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MORE CHARACTERISTIC INVARIANTS OF FOLIATED BUNDLES

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In [1] the author observed that the notion of a Bott connection ([3]) and, more generally, of a connection adapted to a flat partial connection ([7]) gives rise to a number of (primary and secondary) characteristic invariants of the foliated bundle. In particular, for any q-codimensional (smooth) foliation F of a manifold M there exist well-defined invariants $p_i(F) \in H^{2i}(I_F^i)$, $i = 1, \ldots, q$, and $s_{2j+1}(F) \in H^{4j+1}(A^*(M)/I_F^{2j+1})$, $j = 1, \ldots, [q-1/2]$, $I_F \subset A^*(M)$ being the ideal of all the smooth differential forms on M vanishing on leaves of F. These elements (the author called them respectively the Pontryagin classes and the secondary Pontryagin classes of F) play the role of elementary blocks for the exotic characteristic classes of F. In the present paper we construct a larger family of characteristic invariants with values in $H(I_F^a/I_F^b)$, $0 \leqslant a < b$. The elements of that family cover the derived characteristic classes constructed by Kamber and Tondeur ([6], [7]; § 4.50) and, under some natural algebraic conditions, can be described in terms of the primary and secondary characteristic invariants. In conclusion, the Pontryagin classes together with the secondary Pontryagin classes fully determine not only the exotic classes, and the holonomy ring of any leaf ([1]; § 3.23, [4], [8]) but also all the derived characteristic classes.

Throughout this paper all manifolds, bundles, foliations, etc. will be smooth, i.e. of class C^∞.

§ 1. Preliminaries

For any foliation F of a manifold M we accept the following notation:

Q_F—the normal bundle, Q_F^*—the dual normal bundle of F;

LQ_F^*—the manifold of all frames in Q_F^*;

$I_F^k \subset A^*(M)$—the ideal of forms which locally can be described as

$$\sum \varphi^{i_1} \wedge \ldots \wedge \varphi^{i_k} \wedge \psi_{i_1 \ldots i_k},$$

where the forms φ^i are local sections of Q_F^*, $k = 0, 1, \ldots$ The ideals I_F^k, $k = 0, 1, \ldots,$

form a decreasing, consistent with the differential d, filtration of the algebra $I_F^0 = A^*(M)$. Obviously, $I_F^k = 0$ if $k > \operatorname{codim} F$.

By definition ([1], [7]), a foliated bundle with a reduction of structure group (shortly, RF-bundle) is a collection $\big(P(M, G), E, H, \xi\big)$, where $P(M, G)$ is a principal bundle, $E \subset TP$ a flat partial connection (an involutive G-invariant subbundle containing no nonzero vertical vectors), H a closed subgroup of G, and ξ an isotopy class of global sections $M \to P/H$.

Let us note that any section $s\colon M \to P/H$ yields a unique H-reduction $P_s \subset P$ of $P(M, G)$ such that $s(p) = uH$ for $u \in (P_s)_p$, $p \in M$; we have

$$P_s = \{u \in P;\ s(\pi(u)) = uH\},$$

where $\pi\colon P \to M$ is the projection.

Any q-codimensional foliation F of a manifold M gives rise to a canonical RF-bundle $B(F)$ composed of the frame bundle $LQ_F^*(M, \mathrm{Gl}(q))$, the Bott partial connection E, the orthogonal subgroup $\mathrm{O}(q) \subset \mathrm{Gl}(q)$, and of the isotopy class of all sections $M \to LQ_F^*/\mathrm{O}(q)$. Similarly, if F is equipped with an isotopy class t of trivializations of Q_F^*, then one considers also the RF-bundle

$$B(F, t) = \big(LQ_F^*(M, \mathrm{Gl}(q)), E, \{e\}, t\big).$$

In general, any flat partial connection projects onto the tangent bundle of some foliation of the base manifold. Such a foliation will be referred to as the projected one.

Following [7], let us consider a principal bundle $P(M, G)$ foliated by a flat partial connection E. A connection $\omega \in A^1(P)\otimes\mathfrak{g}$ on $P(M, G)$, $\mathfrak{g}$ being the Lie algebra of G, is said to be *adapted* to E if $\omega|E = 0$. If this is the case, then the curvature $\Omega \in A^2(P)\otimes\mathfrak{g}$ lies in $I^1_{\pi^*F}\otimes\mathfrak{g}$, where $\pi\colon P \to M$ denotes the projection and π^*F is the induced foliation of P. Consequently, the canonical G-DG-homomorphism $k(\omega)$ of the Weil algebra $W(G) = \bigwedge \mathfrak{g}^*\otimes S\mathfrak{g}^*$ into $A^*(P)$, defined by the equality

$$\begin{aligned} &k(\omega)(x^1 \wedge \ldots \wedge x^a \otimes y^1 \cdot \ldots \cdot y^b) \\ &\qquad\qquad = (x^1 \circ \omega) \wedge \ldots \wedge (x^a \circ \omega) \wedge (y^1 \circ \Omega) \wedge \ldots \wedge (y^b \circ \Omega), \end{aligned}$$

$a, b = 0, 1, \ldots$, has the very important property:

$$k(\omega)\big(F^k W(G)\big) \subset I^k_{\pi^*F}$$

for

$$F^k W(G) = \bigoplus_{b \geqslant k} \bigwedge \mathfrak{g}^* \otimes S^b\mathfrak{g}^*, \qquad k = 0, 1, \ldots$$

The filtration $W(G) = F^0 \supset F^1 \supset \ldots$ is preserved by the G-DG-structure mappings d, $i(x)$, $\theta(x)$, and $\varrho(g)$ $(x \in \mathfrak{g}, g \in G)$.

