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LORENTZ MANIFOLDS AND GENERAL RELATIVITY THEORY

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The aim of this brief note is to give a summary of how one sets up the mathematics of General Relativity as the study of differential geometry on a restricted type of manifold. The work is based mainly on that of Penrose, Hawking and Geroch and only an outline of the general theme will be given, more details being available in the references.

The success of the Minkowski space-time approach to Special Relativity, together with the principle of equivalence and the principle of covariance, led to a model of General Relativity Theory which consists of a four dimensional differentiable, Hausdorff, real manifold M which carries a Lorentz metric (of signature $+2$) [1]. However, many other properties will be required of the manifold if it is to be a viable model of space-time and it is these extra conditions which will be discussed here. The choice of a manifold as an arena in which to put space-time physics reflects the desire that space-time at the classical (non-quantum) level should be a "continuum". The choice of M as real and four dimensional reflects the intuitive concepts of the three dimensions of locally Euclidean space together with the one dimensional real line for the time axis. The Hausdorff condition is supposed to follow from experience.

In order that the field equations (to be discussed later) be defined, the metric must be at least C^2. However, the exact order of differentiability of the metric is probably not significant and will henceforth assumed to be smooth (C^∞). (One can always consider the manifold to be smooth since any C^r manifold with $r \geqslant 1$ admits a smooth subatlas according to Whitney's theorem [2]). It is therefore important to establish which manifolds admit Lorentz metrics and to this end, the following well known theorems are important.

THEOREM 1. *Let M be a smooth, connected Hausdorff manifold. Then the following conditions are equivalent.*(1)

(1) The connected condition is introduced here because it is required on physical grounds later. However, Theorem 1 is readily modified if the connected condition is dropped.

(i) *M admits a* C^∞ *partition of unity.*

(ii) *M is paracompact.*

(iii) *M is second countable.*

(iv) *M admits a* C^∞ *positive definite Riemannian metric.*

The proof of this theorem can be gathered from the standard texts on manifold theory and differential geometry.

THEOREM 2 [1], [3]. *Let M be a smooth, connected, paracompact manifold. Then the following conditions are equivalent*:

(i) *M admits a* C^∞ *one dimensional distribution.*

(ii) *M admits a* C^∞ *Lorentz metric.*

Now let M be a smooth, connected, paracompact manifold. Then by Theorem 1 above, M admits a C^∞ positive definite metric. If M also admits a C^∞ one dimensional distribution, then a simple construction using the positive definite metric and the distribution yields a C^∞ Lorentz metric for which the members of the distribution determine timelike vectors in an obvious way [1]. Further, the paracompactness condition is necessary as well as sufficient, since all smooth, connected, Hausdorff manifolds which admit a C^r Lorentz metric ($r \geqslant 2$) are necessarily paracompact [4].

The problem now is to consider which manifolds admit one dimensional distributions. The following theorems summarise the situation:

THEOREM 3 (Hopf) [3], [5]. *Let M be a smooth, connected, compact Hausdorff manifold. Then the following conditions are equivalent*:

(i) *M admits a* C^∞ *nowhere zero vector field.*

(ii) *M admits a* C^∞ *one dimensional distribution.*

(iii) *M admits a* C^∞ *Lorentz metric.*

(iv) *The Euler characteristic of M is zero.*

THEOREM 4 (Hirsch [6], Penrose [7]). *If M is a smooth, connected, paracompact, non-compact manifold, then M admits a smooth, nowhere zero vector field. Hence M admits a smooth Lorentz metric by Theorem* 2.

It turns out that there are independent reasons for rejecting compact manifolds as possible models for space-time (to be discussed later) and so from the above results, the conditions on M so far, are that it be a smooth, connected, paracompact, non-compact manifold with a C^∞ Lorentz metric.(2)

One should note here the directional character of a Lorentz metric and consequently how it picks out a null cone at each point. This is reflected in the necessity of M admitting a one dimensional distribution (or *line element* field as it is often

(2) Hawking and Ellis [1] also add the condition that M be *inextendible*, that is M cannot be isometrically embedded into a "larger" space-time M' also satisfying the above properties.

called). One should note also that certain well known results for positive definite metrics fail for manifolds with Lorentz metrics. For example, if M has a positive definite metric g, then if M is compact, it follows that (M, g) is *geodesically complete* and that if (M, g) is geodesically complete, then the exponential map at $p \in M$ maps the tangent space at p, $T_p(M)$, onto the whole of M. Neither of these results is true for a Lorentz manifold (M, γ) where γ is a Lorentz metric on M [8].

