ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XVIII, 2 (1984), p. 225-229

A. STYSZYŃSKI (Wrocław)

ON MINIMAX ESTIMATION OF THE PARAMETER IN THE PÓLYA URN SCHEME

1. Introduction. The minimax estimation procedure is well-known in mathematical statistics. When no prior information about the distribution of the parameter is given, the procedure has its advantages [6] and the minimax estimators may have practical importance. Applying the minimax principle we meet the problem of determining the so-called *least favorable prior distribution of the parameter*. In the present paper we find the least favorable prior distribution by solving an integral equation. Applying the theorems on the existence of the solution of the so-called moment problem, we may establish a general method for the determination of the prior distribution.

Let us formulate the minimax estimation problem for the parameter in the Pólya urn scheme. We consider a random variable X distributed in $\mathscr{X} = \{0, 1, \ldots, n\}$ according to

$$P\{X = k \mid \theta = p\} = \binom{n}{k} \frac{c_k(\gamma p) c_{n-k}(\gamma q)}{c_n(\gamma)}, \quad k \in \mathscr{X}, \tag{1}$$

where $c_0(x) = 1$, $c_k(x) = \prod_{i=0}^{k-1} (x+i)$, $\gamma > 0$, and $p \in [0, 1]$ $(p+q = 1)$ is unknown. The random variable X describes the Pólya urn scheme with $b = Np$ white balls, $c = Nq$ black balls, and $s = N\gamma^{-1}$ white or black balls added each time a white or black ball is drawn. In a fixed sample size experiment we consider the quadratic loss function

$$L[f(X), p] = [f(X) - p]^2,$$

where $f(X)$ is an estimate of the parameter $p \in [0, 1]$.

Now, the risk function is given by

$$R(f, p) = \mathrm{E}_p\{L[f(X), p]\},$$

where E_p denotes the expectation with respect to the probability function in (1). If $G(p)$ is a prior distribution of θ, then the expected risk is given by

$$r(f, G) = \int_0^1 R(f, p)\, dG(p).$$

An estimator f^* satisfying the condition

$$\sup_G r(f^*, G) = \inf_f \sup_G r(f, G)$$

is called a *minimax estimator*. In the sequel we base on the well-known Hodges-Lehmann result [1]: every Bayes estimate for which the risk is constant is a minimax one.

We prove that for some finite positive values of γ there exists a minimax estimator of p in the Pólya urn scheme. In the cases $\gamma = \infty$ (binomial) and $\gamma = -N$ (hypergeometric) the minimax estimators under a quadratic loss function were found by Steinhaus [3] and Trybuła [5], respectively.

2. The constant risk estimator. Assume that a linear estimator $f(X) = aX + b$ of the parameter p in (1) is used. We find that

$$R(f, p) = \left[(an-1)^2 - a^2 n \frac{1+na}{1+a}\right] p^2 + \left[[2b(an-1) + a^2 n \frac{1+na}{1+a}\right] p + b^2,$$

where $a = \gamma^{-1}$.

Assuming that $R(f, p)$ is constant and that $f(k) \in [0, 1]$, $k \in \mathscr{X}$, we obtain

LEMMA 1. *The estimator*

$$f_0(X) = \frac{X + \frac{1}{2}\sqrt{n(na+1)/(a+1)}}{n + \sqrt{n(na+1)/(a+1)}} \tag{2}$$

results in a constant risk.

3. The Bayes estimator and prior distribution. Now, let $G(p)$ be any prior distribution of θ. Since the loss function is quadratic, we can use the general form of the Bayes estimator f^G. We have

$$f^G(k) = \frac{\int_0^1 p\, P\{X = k \mid \theta = p\}\, dG(p)}{\int_0^1 P\{X = k \mid \theta = p\}\, dG(p)}, \qquad k \in \mathscr{X}. \tag{3}$$

Let us introduce the function

$$\beta_x(\gamma p, \gamma q) = \frac{x^{\gamma p-1}(1-x)^{\gamma q-1}}{B(\gamma p, \gamma q)}, \quad x \in [0, 1].$$

Notice that the probability function in (1) has the following representation (see [4]):

$$P\{X = k \mid \theta = p\} = \binom{n}{k} \int_0^1 x^k (1-x)^{n-k} \beta_x(\gamma p, \gamma q)\, dx.$$

Now, we seek a prior distribution $G(p)$ such that the estimator in (3) is equal to the constant risk estimator given in (2). This results in the integral equation

$$\int_0^1 \beta_x(\gamma p, \gamma q)\, dG(p) = \beta_x(\varrho, \varrho), \quad x \in [0, 1], \tag{4}$$

where $\varrho^{-1} = 2[a + \sqrt{(a+1)(a+1/n)}]$.

