolds. We denote by $M_a(g_a)$ $(a = 1, 2)$ any two Riemannian manifolds with metrics g_a. Any vector field X on $M_1 \times M_2$ is decomposed as the sum $X = p_1X + p_2X$ of p_aX, defined by $p_aX = i_{a*}\pi_{a*}X$ for any $X \in \mathscr{X}(M_1 \times M_2)$, which are mutually orthogonal with respect to the metric $\varrho(g_1)+\varrho(g_2)$, say g. The projectors p_a $[p_a^2 = p_a,\ p_ap_b = 0$ $(a \neq b)$, $p_1+p_2 = I$, $g(p_aX, Y) = g(X, p_aY)$ for any $X, Y \in \mathscr{X}(M_1 \times M_2)]$ determine naturally the direct product structure $\{D(p_a)\}$ consisting of m_a-dimensional distributions $D(p_a)$ corresponding to M_a, where $\dim M_a = m_a$. Since each manifold M_a is totally geodesic submanifold of $M_1 \times M_2$, Gauss equations $\nabla_{p_aX}p_aY = p_a\nabla_{p_aX}p_aY$ $(a = 1, 2)$ and Weingarten equations $p_a\nabla_{p_aX}p_bY = 0$ $(a \neq b)$ are obtained for both M_a, where ∇ denotes the Riemannian connection with respect to g. The condition $[p_1X, p_2Y] = 0$ for $X, Y \in \mathscr{X}(M_1 \times M_2)$ implies that p_a are parallel, $\nabla p_a = 0$, and $\nabla_{p_aX}Y = p_a\nabla_XY = p_a\nabla_{p_aX}p_aY$. Accordingly, we can write the covariant derivative ∇_XY of Y with respect to X in the form: $\nabla_XY = \sum_{a=1}^{2} p_a\nabla_{p_aX}p_aY$. It follows that the Riemannian curvature tensor field R is expressible as

$$R(X, Y)Z = \sum_{a=1}^{2} p_aR(p_aX, p_aY)p_aZ,$$

which is equivalent to $R(X, Y, Z, W) = \sum_{a=1}^{2} R(p_aX, p_aY, p_aZ, p_aW)$. Denote by R_a the Riemannian curvature tensor fields with respect to g_a on M_a. Since $g = \varrho(g_1)+$ $+\varrho(g_2)$, we get $R = \varrho(R_1)+\varrho(R_2)$ and, hence, the scalar fields

$$R_a(\pi_{a*}X, \pi_{a*}Y, \pi_{a*}Z, \pi_{a*}W) \circ \pi_a$$

coincide with

$$R(p_aX, p_aY, p_aZ, p_aW)$$

on $M_1 \times M_2$.

THEOREM 6.2. *The product manifold $M_1 \times M_2$ of two Kähler manifolds $M_a(J_a, g_a)$ $(a = 1, 2)$ of constant holomorphic sectional curvatures $\varkappa_a$ is a Kähler manifold on which the Riemannian curvature tensor field R has the form*:

$$R(X, Y, Z, W) = \sum_{a=1}^{2} \frac{\varkappa_a}{4}(D_{\varrho(g_a)}+K_{\varrho(\Omega_a)})(X, Y, Z, W)$$

for any vector fields X, Y, Z and W on $M_1 \times M_2$.

Proof. The curvature tensor fields R_a on M_a are written as

$$R_a(X_a, Y_a, Z_a, W_a) = (\varkappa_a/4)(D_{g_a}+K_{\Omega_a})(X_a, Y_a, Z_a, W_a)$$

for any $X_a, Y_a, Z_a, W_a \in \mathscr{X}(M_a)$. Any vector field X_a on M_a can be regarded as the projection $\pi_{a*}X$ of a vector field X on $M_1 \times M_2$ by π_{a*}. Since $g_a(X_a, Y_a) \circ \pi_a$ $= g_a(\pi_{a*}X, \pi_{a*}Y) \circ \pi_a = (\varrho(g_a))(X, Y)$ and $\Omega_a(X_a, Y_a) \circ \pi_a = (\varrho(\Omega_a))(X, Y)$, we come to the desired expression. ∎

In like manner, we can easily obtain the following results.