At each point $p \in M$, the timelike vectors determined by the Lorentz metric separate into two components in the usual topology on $T_p(M)$. Intuitively one would like to call one of these components the family of future-pointing timelike vectors at p and the others the family of past-pointing timelike vectors at p. The distinguishability of the future direction at p is suggested by local thermodynamical considerations. However, such a choice would have no physical significance if it could not be made consistently and smoothly over M. Further, the existence of a C^∞ one dimensional distribution is not sufficient to ensure that the Lorentz metric constructed from it allows the above choice to be made. Such a choice is available when and only when the distribution is spanned by a nowhere zero (necessarily timelike) C^∞ vector field. In this case the distribution is *orientable* and the space-time is called *time-orientable*. It follows that a space-time is time-orientable if and only if it admits a global nowhere zero C^∞ timelike vector field. It also follows from Theorem 4 that all manifolds admitted so far have time-orientable Lorentz metrics but a *given* Lorentz metric need not be time-orientable [9]. If a space time is not time-orientable, then for each $p \in M$, there exists a closed continuous curve through p (not homotopic to zero) such that if the future direction of time is "carried" smoothly around the curve and back to p it is found to be no longer future pointing at p. (However, if M carries a non time-orientable Lorentz metric, then there exists a two-fold covering manifold M^* which possesses a time-orientable Lorentz metric locally isometric under the usual projection $M^* \to M$ to the original one on M.) Hence if the fundamental group $\pi_1(M)$ of M has no proper subgroups of index two (in particular if M is simply connected) then every smooth one dimensional distribution on M is orientable [10]). One therefore imposes the additional condition that M be time-orientable. Similarly one can discuss *space-orientability*. To do this, let M be a Lorentz manifold with a (timelike) one dimensional distribution d. At each $p \in M$, the three dimensional subspace of $T_p(M)$ orthogonal to d admits two (space) orientations. One calls M *space-orientable* if one can choose this orientation consistently and smoothly over M (the definition being independent of the one dimensional distribution selected). In space-orientable space-times there is a consistent choice of right handedness and one can carry a triad of right handed space axes consistently and smoothly around any closed curve in M. The comments above concerning covering manifolds and the fundamental group of M for time-orientability apply in a similar way for space-orientability. (There is a similar notion of charge-orientability—the ability to define positive charge consistently and smoothly over M [9]). One should note that although the individual conditions of time- and space-orientability are dependent on the Lorentz metric, when combined they imply that M is orientable in the usual sense, and this is a property of the manifold M and not of the Lorentz metric. In fact the properties of M being time-orientable, space-orientable and orientable are such that any two of them imply the third. Experimental results from elementary particle physics and the result from thermodynamics mentioned earlier, together with the C.P.T. theorem from field theory, suggest that the universe is time- and space-orientable [1], [9], [11]. So this condition is usually built into the Lorentz metric for a reasonable model of space-time.

The material content of space-time is described by various types of fields on M upon which physics imposes certain restrictions. These postulates have been fully discussed by Hawking and Ellis [1] and can be summarised as follows:

(a) If $p \in M$ and U is a convex normal neighbourhood of p, then a signal can be sent in U from p to another point $q \in U$ if and only if p and q can be joined by a C^1 curve in U whose tangent vector is never zero and is everywhere non-spacelike. This allows the null cone at p to be determined by observation and the metric at p to be determined to within a conformal factor. It is now seen why the restriction that M is connected was imposed. For if M was not connected, two distinct components of M would have no communication with each other and neither could explore the physics of the other.

(b) The material content of the space-time is represented in the field equations entirely by a symmetric (energy-momentum) tensor T with components T_{ab} ($1 \leqslant a, b \leqslant 4$) depending on the fields, their covariant derivatives and the metric and is such that T is zero on some open set U of M if and only if there are no matter fields in U, and such that $T^b_{a;b} = 0$, where a semi-colon denotes a covariant derivative with respect to the Lorentz metric.

(c) Einstein's field equations (³)

$$R_{ab} - \tfrac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \tag{1}$$

hold on M. In (1), Λ is the *cosmological constant* (⁴) and R_{ab}, g_{ab} and R are, respectively, the Ricci tensor, the Lorentz metric tensor and the Ricci scalar. One notes that $T^b_{a;b} \equiv 0$.

Apart from these postulates one would normally assume that M satisfied the *dominant energy condition* [1], namely that if t^a are the components of a timelike vector at $p \in M$, then $T_{ab}t^at^b \geqslant 0$ and $T^a{}_bt^b$ are the components of a non-spacelike vector at p. This ensures the non-negative nature of energy density and the non-spacelike nature of the local energy flow vector. These conditions restrict the algebraic type of the energy-momentum tensors available [1], [14], [15].

Whereas postulate (a) determines the metric to within a conformal factor, postulate (b) together with the positive energy density assumption can determine the metric to within a constant factor (that is to within the units of measurement)

(³) Here we consider only Einstein's theory of gravitation. For a full review of other theories see [12].

(⁴) There is still dispute about the occurrence of Λ (see [13]).

by the observation of the timelike geodesics followed by small isolated bodies.