LEMMA 1. *The filtration $A^*(P) = I^0_{\pi^*F} \supset I^1_{\pi^*F} \supset \ldots$ is preserved by the canonical G-DG-structure mappings in $A^*(P)$.*

Proof. Let us observe that $\varphi \in I^k_{\pi^*F} \cap A^h(P)$ if and only if $\varphi_u(v_1, \ldots, v_{h-k+1}, \cdot, \ldots \ldots, \cdot) = 0$ whenever all the vectors $v_1, \ldots, v_{h-k+1} \in T_u P$, $u \in P$, project into a tangent space of F, for $h, k = 0, 1, \ldots, h \geqslant k$. In view of this the asserted inclusions

$$i(x)(I^k_{\pi^*F}) \subset I^k_{\pi^*F} \quad \text{and} \quad \varrho(g)(I^k_{\pi^*F}) \subset I^k_{\pi^*F}, \qquad x \in \mathfrak{g},\ g \in G,$$

$k = 0, 1, \ldots,$ follow from the identities

$$\big(i(x)\varphi\big)_u (v_1, \ldots, v_{h-k}, \cdot, \ldots, \cdot) = \varphi(L_{u*}x_e, v_1, \ldots, v_{h-k}, \cdot, \ldots, \cdot),$$

where $\pi_*(L_{u*}x_e) = 0$ and $i(x)\varphi \in A^{h-1}(P)$,

$$\begin{aligned}\big(\varrho(g)\varphi\big)_u (v_1, \ldots, v_{h-k+1}, \cdot, \ldots, \cdot) &= (R_g^*\varphi)_u (v_1, \ldots, v_{h-k+1}, \cdot, \ldots, \cdot) \\ &= R_g^*\big(\varphi_{ug}(R_{g*}v_1, \ldots, R_{g*}v_{h-k+1}, \cdot, \ldots, \cdot)\big),\end{aligned}$$

where $\pi_*(R_{g*}v_i) = \pi_* v_i$ for $i = 1, \ldots, h-k+1$. Here $L_u\colon G \to P$ and $R_g\colon P \to P$ denote the mappings induced from the standard G-action on P.

For any Lie subgroup H of G one defines H-basic elements of any G-DG-algebra A, in particular, of $W(G)$ and of $A^*(P)$, as those $w \in A$ which satisfy the conditions

$$\varrho(h)w = w \quad \text{for} \quad h \in H,$$

and

$$i(x)w = 0 \quad \text{for} \quad x \in \mathfrak{h}$$

(the Lie algebra of H). The H-basic elements in A constitute a DG-algebra A_H. In particular, $A^*(P)_H \subset A^*(P)$ coincides with the image of the monomorphism

$$\pi_H^*\colon A^*(P/H) \to A^*(P),$$

where $\pi_H\colon P \to P/H$ denotes the projection. Consequently, there is a canonical isomorphism

$$A^*(P)_H \cong A^*(P/H).$$

LEMMA 2. *The above isomorphism transfers*

$$(I^k_{\pi^*F})_H \subset A^*(P)_H$$

onto

$$I^k_{\bar\pi^*F} \subset A^*(P/H), \qquad k = 0, 1, \ldots,$$

where $\bar\pi$ denotes the projection $P/H \to M$, and $\bar\pi^*F$ is the induced foliation of P/H.

Proof. As $\pi = \bar\pi \circ \pi_H$, there is $\pi_H^*(\bar\pi^*F) = \pi^*F$ and so

$$\pi_H^*(I^k_{\bar\pi^*F}) \subset I^k_{\pi^*F}.$$

To prove the converse let us take an arbitrary form

$$\varphi \in (I^k_{\pi^*F})_H \cap A^h(P), \qquad h = k, k+1, \ldots$$

There exists $\psi \in A^h(P/H)$ such that $\varphi = \pi_H^*\psi$. Consequently, whenever the first $h-k+1$ of vectors $v_1, \ldots, v_h \in T_{uH}(P/H)$, $u \in P$, are tangent to $\bar\pi^*F$ (that means that $\bar\pi_* v_i$, $i = 1, \ldots, h-k+1$, are tangent to F) there is

$$\psi_{uH}(v_1, \ldots, v_h) = \varphi_u(w_1, \ldots, w_h)$$

for any $w_1, \ldots, w_h \in T_u P$ such that $\pi_{H*} w_i = v_i$, and thus

$$\psi_{uH}(v_1, \ldots, v_{h-k+1}, \cdot, \ldots, \cdot) = 0$$

as for all i's $\pi_* w_i = \bar{\pi}_* v_i$. In conclusion we get $\psi \in I^k_{\bar{\pi}^* F}$.

To end the preliminary section let us consider any filtered differential algebra

$$A = F^0 A \supset F^1 A \supset \ldots$$

with a differential d; by definition, there is

$$d(F^k A) \subset F^k A \quad \text{and} \quad F^k A \cdot F^l A \subset F^{k+l} A$$

for every k and l. We shall accept the abbreviation

$$F^a_b A := F^a A / F^b A \quad \text{for} \quad 0 \leqslant a < b.$$

It is convenient to consider $F^a A$ as $F^a_\infty A$, $F^\infty A := 0$. Each $F^a_b A$, $a < b$, is a differential space and the direct sum

$$\bigoplus_{a<b} H(F^a_b A)$$

carries a canonical structure of an algebra over $\boldsymbol{R}$. Namely, the multiplication in A induces linear mappings

$$\mu = \mu^{ac}_{bd}\colon H(F^a_b A) \otimes H(F^c_d A) \to H(F^{a+c}_m A), \qquad m = \min(a+d, b+c),$$

such that the classical commutativity property is fulfilled.