THEOREM 6.3. *The product manifold $M_1 \times M_2$ of two quasi-Sasakian manifolds $M_a(f_a, E_a, \eta^a, g_a)$ of constant curvatures $\varkappa_a$ is an Hermitian manifold on which the Riemannian curvature tensor field R has the form*:

$$R(X, Y, Z, W) = \Big(\sum_{a=1}^{2} \varkappa_a D_{\varrho(g_a)}\Big)(X, Y, Z, W)$$

for any vector fields X, Y, Z and W on $M_1 \times M_2$.

The product manifolds $S^{2p+1} \times R^{2q+1}$ and $S^{2p+1} \times S^{2q+1}$ are typical examples of Theorem 6.3.

THEOREM 6.4. *The product manifold $M_1 \times M_2$ of two Sasakian (resp. cosymplectic) manifolds $M_a(f_a, E_a, \eta^a, g_a)$ of constant f_a-sectional curvatures $\varkappa_a$ is an Hermitian (resp. Kähler) manifold on which the Riemannian curvature tensor field R has the form*:

$$R(X, Y, Z, W) = \sum_{a=1}^{2} \left(\frac{\varkappa_a + 3\varepsilon}{4} D_{\varrho(g_a)} + \frac{\varkappa_a - \varepsilon}{4} K_{\varrho(F_a), \varrho(\eta^a)}\right)(X, Y, Z, W)$$

for any vector fields X, Y, Z and W on $M_1 \times M_2$, where $\varepsilon = 1$ (resp., $\varepsilon = 0$).

THEOREM 6.5. *The product manifold $M_1 \times M_2$ of a cosymplectic manifold $M_1(f_1, E_1, \eta^1, g_1)$ of constant f_1-sectional curvature $\varkappa_1$ and a Sasakian manifold $M_2(f_2, E_2, \eta^2, g_2)$ of constant f_2-sectional curvature $\varkappa_2$ is an Hermitian manifold on which the Riemannian curvature tensor field R has the form*:

$$R(X, Y, Z, W) = \left\{\frac{\varkappa_1}{4}(D_{\varrho(g_1)} + K_{\varrho(F_1), \varrho(\eta^1)}) + \frac{\varkappa_2 + 3}{4} D_{\varrho(g_2)} + \right.$$
$$\left. + \frac{\varkappa_2 - 1}{4} K_{\varrho(F_2), \varrho(\eta^2)}\right\}(X, Y, Z, W)$$

for any vector fields X, Y, Z and W on $M_1 \times M_2$.

THEOREM 6.6. *The product manifold $M_1 \times M_2$ of a quasi-Sasakian manifold $M_1(f_1, E_1, \eta^1, g_1)$ of constant curvature $\varkappa_1$ and a Kähler manifold $M_2(f_2, g_2)$ of constant holomorphic sectional curvature $\varkappa_2$ is a quasi-Sasakian manifold on which the Riemannian curvature tensor field R has the form*:

$$R(X, Y, Z, W) = \left\{\varkappa_1 D_{\varrho(g_1)} + \frac{\varkappa_2}{4}(D_{\varrho(g_2)} + K_{\varrho(F_2)})\right\}(X, Y, Z, W)$$

for any vector fields X, Y, Z and W on $M_1 \times M_2$.

THEOREM 6.7. *The product manifold $M_1 \times M_2$ of a Sasakian (resp. cosymplectic) manifold $M_1(f_1, E_1, \eta^1, g_1)$ of constant f_1-sectional curvature $\varkappa_1$ and a Kähler manifold $M_2(f_2, g_2)$ of constant holomorphic sectional curvature $\varkappa_2$ is a quasi-Sasakian* *manifold (resp. a cosymplectic manifold) on which the Riemannian tensor field R has the form*:

$$R(X,Y,Z,W) = \left\{\frac{\varkappa_1+3\varepsilon}{4}D_{\varrho(g_1)}+\frac{\varkappa_1-\varepsilon}{4}K_{\varrho(F_1),\varrho(\eta^1)}+\frac{\varkappa_2}{4}(D_{\varrho(g_2)}+K_{\varrho(F_2)})\right\}(X,Y,Z,W)$$

for any vector fields X, Y, Z and W on $M_1\times M_2$, *where* $\varepsilon = 1$ *(resp.* $\varepsilon = 0$).

Theorems 6.2–6.7 have been obtained by making use of the formula

$$R(X, Y, Z, W) = \sum_{a=1}^{2} R(p_aX, p_aY, p_aZ, p_aW)$$

on the Riemannian direct product manifold $M_1 \times M_2$.

