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ON ROBUST ESTIMATION IN THE SIMPLEST EXPONENTIAL MODEL

1. Introduction. Consider the statistical model

$$M_1 = (R_+^1, \mathscr{B}_+^1, \{P_{\lambda,1}: \lambda > 0\}),$$

where R_+^1 is the positive half-line, $\mathscr{B}_+^1$ is the family of Borel subsets of R_+^1, and $P_{\lambda,1}$ is the exponential distribution with probability density function (pdf) $f_{\lambda,1}(x) = \lambda^{-1}\exp(-x/\lambda)$ and the cumulative distribution function (cdf) denoted by $F_{\lambda,1}$. The problem consists in estimating λ.

As in [3], suppose that the model has been conceived as an approximation only and actually the underlying distribution is $P_{\lambda,p}$ with pdf

$$f_{\lambda,p}(x) = \big(\lambda\Gamma(1+1/p)\big)^{-1}\exp\big(-(x/\lambda)^p\big)$$

rather than $P_{\lambda,1}$, the shape parameter p being unknown. The extension of M_1 will be formally described by the mapping π from $\{P_{\lambda,1}: \lambda > 0\}$ into the family of all probability measures on $(R_+^1, \mathscr{B}_+^1)$, defined as

$$\pi(P_{\lambda,1}) = \{P_{\lambda,p}: p_1 \leqslant p \leqslant p_2\}$$

for some p_1 and p_2 ($0 < p_1 \leqslant 1 \leqslant p_2 < 2.16$). Now the problem is to construct an estimate of λ in the model M_1 which would be robust under the extension π. We shall be interested in robustness with respect to two properties of estimates: bias and mean square error.

As in [3], for an estimate T define

$$b_T(\lambda) = \sup_{p_1\leqslant p\leqslant p_2} (\mathrm{E}_{\lambda,p}T - \lambda) - \inf_{p_1\leqslant p\leqslant p_2} (\mathrm{E}_{\lambda,p}T - \lambda),$$

where $\mathrm{E}_{\lambda,p}T$ is the expectation of T under the distribution $P_{\lambda,p}$. Let S be another estimate of λ. We define T as *more bias-robust* (*b-robust* for short) *than* S if $b_T(\lambda) \leqslant b_S(\lambda)$ for all $\lambda > 0$ with strict inequality for at least one value of λ. We define T as the *most b-robust in a class* $\mathscr{T}$ of estimates if T is more b-robust than any $S \in \mathscr{T}$. The following two classes of estimates which are unbiased in the original model M_1 are of special interest: given a sample size n, let

$$\mathscr{T}_n = \Big\{T_n(a) = \sum_{j=1}^{n} a_j X_{j:n}\colon\ a_j \in R^1,\ j = 1, \dots, n;\ \mathrm{E}_{\lambda,1} T_n(a) = \lambda\Big\},$$

$$\mathscr{T}_n^+ = \Big\{T_n(a) = \sum_{j=1}^{n} a_j X_{j:n}\colon a_j \in R_+^1, j = 1, \dots, n; \mathrm{E}_{\lambda,1} T_n(a) = \lambda\Big\},$$

where $X_{1:n} \leqslant X_{2:n} \leqslant \dots \leqslant X_{n:n}$ are order statistics. The condition

$$\mathrm{E}_{\lambda,1} T_n(a) = \lambda \quad \text{for all } \lambda > 0$$

is obviously equivalent to the condition

$$\sum_{j=1}^{n} a_j e_{j:n} = 1, \tag{1.1}$$

where $e_{j:n} = \mathrm{E}_{1,1} X_{j:n}$.

The most b-robust estimate of λ in the class $\mathscr{T}_n^+$ under the extension π has been constructed in [3], and a generalization to some other extensions has been presented in [1].

Keeping in mind the bias of an estimate T as its property under consideration we define the *infinitesimal bias-robustness* (*ib-robustness* for short) of T as

$$\mathrm{ib}_T(\lambda) = \left|\left[\frac{d}{dp}(\mathrm{E}_{\lambda,p} T - \lambda)\right]_{p=1}\right|.$$

Our aim is to construct the most ib-robust estimates in $\mathscr{T}_n^+$ and in $\mathscr{T}_n$, i.e. an estimate T such that $\mathrm{ib}_T(\lambda) \leqslant \mathrm{ib}_S(\lambda)$ for all $S \in \mathscr{T}_n^+$ (respectively, $\mathscr{T}_n$). The results will be presented in Section 2. Observe that

$$\mathrm{ib}_{T_n(a)}(\lambda) = |b_n(a)|\lambda,$$

where $b_n(a) = [(d/dp)\mathrm{E}_{1,p} T_n(a)]_{p=1}$.