It is easy to observe that any homomorphism between two filtered differential algebras induces a homomorphism of the homology algebras described above. The induced homomorphism commutes with the canonical mappings

$$H(F^a_b A) \to H(F^c_d A),$$

defined for $a \geqslant c$ and $b \geqslant d$, and

$$d_*\colon H(F^a_b A) \to H(F^b A).$$

To denote elements of $H(F^a_b A)$ we shall be writing

$$[z]^a_b := [z + F^b A] \quad \text{for} \quad z \in F^a A \cap d^{-1}(F^b A).$$

§ 2. The characteristic homomorphism

THEOREM 1. *For any* RF-*bundle* $B = (P(M, G), E, H, \xi)$ *with a projected foliation* F *the canonical algebra homomorphism*

$$\Phi(B)\colon \bigoplus_{a<b} H\big(F^a_b W(G)_H\big) \to \bigoplus_{a<b} H(I^a_F / I^b_F)$$

induced from the superposition

$$\Phi(B, \omega, s)\colon W(G)_H \xrightarrow{k(\omega)} A^*(P)_H \cong A^*(P/H) \xrightarrow{s^*} A^*(M)$$

is independent of ω, *a connection in* $P(M, G)$ *adapted to* E, *and of* $s \in \xi$.

Remark. The above characteristic homomorphism $\Phi(B)$ cannot be a monomorphism. If $q = \operatorname{codim} F$, then

$$H\big(F^a_b W(G)_H\big) \subset \ker \Phi \quad \text{for} \quad a > q.$$

Moreover, for $a \leqslant q < b$, Φ_b^a coincides with the superposition

$$H(F_b^a W(G)_H) \to H(F_{q+1}^a W(G)_H) \xrightarrow{\Phi} H(I_F^a).$$

The domain of $\Phi(B)$ divided by the above relations is linearly isomorphic with

$$\bigoplus_{0 \leqslant a < b \leqslant q} H(F_b^a W(G)_H).$$

For comparing with each other two adapted connections or two isotopic sections we introduce the trivial extension $\bar{B}$ of any RF-bundle $B = (P(M, G), E, H, \xi)$ through the projection $\boldsymbol{R} \times M \to M$. That is a particular case of the general procedure of inducing RF-bundles by smooth mappings ([1]). $\bar{B}$ consists of the principal G-bundle $\bar{\pi}$: $\boldsymbol{R} \times P \to \boldsymbol{R} \times M$ equipped with the trivial extensions $\bar{E}$ of E and $\bar{\xi}$ of ξ ($\bar{E}$ is the inverse image of E under the projection $T(\boldsymbol{R} \times P) \to TP$; $\bar{\xi}$ is the isotopy class of sections $\mathrm{id}_{\boldsymbol{R}} \times s$: $\boldsymbol{R} \times M \to \boldsymbol{R} \times P/H$, $s \in \xi$). If F is the projected foliation of B, then the projected foliation $\bar{F}$ of $\bar{B}$ has leaves of the form $\boldsymbol{R} \times L$, $L \in F$.

Let dt denote the differential of the appropriate one of the projections $\boldsymbol{R} \times P$, $\boldsymbol{R} \times M \to \boldsymbol{R}$. For the proof of Theorem 1 we shall need the following two lemmas:

LEMMA 3. *If a form $\varphi_1 + dt \wedge \varphi_2$ on $\boldsymbol{R} \times P$ is H-basic and neither φ_1 nor φ_2 involves dt, then both these forms are H-basic.*

Proof. Applying the mappings $i(x)$, $x \in \mathfrak{h}$, and $\varrho(h)$, $h \in H$, to $\varphi = \varphi_1 + dt \wedge \varphi_2$, we get

$$i(x)\varphi = i(x)\varphi_1 - dt \wedge i(x)\varphi_2,$$

and

$$\varrho(h)\varphi = \varrho(h)\varphi_1 + dt \wedge \varrho(h)\varphi_2.$$

Thus, the lemma follows from uniqueness of the decomposition of forms on $\boldsymbol{R} \times P$ into forms not-involving dt.

LEMMA 4. *Let φ be any form in $I_{\bar{F}}^k$, $k = 1, 2, \ldots$*

(i) *If φ decomposes into $\varphi_1 + dt \wedge \varphi_2$, where both φ_1 and φ_2 do not involve dt, then $\varphi_1, \varphi_2 \in I_{\bar{F}}^k$.*

(ii) *The form*

$$\int_{[0,1]} \varphi \in A^*(M)$$

obtained by integration of φ along the unit segment is an element of I_F^k.

Proof. (i): The unit vector field T tangent to the lines $\boldsymbol{R} \times \{p\}$, $p \in M$, is tangent to the foliation $\bar{F}$. Consequently, if $\deg\varphi = h$ and $v_1, \ldots, v_{h-k}$ are any vectors tangent to $\bar{F}$ at a common point, then

$$0 = \varphi(T, v_1, \ldots, v_{h-k}, \cdot, \ldots, \cdot) = \varphi_2(v_1, \ldots, v_{h-k}, \cdot, \ldots, \cdot).$$

Thus $\varphi_2 \in I_{\bar{F}}^k$ and so $\varphi_1 = \varphi - dt \wedge \varphi_2 \in I_{\bar{F}}^k$.

(ii): If $\deg\varphi = h$, then for any vectors $v_1, \ldots, v_h \in T_pM$, $p \in M$, there is

$$\Big(\int_{[0,1]} \varphi\Big)_p (v_1, \ldots, v_h) = \int_{0 \leqslant t \leqslant 1} \varphi(\bar{v}_1, \ldots, \bar{v}_h)(t, p),$$

where

$$(\bar{v}_i)_{(t,p)} = (0_t, v_i) \in T_t\boldsymbol{R} \oplus T_p M = T_{(t,p)}(\boldsymbol{R} \times M)$$

for $t \in \boldsymbol{R}$, $i = 1, \ldots, h$. If, moreover, the vectors $v_1, \ldots, v_{h-k+1}$ are tangent to F, then for all t's $\varphi(\bar{v}_1, \ldots, \bar{v}_{h-k+1}, \cdot, \ldots, \cdot)(t,p)$ is equal to zero and so

$$\Big(\int_{[0,1]} \varphi\Big)(v_1, \ldots, v_{h-k+1}, \cdot, \ldots, \cdot) = 0.$$

This concludes the lemma.

