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THE NUMERICAL TREATMENT OF PERTURBED BIFURCATION IN BOUNDARY VALUE PROBLEMS

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1. Introduction

Bifurcation theory is a study of the branching of solutions of nonlinear equations. Typical applications of bifurcation theory are in elastic and hydrodynamic stability. However, in experiments and in real applications, the sharp transitions of bifurcation rarely occur. Small imperfections, impurities or other inhomogeneities tend to distort these transitions. To analyze mathematically the perturbation of bifurcations caused by imperfections and other impurities the classical bifurcation theory is modified by introducing an additional small perturbation parameter which characterizes the magnitudes of these inhomogeneities.

The principal methods for analyzing such perturbed problems are due to Koiter [4], Keener and Keller [3], Chow *et al.* [1], Matkowsky and Reiss [5], Potier-Ferry [6] and Weber [8]. The approach used here is akin to that of Weber [8] and is based on a transformation of the given problem into an appropriate boundary value problem possessing isolated solutions. Thus, the numerical treatment using standard methods is possible. However the nonisolated solutions are calculated contrary to Weber [8] in dependence of the perturbation parameter. The ansatz used for singular solutions is a generalization of the representation found by Hermann [2].

The advantage of our ansatz consists in the possibility to determine, for a fixed value of the perturbation parameter, the nonisolated solution as well as a solution branch through this singular solution simultaneously. For this purpose an expanded boundary value problem is build up. Initial value methods in shooting techniques with automatic stepsize control are now applicable.

Numerical results illustrating the method are given in the last section.

2. Formulation of the problem

Consider the nonlinear two point boundary value problem given by

$$Ly = f(\lambda, \tau, y), \quad a \leqslant t \leqslant b, \tag{2.1a}$$

$$B[y] = 0, \tag{2.1b}$$

where

$$\begin{gathered} Ly(t) \equiv y'(t) - A(t)y(t), \quad a \leqslant t \leqslant b, \\ B[y] \equiv B_a y(a) + B_b y(b), \quad B_a, B_b \in \boldsymbol{R}^{n,n}, \\ y \in \boldsymbol{C}_n^1[a, b], \quad A \in \boldsymbol{C}_{n\times n}[a, b], \quad \operatorname{rank}[B_a, B_b] = n, \\ f \in \boldsymbol{C}^5(\boldsymbol{R} \times \boldsymbol{R} \times \Omega, \boldsymbol{R}^n), \quad 0 \in \Omega \subset \boldsymbol{R}^n, \quad \Omega \text{ is an open domain}, \\ f(\lambda, 0, 0) = 0 \ \forall \lambda \in \boldsymbol{R}, \quad f(\lambda, \tau, 0) \neq 0 \ \forall \tau \neq 0. \end{gathered} \tag{2.2}$$

λ is an eigenvalue parameter; τ represents a real perturbation parameter characterizing the magnitude of the inhomogeneities of the given problem.

We refer to (2.1) as the perturbed problem and (2.1) with $\tau = 0$ as the corresponding bifurcation problem.

By L_0 we denote the linear differential operator $L - f_y(\lambda_0, 0, 0)$ with domain $\mathscr{D}(L_0) \equiv \{y \in \boldsymbol{C}_n^1[a, b] | \ B[y] = 0\}$.

Assume that

$$\dim \mathscr{N}(L_0) = 1, \quad \mathscr{N}(L_0) = \operatorname{span}\{\varphi_0\}, \quad \|\varphi_0\| = 1. \tag{2.3}$$

It is well known that the nullspace $\mathscr{N}(L_0^*)$ — the adjoint L_0^* of L_0 defined in the usual way (cf. [7]) — is one dimensional

$$\dim \mathscr{N}(L_0^*) = 1, \quad \mathscr{N}(L_0^*) = \operatorname{span}\{\psi_0\}, \quad \|\psi_0\| = 1. \tag{2.4}$$