The mean square error of an estimate T defined as $\mathrm{E}_{\lambda,p}(T-\lambda)^2$ is another property of importance for applications. Define the *infinitesimal mean-square-error robustness* (*iv-robustness* for short) of T as

$$\mathrm{iv}_T(\lambda) = \left|\left[\frac{d}{dp}\mathrm{E}_{\lambda,p}(T-\lambda)^2\right]_{p=1}\right|.$$

Observe that

$$\mathrm{iv}_{T_n(a)}(\lambda) = |w_n(a)|\lambda^2, \tag{1.2}$$

where $w_n(a) = [(d/dp)\mathrm{E}_{1,p}(T_n(a)-1)^2]_{p=1}$. The most iv-robust estimates in $\mathscr{T}_n^+$ and in $\mathscr{T}_n$ will be presented in Section 3.

We close the Introduction with the following two remarks:

1. We are interested in robust estimates in small samples (non-asymptotic theory). The numerical results for some small values of the sample size n will be given. The algorithms of computations for any finite n will become clear.

2. The results depend heavily on the kind of extension of the original model. We shall illustrate the fact considering along with π the extension

$$\pi^*(P_{\lambda,1}) = \{P^*_{\lambda,p}: p_1 \leqslant p \leqslant p_2\},$$

where $P^*_{\lambda,p}$ is the distribution with pdf

$$\big(1/\lambda^p\Gamma(p)\big)x^{p-1}\exp(-x/\lambda),$$

i.e. the gamma distribution. Under a non-infinitesimal approach Bartoszewicz [1] showed that the most b-robust estimate in $\mathscr{T}_n^+$ under π^* is $X_{n:n}/(1+1/2+\ldots+1/n)$ while that under π is $nX_{1:n}$. Some numerical results for the extension π^* will be presented simultaneously with those for π.

We use the following short notation:

$$e_{j:n}(p) = \mathrm{E}_{1,p}X_{j:n}, \quad e_{ij:n}(p) = \mathrm{E}_{1,p}X_{i:n}X_{j:n},$$

$$V_{ij:n}(p) = \mathrm{Cov}_{1,p}(X_{i:n}, X_{j:n}),$$

$$m_{j:n} = \left[\frac{d}{dp}e_{j:n}(p)\right]_{p=1}, \quad m_{ij:n} = \left[\frac{d}{dp}e_{ij:n}(p)\right]_{p=1}.$$

The symbols $e_{j:n}$, $e_{ij:n}$, and $V_{ij:n}$ stand for $e_{j:n}(1)$, $e_{ij:n}(1)$, and $V_{ij:n}(1)$, respectively. The starred symbols, e.g., $P^*_{\lambda,p}$, $\mathrm{E}^*_{\lambda,p}$, $m^*_{j:n}$, have analogous meanings under the extension π^*.

Auxiliary technical results are presented in Section 4.

2. Infinitesimal bias-robust estimate and minimum variance infinitesimal bias-robust estimate in $\mathscr{T}_n$ and in $\mathscr{T}_n^+$. It is well known that the sample mean

$$\bar{X}_n = \sum_{j=1}^{n} X_j/n$$

is a minimum variance unbiased estimate for λ in M_1. Under the distribution $P_{1,p}$ we have

$$\mathrm{E}_{1,p}\bar{X}_n = \frac{\Gamma(2/p)}{\Gamma(1/p)}$$

so that the infinitesimal robustness of $\bar{X}_n$ under the extension π is of the form

$$\mathrm{ib}_{\bar{X}_n}(\lambda) = \lambda\left|\left[\frac{d}{dp}\left(\frac{\Gamma(2/p)}{\Gamma(1/p)}-1\right)\right]_{p=1}\right| = (2-\gamma)\lambda = 1.423\lambda,$$

where $\gamma = 0.577216$ is Euler's constant.