(7) A hypersurface N of a Kähler manifold $P(J, G)$

We consider a hypersurface N in a Kähler manifold $P(J, G)$ with the imbedding map $i\colon N \to P$ and identify N with the image $i(N)$ of N. Let ζ denote the unit normal vector field defined over the hypersurface N. Then we obtain the decomposition law:

$$Ji_*X = i_*fX+\eta(X)\zeta, \qquad J\zeta = -i_*E$$

and

$$g(X, Y) = G(i_*X, i_*Y)$$

for any vector fields X and Y on N, where f is a linear transformation field, η a 1-form, E a global vector field, g the induced metric and i_* denotes the differential of i. The condition that (J, G) is an almost Hermitian structure on P implies that the set (f, E, η, g) of the induced tensor fields defines an almost contact metric structure on N. We have Gauss equation: $\nabla_{i_*X}i_*Y = i_*\nabla_X Y+h(X, Y)\zeta$ and the Weingarten equation: $\nabla_{i_*X}\zeta = -i_*HX$, in this case, ∇ appearing in the right-hand side of the former equation denotes the induced Riemannian connection with respect to g, and h and H are called the *second fundamental tensor fields of type* $(0, 2)$ *and* $(1, 1)$, respectively, with respect to ζ. Both satisfy $h(X, Y) = g(HX, Y)$ and h is symmetric. It follows from the condition: $(\nabla_{i_*X}J)i_*Y = 0$ that $(\nabla_X f)Y = \eta(Y)HX-g(HX, Y)E$ and $(\nabla_X\eta)(Y) = g(fHX, Y)$. Therefore, the torsion tensor field satisfies

$$[f,f](X, Y)+d\eta(X, Y)E = \eta(X)(fH-Hf)Y-\eta(Y)(fH-Hf)X.$$

Thus we have

THEOREM 7.1. *A necessary and sufficient condition for the induced almost contact metric structure* (f, E, η, g) *on a hypersurface N in a Kähler manifold* $P(J, G)$ *to be quasi-Sasakian is that f commutes with the second fundamental tensor field H.*

More precisely we obtain that (i) *if H has the form* $H = I+\lambda\eta\otimes E$ *for some scalar field* λ *on N, then the induced structure is a Sasakian structure, and* (ii) *if H has the form* $H = \lambda\eta\otimes E$ *for some* λ *on N, then the induced structure is a cosymplectic structure on N.*

(8) A submanifold M of codimension 2 of a quasi-Sasakian manifold

Let $\tilde{M}(\tilde{\sigma})$ be an almost contact metric manifold with a structure $\tilde{\sigma} = (\tilde{f}, \tilde{E}, \tilde{\eta}, \tilde{g})$ and let M be a submanifold of codimension 2 of $\tilde{M}(\tilde{\sigma})$ with the imbedding map i: $M \to \tilde{M}$. We identify M with $i(M)$ in $\tilde{M}$. We assume that there exists a normal frame $\{\zeta_1, \zeta_2\}$ defined globally over the submanifold M. Then we obtain the decomposition law for $\tilde{E}, \tilde{f}i_*X, \tilde{f}\zeta_a$ $(a = 1, 2)$:

$$\tilde{E} = i_* E_0 + a\zeta_1 + b\zeta_2, \tag{8.1}$$

$$\tilde{f}i_*X = i_*\phi X + \eta^1(X)\zeta_1 + \eta^2(X)\zeta_2, \tag{8.2}$$

$$\tilde{f}\zeta_1 = -i_*E_1 + c\zeta_2, \tag{8.3}$$

$$\tilde{f}\zeta_2 = -i_*E_2 - c\zeta_1 \tag{8.4}$$

with respect to $\{\zeta_1, \zeta_2\}$, where ϕ denotes a linear transformation field, E_a $(a = 1, 2)$ vector fields, η^a $(a = 1, 2)$ 1-forms, a, b, c scalar fields on M. The induced metric g on M is given by