Moreover, Fredholm's alternative is valid:

$$\mathscr{R}(L_0) = \mathscr{N}(L_0^*)^{\perp}, \tag{2.5}$$

where

$$S^{\perp} \equiv \{y \in \boldsymbol{C}_n[a, b] | \ \langle y, z\rangle = 0 \ \forall z \in S \subset \boldsymbol{C}_n[a, b]\}$$

and

$$\langle y, z\rangle \equiv \int_a^b y(t)^T z(t)\, dt \quad \forall y, z \in \boldsymbol{C}_n[a, b].$$

We require that f satisfies two conditions:

$$\text{(i)} \quad a_1 \equiv \langle \psi_0, f_{y\lambda}(\lambda_0, 0, 0)\varphi_0\rangle \neq 0, \tag{2.6}$$

$$\text{(ii)} \quad b \equiv \langle \psi_0, f_\tau(\lambda_0, 0, 0)\rangle \neq 0. \tag{2.7}$$

3. Nonisolated solutions

The quadruple $(\lambda, \tau, y, \varphi)$ is called *nonisolated* or *singular solution* of (2.1) in correspondence to [3], if the following equations are fulfilled:

$$
\begin{aligned}
&Ly-f(\lambda, \tau, y) = 0, \quad B[y] = 0,\\
&L\varphi-f_y(\lambda, \tau, y)\varphi = 0, \quad B[\varphi] = 0, \quad \varphi \neq 0.
\end{aligned} \tag{3.1}
$$

(2.3) and (2.6) insure for $\tau = 0$ that $(\lambda_0, 0)$ is a bifurcation point, i.e. a family of nontrivial solutions of (2.1) is branching from the trivial solution. Therefore $(\lambda_0, 0, 0, \varphi_0)$ represents a nonisolated solution at the bifurcation point. We look for a curve of nonisolated solutions of (2.1) depending on the perturbation parameter τ and containing the special branching solution $(\lambda_0, 0, 0, \varphi_0)$.

Let the second bifurcation coefficient a_2 be defined as (cf. [2]):

$$
a_2 \equiv \langle \psi_0, f_{yy}(\lambda_0, 0, 0)\varphi_0^2\rangle . \tag{3.2}
$$

3.1. The Case $a_2 \neq 0$. For $a_2 \neq 0$, i.e.

$$
\langle \psi_0, f_{yy}(\lambda_0, 0, 0)\varphi_0^2\rangle \neq 0, \tag{3.3}
$$

we use the ansatz for nonisolated solutions

$$
\begin{aligned}
&y(z) = -\frac{a_1}{a_2}\varphi_0 z+[K(z)\varphi_0+u_1]z^2+v(z)z^3,\\
&\lambda(z) = \lambda_0+z+\lambda_1(z)z^2, \quad \varphi(z) = \varphi_0+\varphi_1 z+\psi(z)z^2,\\
&z^2 = \frac{2ba_2}{a_1^2}\tau, \quad \operatorname{sign}\tau = \operatorname{sign}(b\cdot a_2),\\
&v(z) \in \mathcal{N}(L_0)^{\perp}, \quad \psi(z) \in \mathcal{N}(L_0)^{\perp}.
\end{aligned} \tag{3.4}
$$

u_1 is defined in (3.4) as (unique) solution of the linear boundary value problem

$$
\begin{aligned}
&Lu_1 = -\frac{a_1}{a_2}f_{y\lambda}^0\varphi_0+\frac{a_1^2}{2ba_2}f_\tau^0+\frac{1}{2}\frac{a_1^2}{a_2^2}f_{yy}^0\varphi_0^2,\\
&B[u_1] = 0, \quad \langle \varphi_0, u_1\rangle = 0
\end{aligned} \tag{3.5}
$$

and φ_1 is defined as (unique) solution of the linear boundary value problem

$$
\begin{aligned}
&L\varphi_1 = f_{y\lambda}^0\varphi_0-\frac{a_1}{a_2}f_{yy}^0\varphi_0^2,\\
&B[\varphi_1] = 0, \quad \langle \varphi_0, \varphi_1\rangle = 0.
\end{aligned} \tag{3.6}
$$