Consider an estimate

$$T_n(a) = \sum_{j=1}^{n} a_j X_{j:n}, \qquad a_j \geqslant 0.$$

According to a result in [3], for every p_1 and p_2 $(0 < p_1 \leqslant 1 \leqslant p_2 < 2.16)$ we have

$$n[\mathrm{E}_{1,p_1}X_{1:n} - \mathrm{E}_{1,p_2}X_{1:n}] \leqslant \sum_{j=1}^{n} a_j[\mathrm{E}_{1,p_1}X_{j:n} - \mathrm{E}_{1,p_2}X_{j:n}]$$

for any positive $a = (a_1, a_2, \ldots, a_n)$ satisfying (1.1). Dividing the above inequality by $p_2 - p_1$ and passing to the limits as $p_1 \to 1$ and $p_2 \to 1$ we obtain

$$-nm_{1:n} \leqslant -\left[\frac{d}{dp}\mathrm{E}_{1,p}T_n(a)\right]_{p=1}$$

for all $T_n(a) \in \mathscr{T}_n^+$. By (4.2) the left-hand side value is positive so that

$$|nm_{1:n}| \leqslant \left|\left[\frac{d}{dp}\mathrm{E}_{1,p}T_n(a)\right]_{p=1}\right|,$$

and we get the following result:

The statistic $nX_{1:n}$ is the most ib-robust estimate of λ in the class $\mathscr{T}_n^+$.

By (4.10) the ib-robustness of $nX_{1:n}$ is described as follows:

$$\mathrm{ib}_{nX_{1:n}}(\lambda) = \lambda\left(1 - \lambda + \frac{\log n}{n-1}\right).$$

Some numerical results are given in Table 1. The variance of the estimate $T_n(a)$ in the original model M_1 is given by the obvious formula

$$\mathrm{Var}_{1,1}T_n(a) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j V_{ij:n},$$

where $V_{ij:n}$ can be easily computed by (4.1), and is tabulated along with the ib-robustness b_n of $nX_{1:n}$. For comparison, similar results for the sample mean $\bar{X}_n$ and for infinitesimal robustness under the extension π^* are also presented. In the latter case, by a result given in [1], the most ib-robust estimate of λ in $\mathcal{T}_n^+$ is $X_{n:n}/(1+1/2+\ldots+1/n)$. The symbols in Table 1 have the following meanings: $b_n = b_n(a)$ for the most ib-robust estimate in $\mathcal{T}_n^+$ under π, v_n is the variance of this estimate, $\bar{b}_n = b_n(1/n, \ldots, 1/n)$ (sample mean), $\bar{v}_n = \mathrm{Var}_{1,1}\bar{X}_n$, and

$$v_n^* = \Big(\sum_{k=1}^{n} k^{-2}\Big)\Big/\Big(\sum_{k=1}^{n} k^{-1}\Big)^2.$$

TABLE 1

n	π-extension				π^*-extension			
	b_n	v_n	$\bar{b}_n$	$\bar{v}_n$	b_n^*	v_n^*	$\bar{b}_n^*$	$\bar{v}_n$
2	1.116	1	1.423	1/2	0.871	0.556	1	1/2
3	0.972	1	1.423	1/3	0.816	0.405	1	1/3
4	0.885	1	1.423	1/4	0.757	0.328	1	1/4
5	0.828	1	1.423	1/5	0.724	0.281	1	1/5
∞	0.423	1	1.423	0		0	1	0

To construct the most ib-robust estimate of λ in the class $\mathcal{T}_n$ we have to find $a = (a_1, a_2, \ldots, a_n)$ which minimizes (see (4.11))

$$|b_n(a)| = \Big|\sum_{j=1}^{n} a_j m_{j:n}\Big|$$

under the condition (1.1). Observe that, by (4.1) and (4.10),

$$\det\begin{bmatrix} m_{1:n} & m_{2:n} \\ e_{1:n} & e_{2:n} \end{bmatrix} = \frac{1}{n-1}\left(\frac{\log(n-1)}{n-2} - \frac{\log n}{n-1}\right) \neq 0,$$

so that for $n \geqslant 2$ the system of two linear equations

$$\sum_{j=1}^{n} a_j m_{j:n} = 0, \quad \sum_{j=1}^{n} a_j e_{j:n} = 1 \tag{2.1}$$