$$g(X, Y) = \tilde{g}(i_*X, i_*Y) \tag{8.5}$$

for any vector fields X and Y on M, and we define a 1-form η^0 by

$$\eta^0(X) = \tilde{\eta}(i_*X) \tag{8.6}$$

for any $X \in \mathscr{X}(M)$. It follows from (8.1), (8.5) and (8.6) that $\eta^0(X) = g(X, E_0)$. Since $\tilde{g}(\tilde{f}i_*X, \zeta_a) = -\tilde{g}(i_*X, \tilde{f}\zeta_a)$ $(a = 1, 2)$, then by (8.2)–(8.5) we have $\eta^a(X) = g(X, E_a)$. By means of $\tilde{f}\tilde{E} = 0$ the relations $\phi E_0 = aE_1 + bE_2$, $g(E_0, E_1) = bc$, $g(E_2, E_0) = -ac$ are valid. Applying $\tilde{f}$ to both sides of (8.2), we attain to

$$\phi^2 = -I + \eta^0 \otimes E_0 + \eta^1 \otimes E_1 + \eta^2 \otimes E_2, \quad \eta^1(\phi X) = a\eta^0(X) + c\eta^1(X),$$
$$\eta^2(\phi X) = b\eta^0(X) - c\eta^1(X).$$

Since

$$g(\phi X, Y) = \tilde{g}(i_*\phi X, i_*Y) = \tilde{g}(\tilde{f}i_*X, i_*Y) = -\tilde{g}(i_*X, \tilde{f}i_*Y) = -g(X, \phi Y),$$

then $\eta^a(\phi X) = -g(X, \phi E_a)$ $(a = 0, 1, 2)$. Applying $\tilde{f}$ to (8.3) and to (8.4), we get $\phi E_1 = -aE_0 - cE_2$, $\eta^1(E_1) = 1 - c^2 - a^2$, $\eta^1(E_2) = -ab$; $\phi E_2 = -bE_0 + cE_1$, $\eta^2(E_2) = 1 - b^2 - c^2$. Since $\tilde{E}$ is a unit vector field, it follows from (8.1) that $\eta^0(E_0) = 1 - a^2 - b^2$. Thus we have obtained

$$\phi^2 = -I + \eta^0 \otimes E_0 + \eta^1 \otimes E_1 + \eta^2 \otimes E_2, \tag{8.7}$$

$$\begin{aligned} \phi E_0 &= aE_1 + bE_2, & \eta^0(\phi X) &= -a\eta^1(X) - b\eta^2(X), \\ \phi E_1 &= -aE_0 - cE_2, & \eta^1(\phi X) &= a\eta^0(X) + c\eta^2(X), \\ \phi E_2 &= -bE_0 + cE_1, & \eta^2(\phi X) &= b\eta^0(X) - c\eta^1(X), \end{aligned} \tag{8.8}$$

$$g(\phi X, Y) = -g(X, \phi Y) \tag{8.9}$$

$$\begin{aligned} |E_0|^2 &= 1 - a^2 - b^2, & g(E_0, E_1) &= bc, \\ |E_1|^2 &= 1 - c^2 - a^2, & g(E_1, E_2) &= -ab, \\ |E_2|^2 &= 1 - b^2 - c^2, & g(E_2, E_0) &= -ca. \end{aligned} \tag{8.10}$$

THEOREM 8.1. *Let $\varkappa$ be a scalar field defined by $\varkappa = \sqrt{a^2+b^2+c^2}$ on a submanifold M of codimension 2 of an almost contact metric manifold $\tilde{M}(\tilde{\sigma})$. $\varkappa$ is independent of a choice of the normal frame $\{\zeta_1, \zeta_2\}$ and satisfies $0 \leqslant \varkappa \leqslant 1$.*

Proof. The Cauchy-Schwarz inequalities are applied to E_a, namely,

$$|E_0|^2 \cdot |E_1|^2 - \big(g(E_0, E_1)\big)^2 = (1-a^2)(1-\varkappa^2) \geqslant 0,$$
$$|E_2|^2 \cdot |E_0|^2 - \big(g(E_2, E_0)\big)^2 = (1-b^2)(1-\varkappa^2) \geqslant 0,$$
$$|E_1|^2 \cdot |E_2|^2 - \big(g(E_1, E_2)\big)^2 = (1-c^2)(1-\varkappa^2) \geqslant 0$$

by (8.10). Since g is a positive definite metric, then $a^2+b^2 \leqslant 1$, $b^2+c^2 \leqslant 1$, $c^2+a^2 \leqslant 1$ are valid. Suppose that $\varkappa^2 > 1$. It follows that $a^2 \geqslant 1$, $b^2 \geqslant 1$, $c^2 \geqslant 1$, from which $\varkappa^2 \geqslant 3$. On the other hand, $2\varkappa^2 \leqslant 3$ holds. This is a contradiction. Let $\{\zeta_1', \zeta_2'\}$ be another normal frame defined globally over M. The decomposition law for $\tilde{E}, \tilde{f}\zeta_a'$ are written with respect to $\{\zeta_1', \zeta_2'\}$ as