The superscript zero denotes the corresponding function evaluated at $\lambda = \lambda_0$, $\tau = 0$, $y = 0$, e.g. $f_y^0 \equiv f_y(\lambda_0, 0, 0)$. Using Taylor's theorem and properties (2.2), we obtain for $f(\lambda, \tau, y)$ and $f_y(\lambda, \tau, y)$ expansions of the form

$$(3.7) \qquad f(\lambda, \tau, y) = f_y^0 y + f_\tau^0 \tau + f_{y\lambda}^0 y(\lambda - \lambda_0) + f_{y\tau}^0 y \cdot \tau + \\ + \frac{1}{2} f_{yy}^0 y^2 + f_{\tau\lambda}^0 \tau(\lambda - \lambda_0) + \frac{1}{6} f_{yyy}^0 y^3 + \\ + \frac{1}{2} f_{y\lambda\lambda}^0 y(\lambda - \lambda_0)^2 + R_1(\lambda, \tau, y)$$

and

$$(3.8) \qquad f_y(\lambda, \tau, y) = f_y^0 + f_{y\lambda}^0(\lambda - \lambda_0) + f_{y\tau}^0 \tau + f_{yy}^0 y + f_{yy\lambda}^0 y(\lambda - \lambda_0) + \\ + \tfrac{1}{2} f_{yyy}^0 y^2 + \tfrac{1}{2} f_{y\lambda\lambda}^0 (\lambda - \lambda_0)^2 + R_2(\lambda, \tau, y).$$

The remainders R_1 and R_2 are of the form:

$$(3.9) \qquad R_1 = O(z^4) \quad \text{and} \quad R_2 = O(z^3).$$

Inserting ansatz (3.4) into (3.1) and employing the abbreviation

$$(3.10) \qquad w(z) \equiv -\frac{a_1}{a_2}\varphi_0 + [K(z)\varphi_0 + u_1]z + v(z)z^2$$

yield

$$(3.11a) \quad L_0 v(z) = f_{y\lambda}^0\{K(z)\varphi_0 + u_1 + v(z)z + \lambda_1(z)w(z)\} + \frac{a_1^2}{2ba_2} f_{y\tau}^0 w(z) + \\ + \frac{1}{2} f_{yy}^0 \left\{ -2\frac{a_1}{a_2}\varphi_0 u_1 + \left[u_1^2 - 2\frac{a_1}{a_2}\varphi_0 v(z) + 2u_1 v(z) z + \right.\right. \\ \left.\left. + v(z)^2 z^2 - K(z)^2\varphi_0^2 \right] z + 2K(z)\varphi_0 w(z) \right\} + \\ + \frac{a_1^2}{2ba_2} f_{\tau\lambda}^0 \{1 + \lambda_1(z)z\} + \frac{1}{6} f_{yyy}^0 w(z)^3 + \\ + \frac{1}{2} f_{y\lambda\lambda}^0 \{(1 + \lambda_1(z)z)^2 w(z)\} + \frac{1}{z^3} R_1(\lambda(z), \tau(z), y(z)) \\ \equiv P_1(v(z), \lambda_1(z), K(z); z)$$

and

$$(3.11b) \quad L_0\psi(z) = \left[\lambda_1(z) f_{y\lambda}^0 + \frac{a_1^2}{2ba_2} f_{y\tau}^0 + f_{yy}^0\{K(z)\varphi_0 + u_1 + v(z)z\} + \right. \\ + f_{yy\lambda}^0 w(z) + \lambda_1(z) f_{yy\lambda}^0 w(z) z + \frac{1}{2} f_{yyy}^0 w(z)^2 +$$

$$+\frac{1}{2} f^0_{y\lambda\lambda}\{1+\lambda_1(z)z\}^2+$$

$$\left.+\frac{1}{z^2} R_2\big(\lambda(z), \tau(z), y(z)\big)\right]\cdot[\varphi_0+\varphi_1 z+\psi(z)z^2]+$$

$$+\left[f^0_{y\lambda}-\frac{a_1}{a_2} f^0_{yy}\varphi_0\right]\cdot[\varphi_1+\psi(z)z]$$

$$\equiv Q_1\big(v(z), \lambda_1(z), K(z), \psi(z); z\big).$$

We construct an equivalent boundary value problem of dimension $N\equiv 2n+4$ to solve the problem (3.11) with conventional numerical methods, e.g. with shooting methods