Proof of Theorem 1. It follows from Lemma 1 and Lemma 2 that the differential algebras $A^*(P)_H \cong A^*(P/H)$ are canonically filtered and that the homomorphisms composing $\Phi(B, \omega, s)$ (and so $\Phi(B, \omega, s)$ itself) preserve the filtrations, for any adapted connection ω and any section s. Let us consider any two connections ω_0, ω_1 adapted to E and any two isotopic sections s_0, s_1: $M \to P/H$, $s_0, s_1 \in \xi$. For $t \in \boldsymbol{R}$, we put

$$\omega_t = (1-t)\omega_0 + t\omega_1.$$

The 1-parameter collection $\{\omega_t;\ t \in \boldsymbol{R}\}$ of connections in $P(M, G)$ determines a connection ω in the induced G-bundle $\bar{\pi}$: $\boldsymbol{R} \times P \to \boldsymbol{R} \times M$,

$$\omega_{(t,u)} = (\omega_t)_u \circ \mathrm{pr}_{*(t,u)}, \qquad \mathrm{pr}: \boldsymbol{R} \times P \to P;$$

roughly speaking,

$$\omega = \omega_t \quad \text{over} \quad \{t\} \times P \quad \text{for} \quad t \in \boldsymbol{R}.$$

Since ω_t, $t \in \boldsymbol{R}$, are all adapted to E, ω is adapted to $\bar{E}$.

Similarly, any smooth isotopy $\boldsymbol{R} \ni t \mapsto s_t \in \xi$ determines a section s: $\boldsymbol{R} \times M \to \boldsymbol{R} \times P/H$, $(t, p) \mapsto (t, s_t(p))$ which, being isotopic with $\mathrm{id}_{\boldsymbol{R}} \times s_0$, is an element of $\bar{\xi}$.

For any nonnegative integer a the connections and sections constructed above yield the family of homomorphisms of differential spaces,

$$\Phi^a(\bar{B}, \omega, s): F^a W(G)_H \to I^a_{\bar{F}},$$

and

$$\Phi^a(B, \omega_t, s_t): F^a W(G)_H \to I^a_F, \quad t \in \boldsymbol{R}.$$

Starting from the equalities

$$\begin{aligned} \omega &= \omega_t, \\ \Omega &= \Omega_t + dt \wedge (\omega_1 - \omega_0), \end{aligned} \qquad \text{over} \quad \{t\} \times P,\ t \in \boldsymbol{R},$$

where Ω, Ω_t are the curvature forms of ω and ω_t, respectively, we get for any $z \in F^a W(G)_H$

$$k(\omega)(z) = \varphi_1 + dt \wedge \varphi_2, \tag{1}$$

where $\varphi_1, \varphi_2 \in A^*(\boldsymbol{R} \times P)$ do not involve dt,

$$\varphi_1 = k(\omega_t)(z) \quad \text{over} \quad \{t\} \times P,\ t \in \boldsymbol{R},$$

and both φ_1 and φ_2 are H-basic (Lemma 3). In view of Lemma 4 (i), if we apply the mapping $s^* \circ (\pi_H^*)^{-1}$ to both sides of formula (1), then we conclude

(2) $$\Phi^a(\bar{B}, \omega, s)(z) = \psi_1 + dt \wedge \psi_2,$$

where the forms ψ_1, ψ_2 do not involve dt and belong to $I_{\bar{F}}^a$, and there is

(3) $$\psi_1 = \Phi^a(B, \omega_t, s_t)(z) \quad \text{over} \quad \{t\} \times M,\ t \in \boldsymbol{R},$$

for $z \in F^a W(G)_H$.

Let us now consider any positive integer $b > a$ (possibly $b = \infty$) and let

$$z + F^b W(G)_H \in F_b^a W(G)_H$$

be any cycle. Since $z \in F^a W(G)_H$ and $dz \in F^b W(G)_H$, there is

$$d\Phi^a(\bar{B}, \omega, s)(z) = \Phi^b(\bar{B}, \omega, s)(dz) \in I_{\bar{F}}^b.$$

Consequently, for the forms ψ_1, ψ_2 introduced by (2), we have

$$d\psi_1 - dt \wedge d\psi_2 \in I_{\bar{F}}^b$$

and so, detaching the form dt from $d\psi_1$ and $d\psi_2$, we get

$$d'\psi_1 + dt \wedge (L_T \psi_1 - d'\psi_2) \in I_{\bar{F}}^b,$$

where d' is the partial exterior derivative with respect to M, and L_T is the Lie derivation in direction T. By Lemma 4(i), there is

(4) $$L_T \psi_1 - d'\psi_2 \in I_{\bar{F}}^b.$$

The integration of the above form along the unit segment gives us

$$i_1^* \psi_1 - i_0^* \psi_1 = \int_{[0,1]} L_T \psi_1 = d \int_{[0,1]} \psi_2 + \int_{[0,1]} (L_T \psi_1 - d'\psi_2),$$

where $i_t \colon M \to \{t\} \times M \hookrightarrow \boldsymbol{R} \times M$, $t = 0, 1$, are the injections. In view of (3), (4) and Lemma 4(ii), this means

$$\Phi^a(B, \omega_1, s_1)(z) \sim \Phi^a(B, \omega_0, s_0)(z) \ (\mathrm{mod}\, I_F^b),$$

for $z \in F^a W(G)_H \cap d^{-1} F^b W(G)_H$. Hence the induced mappings

$$\Phi_b^a(B, \omega_0, s_0)_*,\ \Phi_b^a(B, \omega_1, s_1)_* \colon H(F_b^a W(G)_H) \to H(I_F^a / I_F^b)$$

coincide. ∎

Looking through the characteristic invariants of B that constitute the image of $\Phi(B)$ one can easily find all the characteristic classes of B ([6], [7]) among them. These are given by the restriction Φ_{q+1}^0 of $\Phi(B)$ to the algebra $H(F_{q+1}^0 W(G)_H)$, $q = \operatorname{codim} F$.