$$\tilde{E} = i_* E_0' + a'\zeta_1' + b'\zeta_2', \tag{8.11}$$

$$\tilde{f}\zeta_1' = -i_* E_1' + c'\zeta_2', \tag{8.12}$$

$$\tilde{f}\zeta_2' = -i_* E_2' - c'\zeta_1' \tag{8.13}$$

for scalar fields a', b', c' and vector fields E_a'. There exists a certain scalar field θ on N such that

$$\zeta_1 = \sin\theta\, \zeta_1' + \cos\theta\, \zeta_2', \qquad \zeta_2 = -\cos\theta\, \zeta_1' + \sin\theta\, \zeta_2'. \tag{8.14}$$

Accordingly, we obtain $a = a'\sin\theta + b'\cos\theta$, $b = -a'\cos\theta + b'\sin\theta$ by (8.1), (8.11) and (8.14), and $c = c'$ by (8.3), (8.12), (8.13) and (8.14), which imply that $a^2+b^2+c^2 = a'^2+b'^2+c'^2$. ∎

Now we define a vector field $\bar{E}$ and a 1-form $\bar{\eta}$ on M by

$$\bar{E} = cE_0 + bE_1 - aE_2, \qquad \bar{\eta} = c\eta^0 + b\eta^1 - a\eta^2. \tag{8.15}$$

Then we can verify that the following relations hold on M in $\tilde{M}(\tilde{\sigma})$ by (8.7)–(8.10) and (8.15).

$$\begin{gathered}\bar{\eta}(\bar{E}) = \varkappa^2, \quad \bar{\eta}(X) = g(X, \bar{E}), \quad \phi\bar{E} = 0, \quad \bar{\eta}(\phi X) = 0,\\ \phi^3X + \phi X = \eta^0(\phi X)\phi E_0 + \eta^1(\phi X)\phi E_1 + \eta^2(\phi X)\phi E_2,\\ \phi^4X + (\varkappa^2+1)\phi^2X + \varkappa^2X = \bar{\eta}(X)\bar{E},\\ \phi^5X + (\varkappa^2+1)\phi^3X + \varkappa^2\phi X = 0\end{gathered} \tag{8.16}$$

for any vector field X on M.

THEOREM 8.2. *In order that an almost contact metric structure can be induced on a submanifold M of codimension 2 of an almost contact metric manifold $\tilde{M}(\tilde{\sigma})$, it is sufficient that a condition*

either $\varkappa = 0$ on M or $0 < \varkappa \leqslant 1$ on M

holds.

By this assertion we can see that if the scalar field $\varkappa$ is constant on M, then an almost contact metric structure can always be induced on M.

Proof. Suppose that $\varkappa$ vanishes identically on M. This implies $a = b = c = 0$. The set $\{E_0, E_1, E_2\}$ of vector fields E_a is orthonormal at every point of M and $\phi E_a = 0$ by means of (8.8) and (8.10). We put

$$f = \phi + t_0(\eta^1 \otimes E_2 - \eta^2 \otimes E_1) + t_1(\eta^2 \otimes E_0 - \eta^0 \otimes E_2) + t_2(\eta^0 \otimes E_1 - \eta^1 \otimes E_0),$$
$$E = t_0 E_0 + t_1 E_1 + t_2 E_2,$$
$$\eta = t_0 \eta^0 + t_1 \eta^1 + t_2 \eta^2$$

for an arbitrarily chosen and fixed direction ratio $t_0 : t_1 : t_2$ such that $t_0^2 + t_1^2 + t_2^2 = 1$. Then we can verify that the set (f, E, η, g) of tensor fields f, E, η, g defines an almost contact metric structure on M. In this case, when we denote by Φ the associated 2-form with ϕ relative to the metric g, the fundamental 2-form F is written as

$$F = \Phi + t_0 \eta^1 \wedge \eta^2 + t_1 \eta^2 \wedge \eta^0 + t_2 \eta^0 \wedge \eta^1.$$