is consistent and if $a = (a_1, a_2, \ldots, a_n)$ is a solution of (2.1), then $\mathrm{ib}_{T_n(a)}(\lambda) \equiv 0$ and $T_n(a)$ is an absolutely ib-robust estimate. For $n \geqslant 3$ there exist infinitely many a's satisfying (2.1) and we can choose the "best" absolutely ib-robust estimate. We choose the estimate $T_n(a)$ with minimal variance in the original model M_1. To this end we have to find a minimizing $\mathrm{Var}_{1,1}T_n(a)$ under the conditions (2.1). The Lagrange-multiplier technique (with multipliers $2l_1$ and $2l_2$) gives us the following system of linear equations:

$$(2.2)\qquad \begin{gathered} \sum_{j=1}^{n} V_{ij:n}a_j + m_{i:n}l_1 + e_{i:n}l_2 = 0, \quad i = 1, \ldots, n, \\ \sum_{j=1}^{n} a_j m_{j:n} = 0, \quad \sum_{j=1}^{n} a_j e_{j:n} = 1. \end{gathered}$$

Some numerical results for $n = 2$, 3, and 4, along with analogous results for the π^*-extension, are presented in Table 2.

TABLE 2

		π-extension	π^*-extension
$n = 2$	a_1	7.455	-3.383
	a_2	-1.818	1.794
	v_2	11.249	3.851
$n = 3$	a_1	4.739	-2.836
	a_2	2.038	-0.005
	a_3	-1.242	1.064
	v_3	5.105	1.763
$n = 4$	a_1	3.566	-2.568
	a_2	2.068	-0.426
	a_3	0.825	0.307
	a_4	-0.956	0.747
	v_4	3.228	1.116

Studying the data of Tables 1 and 2 we have the impression that "the prize" for better bias-robustness of an estimate is an enlargement of its variance. To reconcile both tendencies one with another, in the next section we study the problem of iv-robust estimates.

Considering the system of linear equations (2.2) it is easy to see that $\mathrm{Var}_{1,1}T_n(a) = -l_2$ and applying Cramer's formulas we obtain

$$\mathrm{Var}_{1,1}T_n(a) = \frac{1}{n}\frac{1}{1-\Delta_n},$$

where

$$\Delta_n = \Big[\sum_{k=1}^{n} k(m_{n-k+1:n} - m_{n-k:n})\Big]^2 \Big/ n\sum_{k=1}^{n} k^2(m_{n-k+1:n} - m_{n-k:n})^2.$$

Studying the following values of Δ_n and Δ_n^*:

n	2	3	4	5	6	7
Δ_n	0.955	0.935	0.921	0.912	0.909	0.900
Δ_n^*	0.870	0.810	0.776	0.753	0.736	0.735

we may conjecture that $\mathrm{Var}_{1,1}T_n(a) = O(n^{-1})$ and $\mathrm{Var}_{1,1}T_n(a^*) = O(n^{-1})$ but we failed in proving any general result as yet.

3. Infinitesimal mean-square-error-robust estimates in $\mathscr{T}_n$ and in $\mathscr{T}_n^+$. To find the most iv-robust estimates in $\mathscr{T}_n$ we have to minimize $|w_n(a)|$ in (1.2) under the condition (1.1). There are two possibilities:

1° the quadratic form

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j m_{ij:n}$$

is non-negative (non-positive) definite and the minimum (maximum) of $w_n(a)$ on the subspace (1.1) is positive (negative);

2° the above quadratic form is non-negative (non-positive) and the minimum (maximum) of $w_n(a)$ on (1.1) is negative (positive) or the quadratic form under consideration is not definite.

In the former case the iv-estimate is given by the solution of the problem

$$w_n(a) = \min \text{ (resp. max)}, \quad \sum_{j=1}^{n} a_j e_{j:n} = 1.$$

In the latter case there exist a's such that

$$w_n(a) = 0 \quad \text{and} \quad \sum_{j=1}^{n} a_j e_{j:n} = 1$$

and we can choose that one which minimizes the variance of the estimate in the original model.

It appears that under the extension π the matrices $(m_{ij:n})_{i,j=1,\ldots,n}$ are negative definite, under the extension π^* the matrices $(m_{ij:n}^*)_{i,j=1,\ldots,n}$ are positive definite, at least for $n = 2, 3, 4, 5$, the maximum of $w_n(a)$ under (1.1) is negative in the former case, and the minimum of $w_n(a)$ is positive in the latter one. Some numerical results are presented in Table 3.