For the induced filtration of the algebra

$$I(G) = W(G)_G$$

of G-invariant polynomials on $\mathfrak{g}$ there is a canonical injection

$$\bigoplus_{a<b} F_b^a I(G) \to \bigoplus_{a<b} H(F_b^a W(G)_H).$$

The appropriate restriction of $\Phi(B)$ is completely determined by the primary characteristic homomorphism

$$\Phi_I(B)\colon\ I(G) \to \bigoplus_a H(I_F^a)$$

that is equal to the superposition

$$\bigoplus_a I^a(G) \to \bigoplus_a F_\infty^a\, I(G) \xrightarrow{\Phi(B)} \bigoplus_a H(I_F^a)$$

(compare with [1]). The algebra homomorphism $\Phi_I(B)$ does not depend on the H-reduction structure of B.

In [1] the author introduced also a notion of secondary characteristic invariants of RF-bundles. Those (precisely, the "left" of them) are homology classes

$$w''(z) = [\lambda(\omega_0, \omega_1)(z) + I_F^k] \in H^{2k-1}(I_F^0/I_F^k), \tag{5}$$

defined for $z \in \ker\big(I^k(G) \to I^k(H)\big)$, $k = 1, 2, \ldots$, where $\lambda(\omega_0, \omega_1)\colon I(G) \to A^*(M)$ is the difference homomorphism ([3]) and ω_0, ω_1 are any two connections in $P(M, G)$ such that ω_0 is reducible through some H-reduction $P_s \hookrightarrow P$, $s \in \xi$, and ω_1 is adapted to the flat partial connection. It is necessary to compare the above secondary invariants with characteristic invariants constructed in the present paper.

PROPOSITION 1. *Let $B = \big(P(M, G), E, H, \xi\big)$ be an arbitrary* RF-*bundle. For any G-invariant polynomial $z \in I^k(G)$, $k = 1, 2, \ldots$, if there exists $z' \in W(G)_H$ such that $z = dz'$, then the secondary invariant $w''(z)$ corresponding to z is equal to $\Phi(B)([z']_k^0)$.*

Proof. If

$$\lambda\colon\ W(G) \to W(G)\otimes W(G) \tag{6}$$

denotes the universal homotopy ([7]),

$$\lambda \circ d_W + d_{W\otimes W} \circ \lambda = 1\otimes \mathrm{id} - \mathrm{id}\otimes 1,$$

and $\mu\colon W(G)\otimes W(G) \to W(G)$ is the multiplication, then for any two connections ω_0, ω_1 in $P(M, G)$ there is

$$\lambda(\omega_0, \omega_1) = (\pi^*)^{-1} \circ \mu \circ \big(k(\omega_0)\otimes k(\omega_1)\big) \circ \lambda|_{I(G)}.$$

Let $z \in I^k(G)$, $k = 1, 2, \ldots$, and $z' \in W(G)_H$ satisfy $z = dz'$. There is $\lambda(z) + d\lambda(z') = 1\otimes z' - z'\otimes 1$ and thus

$$\lambda(\omega_0, \omega_1)z = s^* \circ (\pi_H^*)^{-1} \circ k(\omega_1)z' - s^* \circ (\pi_H^*)^{-1} \circ k(\omega_0)z' - \\ - d\big(s^* \circ (\pi_H^*)^{-1} \circ \mu \circ \big(k(\omega_0)\otimes k(\omega_1)\big) \circ \lambda(z')\big),$$

for any section $s\colon M \to P/H$. The fact that the degree of z' is odd implies that the restriction homomorphism $W(G)_H \to I(H)$ maps z' to 0. On the other hand, if ω_0 is reducible through the inclusion $\bar{s}\colon P_s \hookrightarrow P$, then $\bar{s}^*\omega_0$ is a connection in the reduced H-bundle. Hence, there is

$$s^* \circ (\pi_H^*)^{-1} \circ k(\omega_0)z' = \big((\pi|P_s)^*\big)^{-1} \circ k(\bar{s}^*\omega_0)z' = 0.$$

Consequently, we get

$$\lambda(\omega_0, \omega_1)z \sim s^* \circ (\pi_H^*)^{-1} \circ k(\omega_1)z',$$

which concludes the proposition.

Remark. Let us observe that the ideal

$$I(G, H) := \{z \in I(G);\ z = dz' \text{ for some } z' \in W(G)_H\} \tag{7}$$

is only a subspace of $\ker(I(G) \to I(H))$. It is possible that (in general) there are secondary characteristic invariants which are not covered by $\Phi(B)$. Such a situation is impossible in the case of foliations (see § 3).

For the sake of completeness, let us also note that the homomorphism Φ^0_∞: $H(W(G)_H) \to H^*(M)$ factors into $H(W(G)_H) \to I(H)$ and the Chern–Weil homomorphism $I(H) \to H^*(M)$ of any of the H-bundles $P_s \to M$, $s \in \xi$.

F. W. Kamber and Ph. Tondeur ([6], [7]) have observed that since the homomorphisms $\Phi^0_{q+1}(B, \omega, s)$ are filtration preserving, they induce mappings (independent of ω and s) on all levels of the spectral sequences associated with the corresponding filtered complexes,

$$\Delta_i(B)_*\colon E_i(F^0_{q+1} W(G)_H) \to E_i(A^*(M)), \quad i = 1, 2, \ldots$$

We are now able give another description of derived characteristic—the invariants given by the above homomorphism of spectral sequences.