Suppose that $\varkappa$ satisfies a condition: $0 < \varkappa \leqslant 1$ on M. Now we put

$$f = \frac{1}{\varkappa(\varkappa+1)}\{\phi^3 + (\varkappa^2 + \varkappa + 1)\phi\},$$
$$(8.17) \qquad E = \frac{1}{\varkappa}\bar{E}, \qquad \eta = \frac{1}{\varkappa}\bar{\eta}.$$

It enables us to see that the set (f, E, η, g) of tensor fields f, E, η, g defines an almost contact metric structure on the submanifold M. In fact,

$$f^2 X = \{\varkappa^2(\varkappa+1)^2\}^{-1}\{\phi^6 X + 2(\varkappa^2+\varkappa+1)\phi^4 X + (\varkappa^2+\varkappa+1)^2\phi^2 X\}$$
$$= \varkappa^{-2}\{\phi^4 X + (\varkappa^2+1)\phi^2 X\} = -X + \eta(X)E$$

by virtue of (8.15) and (8.16). ∎

Next we consider the submanifold M in a quasi-Sasakian manifold $\tilde{M}(\tilde{\sigma})$. We denote by $\tilde{A}$ the indicator tensor field of the quasi-Sasakian structure $\tilde{\sigma}$. The decomposition law for $\tilde{A} i_* X$ with respect to $\{\zeta_1, \zeta_2\}$ is written as

$$(8.18) \qquad \tilde{A} i_* X = i_* QX + q^1(X)\zeta_1 + q^2(X)\zeta_2$$

for any vector field X on M, where Q denotes a linear transformation field and q^i $(i = 1, 2)$ are 1-forms on M. Furthermore, we have the equations

$$(8.19) \qquad \begin{aligned} &\nabla_{i_* X} i_* Y = i_* \nabla_X Y + h(X, Y)\zeta_1 + k(X, Y)\zeta_2, \\ &\nabla_{i_* X}\zeta_1 = -i_* HX + \omega(X)\zeta_2, \quad \nabla_{i_* X}\zeta_2 = -i_* KX - \omega(X)\zeta_1, \end{aligned}$$

where h and k (resp., H and K) are the second fundamental tensor fields of type (0, 2) (resp., (1, 1)) and ω the third fundamental 1-form with respect to $\{\zeta_1, \zeta_2\}$. h and k are symmetric and satisfy $h(X, Y) = g(HX, Y)$, $k(X, Y) = g(KX, Y)$ for any $X, Y \in \mathfrak{X}(M)$. Since the equation

$$(\nabla_{i_* X}\tilde{f}) i_* Y = \tilde{\eta}(i_* Y)\tilde{A} i_* X - \tilde{g}(\tilde{A} i_* X, i_* Y)\tilde{E}$$

holds by means of Theorem 3.1, then the left-hand side of this reduces to

$$
\begin{aligned}
&(\tilde{\nabla}_{i_*X}\tilde{f})i_*Y = \tilde{\nabla}_{i_*X}\tilde{f}i_*Y-\tilde{f}\tilde{\nabla}_{i_*X}i_*Y\\
&= \tilde{\nabla}_{i_*X}(i_*\phi Y+\eta^1(Y)\zeta_1+\eta^2(Y)\zeta_2)-\tilde{f}(i_*\nabla_X Y+h(X, Y)\zeta_1+k(X, Y)\zeta_2)\\
&= i_*\nabla_X\phi Y+h(X,\phi Y)\zeta_1+k(X,\phi Y)\zeta_2+\big(X(\eta^1(Y))\big)\zeta_1+\eta^1(Y)\{-i_*HX+\omega(X)\zeta_2\}+\\
&\quad+\big(X(\eta^2(Y))\big)\zeta_2+\eta^2(Y)\{-i_*KX-\omega(X)\zeta_1\}-i_*\phi\nabla_X Y-\eta^1(\nabla_X Y)\zeta_1-\eta^2(\nabla_X Y)\zeta_2-\\
&\quad-h(X, Y)\{-i_*E_1+c\zeta_2\}-k(X, Y)\{-i_*E_2-c\zeta_1\}\\
&= i_*\{(\nabla_X\phi)Y-\eta^1(Y)HX-\eta^2(Y)KX+h(X, Y)E_1+k(X, Y)E_2\}+\\
&\quad+\{h(X,\phi Y)+(\nabla_X\eta^1)(Y)-\omega(X)\eta^2(Y)+ck(X, Y)\}\zeta_1+\\
&\quad+\{k(X,\phi Y)+(\nabla_X\eta^2)(Y)+\omega(X)\eta^1(Y)-ch(X, Y)\}\zeta_2
\end{aligned}
$$