TABLE 3

		iv-robust estimates $T_n(a)$ in $\mathcal{T}_n$ under	
		π-extension	π^*-extension
	a_1	0.771	0.060
	a_2	0.410	0.647
$n = 2$	w_2	-1.870	0.443
	v_2	0.516	0.532
	$\bar{w}_2$	-1.923	0.500
	a_1	0.653	-0.034
	a_2	0.394	0.141
$n = 3$	a_3	0.248	0.487
	w_3	-1.225	0.279
	v_3	0.350	0.376
	$\bar{w}_3$	-1.281	0.333
	a_1	0.607	-0.061
	a_2	0.358	0.048
	a_3	0.253	0.150
$n = 4$	a_4	0.175	0.396
	w_4	-0.909	-0.202
	v_4	0.267	0.288
	$\bar{w}_4$	-0.961	0.250
	a_1	0.557	-0.070
	a_2	0.340	0.010
	a_3	0.243	0.073
	a_4	0.185	0.145
$n = 5$	a_5	0.135	0.335
	w_5	-0.723	0.158
	v_5	0.215	0.234
	$\bar{w}_5$	-0.769	0.200

$w_n = w_n(a)$ for the most iv-robust estimate,
v_n — the variance of this estimate,
$\bar{w}_n = w_n(1/n, \ldots, 1/n)$.

The numerical results show that under the π-extension the most iv-robust estimate in $\mathcal{T}_n$ is identical with the most iv-robust estimate in $\mathcal{T}_n^+$, at least for $n = 2, 3, 4, 5$. This is not the case under the π^*-extension.

4. Technical results. Simple calculations (see, e.g., [2], Chapter VIII.9) give us

$$e_{i:n} = \sum_{k=1}^{i} \frac{1}{n-i+k}, \quad V_{ij:n} = \sum_{k=1}^{i\wedge j} \left(\frac{1}{n-(i\wedge j)+k}\right)^2, \tag{4.1}$$

where $i \wedge j$ denotes the smaller of two numbers i and j.

The family of exponential-power distributions $\{P_{1,p}: 0 < p < 2.16\}$ is monotonic: if $0 < p < q < 2.16$, then $F_{1,p}(x) < F_{1,q}(x)$ for all $x > 0$. It follows that

(4.2) $e_{i:n}(p)$ is strictly decreasing in $p \in (0, 2.16)$.

To compute

$$m_{i:n} = \left[\frac{d}{dp} e_{i:n}(p)\right]_{p=1} \quad \text{and} \quad m_{ij:n} = \left[\frac{d}{dp} e_{ij:n}(p)\right]_{p=1}$$

we use the formulas

$$\text{(4.3)} \quad \begin{aligned} e_{i:n}(p) &= \frac{n!}{(i-1)!(n-1)!} \int_0^1 F_{1,p}^{-1}(t) \cdot t^{i-1}(1-t)^{n-i} dt \\ e_{ij:n}(p) &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \times \\ &\quad \times \int_0^1 du \int_u^1 dv \cdot F_{1,p}^{-1}(u) F_{1,p}^{-1}(v) \cdot u^{i-1}(v-u)^{j-i-1}(1-v)^{n-j} \end{aligned}$$

changing the order of integration and differentiation. To justify the procedure we have to prove the uniform integrability of the following families of functions for some p_1 and p_2 such that $0 < p_1 \leqslant 1 \leqslant p_2 < 2.16$:

$$\text{(4.4)} \qquad \left\{F_{1,p}^{-1}(x) \frac{d}{dp} F_{1,p}^{-1}(x):\ p_1 \leqslant p \leqslant p_2\right\},$$

$$\text{(4.5)} \qquad \left\{\frac{d}{dp} F_{1,p}^{-1}(x): p_1 \leqslant p \leqslant p_2\right\},$$

$$\text{(4.6)} \qquad \left\{F_{1,p}^{-1}(x) \frac{d}{dp} F_{1,p}^{-1}(y): p_1 \leqslant p \leqslant p_2\right\},$$

$$\text{(4.7)} \qquad \left\{\frac{d}{dp} F_{1,p}^{-1}(x) \cdot F_{1,p}^{-1}(y): p_1 \leqslant p \leqslant p_2\right\}.$$

The functions in (4.4) and (4.5) are defined on the interval $0 < x < 1$ and those in (4.6) and (4.7) on the set $\{(x, y): 0 < x < y < 1\}$. The proof is as follows.