PROPOSITION 2. *For each $i = 1, 2, \ldots$ there exist canonical surjective algebra homomorphisms*

$$\bigoplus_p H(F^p_{p+i} W(G)_H) \to E_i(F^0_{q+1} W(G)_H),$$

and

$$\bigoplus_p H(I^p_F/I^{p+i}_F) \to E_i(A^*(M))$$

that make commutative the diagram

$$\begin{array}{ccc} \bigoplus_p H(F^p_{p+i} W(G)_H) & \xrightarrow{\Phi(B)} & \bigoplus_p H(I^p_F/I^{p+i}_F) \\ \downarrow & & \downarrow \\ E_i(F^0_{q+1} W(G)_H) & \xrightarrow{\Delta_i(B)_*} & E_i(A^*(M)) \end{array}$$

whenever F, $\operatorname{codim} F = q$, *is the projected foliation of $B = (P(M, G), E, H, \xi)$.*

Proof. Let us observe that for any filtered differential algebra $A = F^0 A \supset F^1 A \supset \ldots$ there is a canonical surjection

$$H(F^p_{p+i} A) \to E^p_i(A)$$

which coincides with the projection

$$Z^p_i/d(F^p) + F^{p+i} \to Z^p_i/Z^{p+1}_{i-1} + D^p_{i-1},$$

where $Z^p_i = F^p \cap d^{-1}(F^{p+i})$, $F^{p+i} \subset Z^{p+1}_{i-1} = F^{p+1} \cap d^{-1}(F^{p+i})$, and $d(F^p) \subset D^p_{i-1} = F^p \cap d(F^{p-i+1})$, $i \geq 1$, $p = 0, 1, \ldots$ Moreover, for any $q \geq 0$ the terms E^p_i of the spectral sequence corresponding to the induced filtration $F^0_{q+1} \supset F^1_{q+1} \supset \ldots$ are equal to

$$\bar{E}^p_i = Z^p_i + F^{q+1}/Z^{p+1}_{i-1} + D^p_{i-1} + F^{q+1} = Z^p_i/Z^{p+1}_{i-1} + D^p_{i-1} + Z^p_i \cap F^{q+1}$$

and so there are canonical surjections

$$H(F^p_{p+i}) \to \bar{E}^p_i, \quad i \geqslant 1.$$

To end the proof it now suffices to apply the above results to the filtrations of $A^*(M)$ and $W(G)_H$, respectively.

§ 3. Computations and a structure theorem

Let us assume that the homogeneous space G/H is reductive. Equivalently, there exists an H-equivariant projection $\vartheta\colon \mathfrak{g} \to \mathfrak{h}$ of the corresponding Lie algebras. Any such projection determines an algebraic H-connection in $W(G)$ and extends to a unique H-DG-algebra homomorphism

$$k(\vartheta)\colon\ W(H) \to W(G).$$

This homomorphism yields a homotopy equivalence

$$k(\vartheta)_H\colon\ I(H) \to W(G)_H$$

with homotopy inverse $W(i)$, $i\colon\ H \hookrightarrow G$, $\mathfrak{h} \hookrightarrow \mathfrak{g}$. In particular, the mapping $\big(k(\vartheta)_H\big)_*$ being equal to $(W(i)_H)_*^{-1}\colon I(H) \to H\big(W(G)_H\big)$ is independent of ϑ. Moreover, there is

$$I(G, H) = \ker\big(I(G) \to I(H)\big)$$

(see (7)). The universal homotopy (6) yields us a direct description of a linear mapping

$$\delta_\vartheta\colon\ I(G, H) \to W(G)_H$$

satisfying $d \circ \delta_\vartheta = \mathrm{id}$. Namely, we may put

$$\delta_\vartheta = \mu \circ \big(k(\vartheta) \circ W(i) \otimes \mathrm{id}\big) \circ \lambda | I(G, H).$$

Let us observe that for any $0 \neq z \in I^k(G, H)$, $k \geqslant 1$, $\delta_\vartheta z$ determines a nonzero element of $H\big(F^0_k W(G)_H\big)$. Moreover, the mapping

$$\delta\colon\ I^k(G, H) \ni z \mapsto [\delta_\vartheta z]^0_k \in H^{2k-1}\big(F^0_k W(G)_H\big)$$

is independent of the choice of ϑ and, by Proposition 1, for any RF-bundle B the superposition $\Phi(B) \circ \delta$ coincides with the secondary characteristic homomorphism (5).

The above constructions may be applied to the linear group $\mathrm{Gl}(q)$ and its orthogonal and identity subgroups, $\mathrm{O}(q)$ and $\{e\}$. In particular, among the invariants of any foliation F we can find the Pontryagin classes and the secondary Pontryagin classes of F ([1]). By the definition and by Proposition 2, if we put

$$\det(I + yA) = \sum_{i \geqslant 0} y^i p_i^{(q)}(A) \quad \text{for} \quad A \in \mathfrak{gl}(q),$$

$q = \operatorname{codim} F$, $y \in \boldsymbol{R}$, then:

the (primary) Pontryagin classes of F, $p_i(F) \in H^{2i}(I_F^i)$, are equal to

$$\Phi_I(B(F))(p_i^{(q)}) = \Phi(B(F))[p_i^{(q)}]_{q+1}^i,$$

for $i = 0, 1, \ldots$;

the (secondary) Pontryagin classes of F, $s_{2j+1}(F) \in H^{4j+1}(I_F^0/I_F^{2j+1})$, are equal to $\Phi(B(F))\,\delta p_{2j+1}^{(q)}$, for $j = 0, 1, \ldots$; and, if F is equipped with a trivialization t of its (dual) normal bundle,

the (secondary) Pontryagin classes of (F, t), $s_i(F, t) \in H^{2i-1}(I_F^0/I_F^i)$, are equal to $\Phi(B(F, t))\,\delta p_i^{(q)}$, for $i = 1, 2, \ldots$

Recall the well-known fact that there is

$$I(\mathrm{Gl}(q)) = \boldsymbol{R}[p_1^{(q)}, \ldots, p_q^{(q)}]$$

and that $I(\mathrm{Gl}(q), \mathrm{O}(q))$ is the ideal generated by the $p_i^{(q)}$'s with i odd. It is also worth noticing that there is

$$s_{2j+1}(F, t) = s_{2j+1}(F) \quad \text{for all } j\text{'s}.$$

The importance of the (primary and secondary) Pontryagin classes of foliations lies in their multiplicative properties (cf. [1], [2]) and in the fact that they generate all the characteristic invariants of foliations. To show this (see the Corollary of Theorem 2) we shall recall and slightly modify a construction described in [7]; Ch.V.