$$\begin{aligned}
L_0 v(z) &= P_1\big(v(z), \lambda_1(z), K(z); z\big), \quad B[v(z)]=0,\\
L_0\psi(z) &= Q_1\big(v(z), \lambda_1(z), K(z), \psi(z); z\big), \quad B[\psi(z)]=0,\\
&\lambda_1(z)'=0, \quad K(z)'=0, \qquad\qquad (3.12)\\
\mu_1'-\varphi_0^T v(z) &= 0, \quad \mu_1(a)=\mu_1(b)=0,\\
\mu_2'-\varphi_0^T \psi(z) &= 0, \quad \mu_2(a)=\mu_2(b)=0.
\end{aligned}$$

We can now formulate the main result of this section.

THEOREM 3.1. *Let L, f and λ_0 satisfy conditions* (2.2), (2.3), (2.6), (2.7) *and* (3.3). *Then there is a positive constant τ_0 such that for each τ with $|\tau|\leqslant\tau_0$ and* $\operatorname{sign}\tau=\operatorname{sign}(b\cdot a_2)$ *there exists an isolated solution* $\big(v(z), \psi(z), K(z), \lambda_1(z), \mu_1(z), \mu_2(z)\big)$ *of the boundary value problem* (3.12).

It is possible to construct a continuous family of nonisolated solutions of (2.1) *with ansatz* (3.4) *containing the special singular solution* $(\lambda_0, 0, 0, \varphi_0)$.

Proof. For $z\to 0$ problem (3.12) is reduced to

$$L_0 v(0)+\lambda_1(0)\frac{a_1}{a_2} f^0_{y\lambda}\varphi_0+K(0)\left\{\frac{a_1}{a_2} f^0_{yy}\varphi_0^2-f^0_{y\lambda}\varphi_0\right\}$$

$$=f^0_{y\lambda}u_1-\frac{a_1^3}{2\cdot b\cdot a_2^2} f^0_{y\tau}\varphi_0-\frac{a_1}{a_2} f^0_{yy}\varphi_0 u_1+\frac{a_1^2}{2\cdot b\cdot a_2} f^0_{\tau\lambda}-$$

$$-\frac{a_1^3}{6\cdot a_2^3} f^0_{yyy}\varphi_0^3-\frac{a_1}{2\cdot a_2} f^0_{y\lambda\lambda}\varphi_0,$$

$$L_0\psi(0)-\lambda_1(0) f^0_{y\lambda}\varphi_0-K(0) f^0_{yy}\varphi_0^2$$

$$=\frac{a_1^2}{2\cdot b\cdot a_2} f^0_{y\tau}\varphi_0+f^0_{yy}u_1\varphi_0-\frac{a_1}{a_2} f^0_{yy\lambda}\varphi_0^2+\frac{1}{2}\frac{a_1^2}{a_2^2} f^0_{yyy}\varphi_0^3+$$

$$+\frac{1}{2} f^0_{y\lambda\lambda}\varphi_0+f^0_{y\lambda}\varphi_1-\frac{a_1}{a_2} f^0_{yy}\varphi_0\varphi_1,$$

$$
\begin{aligned}
\lambda_1(0)' &= 0, & K(0)' &= 0, \\
\mu_1' - \varphi_0^T v(0) &= 0, & \mu_2' - \varphi_0^T \psi(0) &= 0, \\
B[v(0)] &= 0, & B[\psi(0)] &= 0, \\
\mu_1(a) = \mu_1(b) &= 0, & \mu_2(a) = \mu_2(b) &= 0.
\end{aligned} \tag{3.13}
$$