Differentiation (with respect to p) of the identity

$$\frac{1}{\Gamma(1+1/p)} \int_0^{F_{1,p}^{-1}(x)} \exp(-t^p)\,dt = x$$

gives us

(4.8)

$$\frac{d}{dp} F_{1,p}^{-1}(x) = \exp\{[F_{1,p}^{-1}(x)]^p\} \int_0^{F_{1,p}^{-1}(x)} \left[t^p \log t - \frac{\psi(1+1/p)}{p^2}\right] \exp(-t^p)\,dt$$

$$= -\exp\{[F_{1,p}^{-1}(x)]^p\} \int_{F_{1,p}^{-1}(x)}^{\infty} \left[t^p \log t - \frac{\psi(1+1/p)}{p^2}\right] \exp(-t^p)\,dt,$$

where

$$\psi(t) = \frac{d}{dt} \log \Gamma(t)$$

is Euler's ψ-function. For any p we have

$$\frac{d}{dp} F_{1,p}^{-1}(0) = 0.$$

The integral

$$\int_0^y \left[t^p \log t - \frac{\psi(1+1/p)}{p^2}\right] \exp(-t^p)\,dt$$

is equal to zero for $y = 0$, is decreasing for $y \in (0, y_p)$, where y_p is the unique solution of the equation

$$t^p \log t = \frac{\psi(1+1/p)}{p^2},$$

and is increasing up to zero for $y \in (y_p, 1)$. It follows that

$$\frac{d}{dp} F_{1,p}^{-1}(x) < 0 \quad \text{for all } x > 0.$$

We have also

$$\frac{d}{dp} F_{1,p}^{-1}(x) \to -\infty \quad \text{as } x \to 1.$$

Using (4.8) and introducing a new variable $y = F_{1,p}^{-1}(x)$, for any $\delta \in (0,1)$ we have

$$(4.9)\qquad \int_{\delta}^{1}\left[F_{1,p}^{-1}(x)\frac{d}{dp}F_{1,p}^{-1}(x)\right]dx$$

$$= -\int_{F_{1,p}^{-1}(\delta)}^{\infty} dy \int_{y}^{\infty} dt\cdot y\left[t^{p}\log t - \frac{\psi(1+1/p)}{p^{2}}\right]\frac{\exp(-t^{p})}{\Gamma(1+1/p)}.$$

Let t_1 be the (unique) solution of the equation

$$t^{p_1}\log t = \frac{\psi(1+1/p_1)}{p_1^2}.$$

The right-hand side value is positive, and hence $t_1 > 1$. It follows that for all $t \geqslant t_1$ and $p \in [p_1, p_2]$

$$t^{p_2}\log t - \frac{\psi(1+1/p_2)}{p_2^2} \geqslant t^{p}\log t - \frac{\psi(1+1/p)}{p^2}$$

$$\geqslant t^{p_1}\log t - \frac{\psi(1+1/p_1)}{p_1^2} \geqslant 0.$$

Let $\delta_1 \in (0,1)$ be a number such that $F_{1,p_2}^{-1}(\delta_1) \geqslant t_1$. Then $F_{1,p}^{-1}(\delta_1 \geqslant F_{1,p_2}^{-1}(\delta_1)$ for $p \in [p_1, p_2]$. By (4.9), observing that

$$\Gamma(1+1/p_2) \leqslant \Gamma(1+1/p_1) \quad\text{and}\quad \exp(-t^{p}) \leqslant \exp(-t^{p_1}),$$

we obtain

$$\left|\int_{\delta_1}^{1}\left[F_{1,p}^{-1}(x)\frac{d}{dp}F_{1,p}^{-1}(x)\right]dx\right|$$

$$\leqslant \int_{F_{1,p_2}^{-1}(\delta_1)}^{\infty} dy \int_{y}^{\infty} dt\cdot y\left[t^{p_2}\log t - \frac{\psi(1+1/p_2)}{p_2^2}\right]\frac{\exp(-t^{p_1})}{\Gamma(1+1/p_2)}.$$