Throughout the rest of this section we shall assume that the Lie groups G, H satisfy the following additional conditions:

(8) (i) $I(G) = I(G_0) = I(\mathfrak{g})$ for the connected component G_0 of G;

(ii) H has finitely many components;

(iii) $(\mathfrak{g}, \mathfrak{h})$ is a reductive special Cartan pair; (compare [7]; Ch.V–VI).

Under the above restrictions on G and H one can find invariant polynomials

$$c_1, \ldots, c_r \in I(G), \quad \deg c_1 \leqslant \ldots \leqslant \deg c_r$$

such that $I(G) = \boldsymbol{R}[c_1, \ldots, c_r]$ and $I(G, H) = \ker(I(G) \to I(H))$ is the ideal generated by some c_{α_i}, $i = 1, \ldots, r'$, $\alpha_1 < \ldots < \alpha_{r'}$. Let $y_i \in (\bigwedge \mathfrak{g}^*)_H$ denote the suspension of c_{α_i}, $i = 1, \ldots, r'$. Following Kamber and Tondeur ([5], [7]), let us consider the differential filtered algebra

$$\hat{A} = \bigwedge(y_1, \ldots, y_r)\otimes I(G);$$
$$dy_i = c_{\alpha_i} \quad \text{for} \quad i = 1, \ldots, r', \quad d|I(G) \equiv 0,$$
$$F^a\hat{A} = \bigwedge(y_1, \ldots, y_r)\otimes F^a I(G).$$

For any H-equivariant projection $\vartheta\colon \mathfrak{g} \to \mathfrak{h}$ there is a unique algebra homomorphism

$$\iota\colon \hat{A} \to W(G)_H$$

such that

$$\iota(y_i) = \delta_\vartheta c_{\alpha_i}, \quad i = 1, \ldots, r',$$

and

$$\iota|I(G) \text{ is the inclusion.}$$

Since ι preserves the filtrations and commutes with the differentials, it determines canonically a graded homorphism of algebras

$$\iota_*: \bigoplus_{a<b} H(F_b^a \hat{A}) \to \bigoplus_{a<b} H\big(F_b^a W(G)_H\big).$$

Kamber and Tondeur proved that for any $b \geqslant 1$ the algebra homomorphism

$$\mu \circ \big(\iota_{*b}^{\,0} \otimes (k(\vartheta)_H)_*\big)\colon H(F_b^0 \hat{A}) \otimes I(H) \to H\big(F_b^0 W(G)_H\big),$$

μ being the multiplication, is surjective. In fact, they observed that the above mapping factors through an isomorphism

$$H(F_b^0 \hat{A}) \otimes_{I(G)} I(H) \cong H\big(F_b^0 W(G)_H\big).$$

We shall generalize slightly the above result.

LEMMA 5. *Under assumptions* (8) *the graded homomorphism of algebras*

$$\mu \circ \big(\iota_* \otimes (k(\vartheta)_H)_*\big)\colon \bigoplus_{a<b} H(F_b^a \hat{A}) \otimes I(H) \to \bigoplus_{a<b} H\big(F_b^a W(G)_H\big)$$

is surjective.

Proof. Let $a < b$ be any two positive integers. There is a commuting diagram of differential spaces

$$\begin{array}{ccccccccc} 0 & \to & F_b^a W(G)_H & \to & F_b^0 W(G)_H & \to & F_a^0 W(G)_H & \to & 0 \\ & & \uparrow{\scriptstyle \iota_b^a} & & \uparrow{\scriptstyle \iota_b^0} & & \uparrow{\scriptstyle \iota_a^0} & & \\ 0 & \to & F_b^a \hat{A} & \to & F_b^0 \hat{A} & \to & F_a^0 \hat{A} & \to & 0 \end{array}$$

with exact (and canonical) rows. Multiplying the corresponding long exact sequence by the single column

$$\begin{array}{c} H\big(W(G)_H\big) \\ \uparrow{\scriptstyle (k(\vartheta)_H)_*} \\ I(H), \end{array}$$

we get the following diagram

$$\begin{array}{ccccccc} \ldots \to & H(F_a^0) & \xrightarrow{d_*} & H(F_b^a) & \to & H(F_b^0) & \to \ldots \\ & \uparrow & & \uparrow & & \uparrow & \\ \ldots \to & H(F_a^0) \otimes H(F^0) & \xrightarrow{d} & H(F_b^a) \otimes H(F^0) & \to & H(F_b^0) \otimes H(F^0) & \to \ldots \\ & \uparrow & & \uparrow & & \uparrow & \\ \ldots \to & H(\hat{F}_a^0) \otimes I(H) & \xrightarrow{d_* \otimes \mathrm{id}} & H(\hat{F}_b^a) \otimes I(H) & \to & H(\hat{F}_b^0) \otimes I(H) & \to \ldots, \end{array}$$

which commutes and has exact rows. In the above, we accepted the abbreviations $F_j^i := F_j^i W(G)_H$, $\hat{F}_j^i := F_j^i \hat{A}$.

By the well-known "five lemma", surjectivity of $\mu \circ \big(\iota_{*b}^{\,a} \otimes (k(\vartheta)_H)_*\big)$ follows from surjectivity of $\mu \circ \big(\iota_{*i}^{\,0} \otimes (k(\vartheta)_H)_*\big)$, $i = a, b$.