by using (8.2)–(8.4) and (8.19), and the right-hand side of the above equation reduces to

$$
\begin{aligned}
\tilde{\eta}(i_*Y)\tilde{A}i_*X-\tilde{g}(\tilde{A}i_*X, i_*Y)\tilde{E} &= \eta^0(Y)\{i_*QX+q^1(X)\zeta_1+q^2(X)\zeta_2\}-\\
&\qquad -g(QX, Y)\{i_*E_0+a\zeta_1+b\zeta_2\}\\
&= i_*\{\eta^0(Y)QX-g(QX, Y)E_0\}+\{q^1(X)\eta^0(Y)-\\
&\qquad -ag(QX, Y)\}\zeta_1+\{q^2(X)\eta^0(Y)-bg(QX, Y)\}\zeta_2
\end{aligned}
$$

by using (8.18). Comparing the tangential parts to M of both sides of quantities mentioned above we obtain the covariant derivative $\nabla_X\phi$ of ϕ

$$
\begin{aligned}
(8.20)\qquad (\nabla_X\phi)Y = \eta^0(Y)QX+\eta^1(Y)HX+\eta^2(Y)KX-g(QX, Y)E_0-\\
-g(HX, Y)E_1-g(KX, Y)E_2
\end{aligned}
$$

for any $X, Y\in\mathfrak{X}(M)$.

We would like to close our discussion here by showing the following assertion.

THEOREM 8.3. *Suppose that $\varkappa = 1$ on a submanifold M of codimension 2 of a quasi-Sasakian manifold $\tilde{M}(\tilde{\sigma})$. If the fundamental tensor field f of the induced almost contact metric structure (f, E, η, g) on M commutes with $cQ+bH-aK$, then (f, E, η, g) is a quasi-Sasakian structure.*

Proof. The assumption: $\varkappa = 1$ implies that $|E_0|\cdot|E_1| = g(E_0, E_1)$, $|E_1|\cdot|E_2| = g(E_1, E_2)$, $|E_2|\cdot|E_0| = g(E_2, E_0)$, which show that E_0, E_1, E_2 are linearly dependent to one another. By a simple calculation we may obtain $E_0 = cE$, $E_1 = bE$, $E_2 = -aE$ for a unit vector field E and $\eta^0 = c\eta$, $\eta^1 = b\eta$, $\eta^2 = -a\eta$ for its associated 1-form η and, in addition, $f = \phi$ by means of (8.16) and (8.17). Hence, (8.20) reduces to

$$(\nabla_X f)Y = \eta(Y)\psi X-g(\psi X, Y)E,$$

where we have put

$$\psi = cQ+bH-aK.$$

Since

$$g(QX, Y) = \tilde{g}(i_*QX, i_*Y) = \tilde{g}(\tilde{A}i_*X, i_*Y) = \tilde{g}(i_*X, \tilde{A}i_*Y) = g(X, QY),$$

ψ is symmetric and commutes with f. By Theorem 3.1 we can conclude that (f, E, η, g) is quasi-Sasakian. ∎

A submanifold M is called *$\tilde{f}$-invariant* if $\tilde{f}T_x(M)$ is contained in $T_x(M)$ at every point x of M, namely, if $\tilde{f}i_*X = i_*\phi X$ holds identically. Hence, $a = b = 0$ and $c^2 = 1$ (accordingly, $\varkappa = 1$) are valid on the $\tilde{f}$-invariant submanifold M, on which f must necessarily commute with ψ by using the condition $\tilde{f}\tilde{A}i_*X = \tilde{A}\tilde{f}i_*X$.

COROLLARY 8.4. *An $\tilde{f}$-invariant submanifold M of codimension 2 of a quasi-Sasakian (resp. Sasakian, cosymplectic) manifold $\tilde{M}(\tilde{\sigma})$ admits a quasi-Sasakian (resp. Sasakian, cosymplectic) structure.*

References

[1] D. E. Blair, *The theory of quasi-Sasakian structures*, J. Differential Geometry 1 (1967), 331–345.

[2] E. Calabi and B. Eckmann, *A class of compact complex manifolds which are not algebraic*, Ann. of Math. 58 (1953), 494–500.