The integral

$$\int_{0}^{\infty} dy \int_{y}^{\infty} dt\cdot y\left[t^{p_2}\log t - \frac{\psi(1+1/p_2)}{p_2^2}\right]\frac{\exp(-t^{p_1})}{\Gamma(1+1/p_2)}$$

$$= \int_{0}^{\infty} dt \int_{0}^{t} dy\cdot y\left[t^{p_2}\log t - \frac{\psi(1+1/p_2)}{p_2^2}\right]\frac{\exp(-t^{p_1})}{\Gamma(1+1/p_2)}$$

$$= \frac{1}{2p_1\Gamma(1+1/p_2)}\left[\frac{1}{p_1}\Gamma\left(\frac{p_2+3}{p_1}\right)\psi\left(\frac{p_2+3}{p_1}\right) - \frac{1}{p_2^2}\Gamma\left(\frac{3}{p_1}\right)\psi\left(1+\frac{1}{p_2}\right)\right]$$

is finite so that for every $\varepsilon > 0$ there exists $\delta_2 = \delta_2(\varepsilon, p_1, p_2)$ such that for $\delta > \delta_2$ we have

$$\int_{F_{1,p_2}^{-1}(\delta)}^{\infty} dy \int_{y}^{\infty} dt \cdot y \left[t^{p_2} \log t - \frac{\psi(1+1/p_2)}{p_2^2} \right] \frac{\exp(-t^{p_1})}{\Gamma(1+1/p_2)} < \varepsilon .$$

Let $\delta' = \max\{\delta_1, \delta_2\}$. If $\delta > \delta'$, then

$$\int_{\delta}^{1} \left[F_{1,p}^{-1}(x) \frac{d}{dp} F_{1,p}^{-1}(x) \right] dx < \varepsilon ,$$

which proves the uniform integrability of (4.4). The uniform integrability of (4.5) follows from the fact that for x large enough we have

$$\left| \frac{d}{dp} F_{1,p}^{-1}(x) \right| \leqslant \left| F_{1,p}^{-1}(x) \frac{d}{dp} F_{1,p}^{-1}(x) \right| .$$

To prove the uniform integrability of (4.6) it is enough to show that for each $\varepsilon > 0$ there exists $\delta' \doteq \delta'(\varepsilon, p_1, p_2) \in (0, 1)$ such that

$$\left| \int_{0}^{1} \left(\int_{\max\{\delta, x\}}^{1} \frac{d}{dp} F_{1,p}^{-1}(y)\, dy \right) F_{1,p}^{-1}(x)\, dx \right| < \varepsilon$$

for $\delta > \delta'$ and for all $p \in [p_1, p_2]$. By the inequalities

$$\frac{d}{dp} F_{1,p}^{-1}(y) < 0 ,$$

$$\int_{0}^{1} F_{1,p}^{-1}(x)\, dx \leqslant \int_{0}^{1} F_{1,p_1}^{-1}(x)\, dx = \mathrm{E}_{1,p_1} X ,$$

we have

$$\left| \int_{0}^{1} \left(\int_{\max\{\delta, x\}}^{1} \frac{d}{dp} F_{1,p}^{-1}(y)\, dy \right) F_{1,p}^{-1}(x)\, dx \right|$$

$$= \int_{0}^{1} \left(\int_{\max\{\delta, x\}}^{1} \left| \frac{d}{dp} F_{1,p}^{-1}(y) \right| dy \right) F_{1,p}^{-1}(x)\, dx$$

$$\leqslant \int_{0}^{1} F_{1,p}^{-1}(x)\, dx \int_{\delta}^{1} \left| \frac{d}{dp} F_{1,p}^{-1}(y) \right| dy = \mathrm{E}_{1,p_1} X \int_{\delta}^{1} \left| \frac{d}{dp} F_{1,p}^{-1}(x) \right| dx$$

and the uniform integrability of (4.6) follows from that of (4.5).

The uniform integrability of (4.7) can be proved similarly by observing that

$$\int_0^1 \left| \frac{d}{dp} F_{1,p}^{-1}(x) \right| dx$$

$$= \frac{1}{p^2\Gamma(1+1/p)} \Gamma\left(1+\frac{2}{p}\right)\psi\left(1+\frac{2}{p}\right) - \frac{1}{p^2}\psi\left(1+\frac{1}{p}\right) \mathrm{E}_{1,p}X,$$

which is bounded on the interval $[p_1, p_2]$.