Postulate (a) covers causality and the nature of local signal propagation in General Relativity. Now one can consider *global causality.* Allowing the usual interpretation of "free will" in the universe, one then requires that M admit no closed timelike curves because of the obvious contradictions this would involve. However, this condition is not sufficient from the physical viewpoint since a space-time M with no closed timelike curves might contain "almost closed" timelike curves such as those discussed in more detail in [1], [8], [16]. Arguments from quantum theory and the uncertainty principle [1] suggest that the Lorentz metric on M should be "stable" in its not admitting closed timelike curves. By this is meant that the Lorentz metric γ on M admits no closed timelike curves and that γ is contained in a neighbourhood U in the C^0 open topology on the bundle L of all C^r Lorentz metrics ($r \geqslant 1$) on M, no member of which admits closed timelike curves [1], [17], [18]. Such a Lorentz metric is called *stably causal.* It can now be seen why compact manifolds were rejected as realistic models of space-time since a smooth compact four dimensional manifold admitting a Lorentz metric necessarily also admits closed timelike curves [19], [20], [8].

There has been much recent discussion of the topological aspects of General Relativity. The usual manifold topology on M is a "homogeneous" topology and reflects the locally $\boldsymbol{R}^4$ nature of M rather than its Lorentz metric structure, which has a directional character. One might therefore consider other topologies for M which are not homogeneous and from which one can deduce the null cone structure at each point. Work on this subject has been pioneered by Zeeman [21] who proposed a topology for *Minkowski space* with the above mentioned property and which is the finest topology on M which induces the real line topology on each straight timelike line and the three dimensional Euclidean topology on each spacelike hyperplane. Zeeman called it the *fine topology* for Minkowski space and in fact it is strictly finer than the usual topology for Minkowski space. It has the further properties that its homeomorphism group is that group generated by the inhomogeneous Lorentz group together with the dilatations of M and that when M has the fine topology, all strictly order preserving (see [21]) continuous maps $[0, 1] \to M$ have piecewise linear images, consisting of a finite number of straight timelike line segments, as one might expect of the world line of a free particle undergoing a finite number of collisions. However, the fine topology is somewhat complicated, failing to satisfy normality, local compactness and first countability. Zeeman's work has been extended to General Relativity by Göbel [22], [23] and has recently been improved both from the physical and mathematical (topological) viewpoint by Hawking, King and McCarthy [24].

In this brief note, many aspects of space-time structure have necessarily been omitted. For example no mention has been made of spinor structure (see Geroch [4], [25]) or causal structure (see Penrose and Kronheimer and Penrose [8], [26]). Many other useful points and references can be found in Hawking and Ellis [1] and in Sachs and Wu [27].

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References

[1] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-time*, Cambridge University Press, 1973.
[2] H. Whitney, Ann. of Math. 37 (1936), 645 [see also J. R. Munkres, *Elementary Differential Topology*, Princeton University Press, 1966].
[3] N. Steenrod, *The Topology of Fibre Bundles*, Princeton University Press, 1951.
[4] R. P. Geroch, J. Mathematical Phys. 9 (1968), 1739.
[5] P. Alexandroff and H. Hopf, *Topologie*, J. Springer, 1935.
[6] M. Hirsch, Ann. of Math. 73 (1961), 566.
[7] R. Penrose, in: *Battelle Recontre* (ed. C. M. De Witt and J. A. Wheeler), Benjamin, New York 1968, 121.
[8] —, *Techniques of Differential Topology in Relativity*, A.M.S. Colloquium Publications (1972).
[9] R. P. Geroch, in: *Proceedings of the International School of Physics "Enrico Fermi"* (ed. R. K. Sachs), Academic Press, 1971, 71.
[10] L. Markus, Ann. of Math. 62 (1955), 411.
[11] R. P. Geroch and G. T. Horowitz, *Global Structure of Spacetimes*; in: *General Relativity* (ed. S. W. Hawking and W. Israel), Cambridge University Press, 1979, 212.
[12] C. M. Will, *The Confrontation between Gravitation Theory and Experiment*; in: *General Relativity* (ed. S. W. Hawking and W. Israel), Cambridge University Press, 1979.
[13] V. Petrosian, in: *Confrontation of Cosmological Models with Observational Data* (ed. M. S. Longair), 1974.
[14] J. Plebański, Acta Phys. Polon. 26 (1964), 963.
[15] G. S. Hall, J. Phys. A9 (1976), 541 [see also other article in these proceedings, pp. 53–73].
[16] B. Carter, General Relativity and Gravitation 1 (1971), 349.
[17] S. W. Hawking, Proc. Roy. Soc. A308 (1968), 433.
[18] —, General Relativity and Gravitation 1 (1971), 393.
[19] R. W. Bass and L. Witten, Rev. Mod. Phys. 29 (1957), 452.
[20] R. P. Geroch, J. Mathematical Phys. 8 (1967), 782.
[21] E. C. Zeeman, Topology 6 (1967), 161.
[22] R. Göbel, Comm. Math. Phys. 46 (1976), 289.
[23] —, J. Mathematical Phys. 17 (1976), 845.
[24] S. W. Hawking, A. R. King and P. J. McCarthy, ibid. 17 (1976), 174.
[25] R. P. Geroch, ibid. 11 (1970), 343.
[26] E. H. Kronheimer and R. Penrose, Proc. Cambridge Philos. Soc. 63 (1967), 481.
[27] R. K. Sachs and H. Wu, *General Relativity for Mathematicians*, Springer-Verlag, 1977.

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