[3] S. I. Goldberg and K. Yano, *On normal globally framed f-manifolds*, Tôhoku Math. J. 22 (1970), 362–370.

[4] J. W. Gray, *Some global properties of contact structures*, Ann. of Math. 69 (1959), 421–450.

[5] S. Ishihara, *Normal structure f satisfying $f^3+f=0$*, Kōdai Math. Sem. Rep. 18 (1966), 36–47.

[6] S. Kanemaki, *Twisted f-structures* I, II, Tensor N.S. 27 (1973), 63–69; 70–72.

[7] —, *On submanifolds of codimension 2 of almost contact metric manifolds* I, II, ibid. 27 (1973), 281–286; 32 (1978), 323–331.

[8] —, *Products of f-manifolds*, TRU Math. 10 (1974), 11–17.

[9] —, *Hermitian manifolds with structure satisfying $d\Omega = 2\varepsilon\wedge\Omega$*, ibid. 12, 2 (1976), 13–18.

[10] —, *Quasi-Sasakian manifolds*, Tôhoku Math. J. 29 (1977), 227–233.

[11] —, *Almost contact metric manifolds with global fields of k-frames*, TRU Math. 13, 1 (1977), 43–52.

[12] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* I, II, Wiley (Interscience), New York 1963, 1969.

[13] G. D. Ludden, *Submanifolds of cosymplectic manifolds*, J. Differential Geometry 4 (1970), 237–244.

[14] A. Morimoto, *On normal almost contact structures*, J. Math. Soc. Japan 15 (1963), 420–436.

[15] —, *On normal almost contact structures with a regularity*, Tôhoku Math. J. 16 (1964), 90–104.

[16] H. Nakagawa, *f-structures induced on submanifolds in spaces, almost Hermitian or Kaehlerian*, Kōdai Math. Sem. Rep. 18 (1966), 161–183.

[17] K. Ogiue, *On almost contact manifolds admitting axiom of planes or axiom of free mobility*, ibid. 16 (1964), 223–232.

[18] M. Okumura, *Compact orientable submanifold of codimension 2 in an odd dimensional sphere*, Tôhoku Math. J. 20 (1968), 8–20.

[19] —, *On contact metric immersion*, Kōdai Math. Sem. Rep. 20 (1968), 389–409.

[20] S. Sasaki, *On differentiable manifolds with certain structures which are closely related to almost contact structure* I, Tôhoku Math. J. 12 (1960), 459–476.

[21] —, *Almost contact manifolds*, Lecture Note (1965), Mathematical Institute, Tôhoku University.

[22] S. Sasaki and Y. Hatakeyama, *On differentiable manifolds with certain structures which are closely related to almost contact structure* II, Tôhoku Math. J. 13 (1961), 281–294.

[23] S. Sasaki and Y. Hatakeyama, *On differentiable manifolds with contact metric structures*, J. Math. Soc. Japan 14 (1962), 249–271.

[24] S. Tanno, *Sasakian manifolds with constant ϕ-holomorphic sectional curvature*, Tôhoku Math. J. 21 (1969), 501–507.

[25] —, *Quasi-Sasakian structures of rank* $2p+1$, J. Differential Geometry 5 (1971), 317–324.

[26] Y. Tashiro, *On contact structure of hypersurfaces in complex manifolds*, I, Tôhoku Math. J. 15 (1963), 62–78.

[27] A. G. Walker, *Connexions for parallel distributions in the large*, Quart. J. Math. (Oxford) (2) 6 (1955), 301–308; II, 9 (1958), 221–231.

[28] —, *Almost-product structures*, Proc. Symp. Pure Math. III (1961), 94–100.

[29] K. Yano, *On a structure defined by a tensor field* f *of type* (1,1) *satisfying* $f^3+f=0$, Tensor N.S. 14 (1963), 99–109.

[30] —, *Differential Geometry on complex and almost complex spaces*, Pergamon Press, New York 1965.

[31] K. Yano and S. Ishihara, *On a problem of Nomizu–Smyth on a normal contact Riemannian manifold*, J. Differential Geometry 3 (1969), 45–58.

[32] K. Yano and I. Mogi, *On real representations of Kaehlerian manifolds*, Ann. of Math. 61 (1955), 170–189.

Presented to the Semester
Differential Geometry
(September 17–December 15, 1979)