Let us examine the existence of a solution of equation (4). First, we write (4) in the form

$$x^{\varrho}(1-x)^{\varrho} = \int_0^1 x^{\gamma p}(1-x)^{\gamma(1-p)}\, dG(p),$$

where $x \in [0, 1]$ and G is a non-negative measure depending on γ. Putting $e^{-u} = x(1-x)^{-1}$, we obtain

$$e^{-\varrho u}(1+e^{-u})^{\gamma-2\varrho} = \int_0^1 e^{-\gamma u p}\, dG(p).$$

Introducing $\varphi(u) = e^{-\varrho u}(1+e^{-u})^{\gamma-2\varrho}$, we finally have

$$\varphi(u) = \int_0^{\gamma} e^{-uz}\, dF(z),$$

where F is a measure with support on $[0, \gamma]$ corresponding to the measure G. Now, in order that $\varphi(u)$ be the Laplace-Stieltjes transform of F we should require that $\gamma - 2\varrho = m$ (see [2]), where m is a positive integer. In that case we obtain $m + \varrho = \gamma - \varrho < \gamma$ and

$$\varphi(u) = e^{-\varrho u} + \sum_{l=1}^{m} \binom{m}{l} e^{-(l+\varrho)u}$$

as well as

$$\varphi(u) = \int_0^{\gamma-\varrho} e^{-uz}\, dF(z).$$

Hence we have

LEMMA 2. *If $\gamma-2\varrho$ is a positive integer, then equation* (4) *has a solution G^* given by*

$$P\left\{\theta=\frac{\varrho+l}{\gamma}\right\}=\binom{m}{l}\frac{B(\varrho+l,\,\varrho+m-l)}{B(\varrho,\,\varrho),},\qquad l=0,1,\ldots,m,$$

where $m=\gamma-2\varrho$.

Now, we check that the Bayes estimator f^{G^*} given by (3) is a constant risk estimator. From (3) and (1) we obtain

$$f^{G^*}(k)=\gamma^{-1}\left[\frac{\int_0^1 c_{k+1}(\gamma p)c_{n-k}(\gamma q)\,dG^*(p)}{\int_0^1 c_k(\gamma p)c_{n-k}(\gamma q)\,dG^*(p)}-k\right].$$

From (4) we infer that

$$c_{n+1}(2\varrho)\int_0^1 c_{k+1}(\gamma p)c_{n-k}(\gamma q)\,dG^*(p)=c_{n+1}(\gamma)c_{k+1}(\varrho)c_{n-k}(\varrho)$$

and

$$c_n(2\varrho)\int_0^1 c_k(\gamma p)c_{n-k}(\gamma q)\,dG^*(p)=c_n(\gamma)c_k(\varrho)c_{n-k}(\varrho).$$

Using the last equalities and the definition of ϱ we have

$$f^{G^*}(k)=\frac{(\gamma-2\varrho)k+\varrho(\gamma+n)}{\gamma(2\varrho+n)}=\frac{k+\frac{1}{2}\sqrt{n(na+1)/(a+1)}}{n+\sqrt{n(na+1)/(a+1)}}=f_0(k),\qquad k\in\mathscr{X}.$$

Thus, by Lemmas 1 and 2 we have

THEOREM. *If $\gamma-2\varrho$ is a positive constant integer, then the estimator f_0 given in* (2) *is a minimax estimator of the parameter p in the Pólya urn scheme.*

As an example, let us consider the Pólya urn scheme with $s=1$, $N=7$, and $n=2$. We find that $\gamma=7$ and $\varrho=0.5$. The condition of Lemma 2 is satisfied and $m=\gamma-2\varrho=6$.

References

[1] J. L. Hodges and E. L. Lehmann, *Some problems in minimax point estimation*, Ann. Math. Statist. 21 (1950), p. 182-197.

[2] J. Keilson and F. W. Steutel, *Mixtures of distributions, moment inequalities and measures of exponentiality and normality*, Ann. Probability 2 (1974), p. 112-131.

[3] H. Steinhaus, *The problem of estimation*, Ann. Math. Statist. 28 (1957), p. 633-648.

[4] L. Takács, *Combinatorial methods in the theory of stochastic processes*, New York 1967.

[5] S. Trybuła, *Some problems of simultaneous minimax estimation*, Ann. Math. Statist. 29 (1958), p. 245-253.

[6] S. Zacks, *The theory of statistical inference*, New York 1971.

INSTITUTE OF MATHEMATICS
WROCŁAW TECHNICAL UNIVERSITY
50-370 WROCŁAW

Received on 23. 5. 1980