Differentiating (4.3) we obtain

$$m_{i:n} = \frac{n!}{(i-1)!(n-i)!} \int_0^1 \left[\frac{d}{dp} F_{1,p}^{-1}(t)\right]_{p=1} t^{i-1}(1-t)^{n-i} dt \tag{4.10}$$

$$= \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{i-1} (-1)^k \binom{i-1}{k} A^{(1)}(n-i+k),$$

where

$$A^{(1)}(l) = \int_0^1 \left[\frac{d}{dp} F_{1,p}^{-1}(t)\right]_{p=1} (1-t)^l dt, \qquad l = 0, 1, \ldots, n-1.$$

By the same reasoning we obtain

$$m_{ii:n} = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{i-1} (-1)^k \binom{i-1}{k} A^{(2)}(n-i+k),$$

where

$$A^{(2)}(l) = \int_0^1 \left(\frac{d}{dp} [F_{1,p}^{-1}(t)]^2\right)_{p=1} (1-t)^l dt, \qquad l = 0, 1, \ldots, n-1,$$

and

$$m_{ij:n} = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \sum_{r=0}^{i-1} \sum_{q=0}^{j-i-1} (-1)^{r+q} \binom{i-1}{r} \binom{j-i-1}{q} \times$$

$$\times A(j-i-q+r-1,\ n-j+q),$$

where

$$A(k,l) = \int_0^1 dx \int_x^1 dy \left(\frac{d}{dp}\,[F_{1,p}^{-1}(x)F_{1,p}^{-1}(y)]\right)_{p=1}(1-x)^k(1-y)^l,$$

$$k,l = 0,1,\ldots,n-1.$$

Observe that

$$A(k,l) = \frac{1}{(k+1)^2}A^{(1)}(l)+\left[\frac{1}{(l+1)^2}-\frac{1}{(k+1)^2}\right]A^{(1)}(k+l+1)+$$
$$+\frac{1}{2}\left[\frac{1}{l+1}-\frac{1}{k+1}\right]A^{(2)}(k+l+1)$$

so that, given an extension π, it is enough to tabulate the values of $A^{(s)}(l)$, $s = 1,2$.

Under the extension π we have

$$F_{1,p}(t) = \frac{1}{\Gamma(1+1/p)}\int_0^t \exp(-x^p)\,dx$$

and

$$\left[\frac{d}{dp}F_{1,p}^{-1}(t)\right]_{p=1} = 1-\gamma-\frac{1}{1-t}\int_{\log 1/(1-t)}^{\infty}(u\log u)e^{-u}du.$$

Hence, after some simple calculations, we get

$$A^{(1)}(l) = \begin{cases} \gamma-2 & \text{if } l = 0, \\ -\dfrac{(1-\gamma)l+\log(l+1)}{l(l+1)^2} & \text{if } l \geqslant 1 \end{cases}$$

and

$$A^{(2)}(l) = \begin{cases} -4(1-\gamma)-5 & \text{if } l = 0, \\ -\dfrac{2}{l(l+1)^3}\left[2(1-\gamma)l-1+\dfrac{(3l+1)\log(l+1)}{l}\right] & \text{if } l \geqslant 1. \end{cases}$$

Analogous results for the π^*-extension are

$$A^{*(1)}(l) = \begin{cases} 1 & \text{if } l = 0, \\ \dfrac{\log(l+1)}{l(l+1)} & \text{if } l \geqslant 1 \end{cases}$$

and

$$A^{*(2)}(l) = \begin{cases} 3 & \text{if } l = 0, \\ \dfrac{2}{l^2(l+1)^2}[(2l+1)\log(l+1)-l] & \text{if } l \geqslant 1. \end{cases}$$

The simple formulas

$$b_n(a) = \left[\frac{d}{dp}\,\mathrm{E}_{1,p}T_n(a)\right]_{p=1} = \sum_{j=1}^{n} a_j m_{j:n} \tag{4.11}$$

and

$$w_n(a) = \left[\frac{d}{dp}\mathrm{E}_{1,p}\big(T_n(a)-1\big)^2\right]_{p=1} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j m_{ij:n} - 2b_n(a)$$

enable us to construct the most ib-robust and iv-robust estimates.

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