Let us remind the $\mathbf{R}$-basis of $H(F_k^0 \hat{A})$, $k = 1, 2, \ldots$, distinguished in [5], [7]. The basis consists of the homology classes of

$$y_{(i)} \otimes c_{(j)} = y_{i_1} \wedge \ldots \wedge y_{i_s} \otimes c_1^{j_1} \ldots c_r^{j_r}$$

such that for $2p = \deg c_{(j)}$ there is

(9) (i) $0 \leqslant p < k$;

(ii) $1 \leqslant i_1 < \ldots < i_s \leqslant r$;

(iii) $\deg y_{i_1}+1 \geqslant 2(k-p)$ if $(i) \neq \emptyset$;

(iv) $j_{\alpha_l} = 0$ for $l < i_1$ if $(i) \neq \emptyset$ and $j_{\alpha_l} = 0$ for all l if $(i) = \emptyset$.

The property $dy_i = c_{\alpha_i}$ states that there are well-defined homology classes

$$[y_i] = [y_i]^0_{h_i} \in H(F^0_{h_i}\hat{A}),$$

where $\deg y_i = 2h_i-1$, $i = 1, \ldots, r'$. Similarly, for each c_i such that $2k_i = \deg c_i < 2k$, there is a homology class

$$[c_i]_k = [c_i]^{k_i}_k \in H(F^{k_i}_k\hat{A}).$$

If the monomial $y_{(i)}\otimes c_{(j)}$ fulfils conditions (9), then we have

$$(10) \qquad [y_{i_1}] \ldots [y_{i_s}]\,([c_1]_k)^{j_1} \ldots ([c_r]_k)^{j_r} = [y_{(i)}\otimes c_{(j)}]^p_m,$$

where $m = k$ if $(j) \neq \emptyset$ and $m = h_{i_1} \geqslant k$ if $(j) = \emptyset$.

In any case the canonical mapping $H(F^p_m\hat{A}) \to H(F^0_k\hat{A})$ transfers the above product to the element of the distinguished basis.

Let us observe that for $a < b$ the homomorphism $H(F^0_b\hat{A}) \to H(F^0_a\hat{A})$ maps $[y_{(i)}\otimes c_{(j)}]^0_b$ to zero if $2a \leqslant \deg c_{(j)} < 2b$ and to the element $[y_{(i)}\otimes c_{(j)}]^0_a$ of the distinguished basis of $H(F^0_a\hat{A})$ if $\deg c_{(j)} < 2a$. Thus, the linear space

$$\ker\big(H(F^0_b\hat{A}) \to H(F^0_a\hat{A})\big)$$

is freely spanned by $[y_{(i)}\otimes c_{(j)}]^0_b$ with $2a \leqslant \deg c_{(j)} < 2b$. Taking adventage of the long exact homology sequence that involves $H(F^a_b\hat{A})$, $H(F^0_a\hat{A})$ and $H(F^0_b\hat{A})$ we finally get

LEMMA 6. *For any positive integers a, b, $a < b$, there is*

$$H(F^a_b\hat{A}) = \operatorname{im}\big(d_*\colon H(F^0_a\hat{A}) \to H(F^a_b\hat{A})\big)\oplus$$
$$\oplus\operatorname{Lin}\{[y_{(i)}\otimes c_{(j)}]^a_b;\ 2a \leqslant \deg c_{(j)} < 2b \text{ and } \deg y_{i_1}+1 \geqslant 2b-\deg c_{(j)}\}.$$

We are now able to prove that under the algebraic conditions (8) all the characteristic invariants of an RF-bundle can be constructed from its primary and secondary characteristic invariants.

THEOREM 2. *Let G be a Lie group and $H \subset G$, a closed subgroup such that conditions* (8) *hold. For any* RF-*bundle $B = \big(P(M, G), E, H, \xi\big)$ with a projected foliation F, the characteristic homomorphism $\Phi(B)$ is completely determined by*

$$\Phi(B)|I(G)\cup I(H)\cup\delta I(G, H).$$

Precisely, if $I(G) = R[c_1, \ldots, c_r]$ and the ideal $I(G, H) = \ker\big(I(G) \to I(H)\big)$ is generated by c_{α_i}, $i = 1, \ldots, r'$, then $\Phi(B)$ is determined by the collection:

$$\Phi_I(B)(c_i) = \Phi(B)([c_i]^h_\infty) \in H^{2h}(I^h_F), \quad \textit{where} \quad 2h = \deg c_i, \quad i = 1, \ldots, r;$$

the characteristic homomorphism $\Phi(B)\circ(k(\vartheta)_H)_*\colon I(H)\to H^*(M)$ *of any of the isomorphic H-reductions* $P_s\to M$, $s\in\xi$; *and*

$$\Phi(B)\,\delta c_{\alpha_i}\in H^{2h-1}\big(A^*(M)/I_F^h\big),\quad \textit{where}\quad 2h=\deg c_{\alpha_i},\quad i=1,\dots,r'.$$

Proof. In view of Lemma 5, $\Phi=\Phi(B)$ is uniquely determined by the superposition

$$\Phi\circ\mu\circ\big(\iota_*\otimes(k(\vartheta)_H)_*\big)=\mu\circ\big(\Phi\circ\iota_*\otimes\Phi\circ(k(\vartheta)_H)_*\big).$$

Consequently, we may restrict ourselves to the mapping $\Phi\circ\iota_*$, which is induced from a filtered algebra homomorphism $\hat{A}\to A^*(M)$. By Lemma 6, the Theorem follows from (10) and general properties of the maps induced from homomorphisms of filtered differential algebras.

COROLLARY. *For any foliation F (resp., any framed foliation* (F,t)*) its Pontryagin classes together with its secondary Pontryagin classes completely determine the characteristic homomorphism* $\Phi(B(F))$ *(resp.,* $\Phi(B(F,t))$*).*

In particular, the Pontryagin classes and the secondary Pontryagin classes of a foliation generate the ring of exotic classes, and all the derived characteristic classes.

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