Let $(v^0, \psi^0, K^0, \lambda_1^0, \mu_1^0, \mu_2^0)$ be a solution of the homogeneous problem (3.13). Then clearly

$$L_0 v^0 + \lambda_1^0 \frac{a_1}{a_2} f_{y\lambda}^0 \varphi_0 + K^0 \left\{\frac{a_1}{a_2} f_{yy}^0 \varphi_0^2 - f_{y\lambda}^0 \varphi_0\right\} = 0,$$

$$L_0 \psi^0 - \lambda_1^0 f_{y\lambda}^0 \varphi_0 - K^0 f_{yy}^0 \varphi_0^2 = 0,$$

$$\lambda_1^{0\prime} = 0, \quad K^{0\prime} = 0, \quad \mu_1^{0\prime} - \varphi_0^T v^0 = 0, \quad \mu_2^{0\prime} - \varphi_0^T \psi^0 = 0,$$

$$B[v^0] = 0, \quad B[\psi^0] = 0, \quad \mu_1^0(a) = \mu_1^0(b) = 0, \quad \mu_2^0(a) = \mu_2^0(b) = 0.$$

Since

$$L_0 v^0 \in \mathscr{R}(L_0), \quad L_0 \psi^0 \in \mathscr{R}(L_0) \quad \text{and} \quad \mathscr{R}(L_0) = \mathscr{N}(L_0^*)^\perp$$

it follows that

$$
\begin{aligned}
0 &= \langle \psi_0, L_0 v^0\rangle = -\left\langle \psi_0, \lambda_1^0 \frac{a_1}{a_2} f_{y\lambda}^0 \varphi_0 + K^0 \left\{\frac{a_1}{a_2} f_{yy}^0 \varphi_0^2 - f_{y\lambda}^0 \varphi_0\right\}\right\rangle, \\
0 &= \langle \psi_0, L_0 \psi^0\rangle = \langle \psi_0, \lambda_1^0 f_{y\lambda}^0 \varphi_0 - K^0 f_{yy}^0 \varphi_0^2\rangle.
\end{aligned} \tag{3.14}
$$

The first equation in (3.14) implies

$$\lambda_1^0 = 0.$$

We thus obtain

$$0 = \langle \psi_0, L_0 \psi^0\rangle = K^0 \langle \psi_0, f_{yy}^0 \varphi_0^2\rangle = K^0 \cdot a_2.$$

Supposition (3.3) yields

$$K^0 = 0.$$

Now, because of assumption (2.3) we get

$$\psi^0 = C_1 \varphi_0 \quad \text{and} \quad v^0 = C_2 \varphi_0; \quad C_1, C_2 \in \mathbf{R}.$$

Using the problems

$$\mu_1^{0\prime} - \varphi_0^T v^0 = 0, \quad \mu_1^0(a) = \mu_1^0(b) = 0,$$

$$\mu_2^{0\prime} - \varphi_0^T \psi^0 = 0, \quad \mu_2^0(a) = \mu_2^0(b) = 0,$$

we can see that $C_1 = 0$ and $C_2 = 0$, i.e.

$$v^0 = 0, \quad \psi^0 = 0, \quad \mu = 0, \quad \mu_2^0 = 0.$$

Let $\mathscr{L}$ denote the differential operator defining the homogeneous equation (3.13). This problem has only the trivial solution. Hence, the nullspace of $\mathscr{L}^*$ is one dimensional, i.e. $\mathscr{R}(\mathscr{L}) = C_n[a, b]$ and $\mathscr{L}$ is invertible.

Finally, from the implicit function theorem, we obtain the statements of Theorem 3.1. ■

3.2. The Case $a_2 = 0$. Consider now vanishing of the second bifurcation coefficient

$$a_2 \equiv \langle \psi_0, f_{yy}(\lambda_0, 0, 0)\varphi_0^2 \rangle = 0. \tag{3.15}$$

Let $\tilde{u}_1$ be (unique) solution of the linear boundary value problem

$$L_0 \tilde{u}_1 = \tfrac{1}{2} f_{yy}^0 \varphi_0^2, \quad B[\tilde{u}_1] = 0, \quad \langle \varphi_0, \tilde{u}_1 \rangle = 0. \tag{3.16}$$

Define $C_2 \in \boldsymbol{R}$ as

$$C_2 \equiv - \frac{a_1}{3\langle \psi_0, f_{yy}^0 \tilde{u}_1 \varphi_0 + \frac{1}{6} f_{yyy}^0 \varphi_0^3 \rangle} \tag{3.17}$$

on the condition that

$$\langle \psi_0, f_{yy}^0 \tilde{u}_1 \varphi_0 + \tfrac{1}{6} f_{yyy}^0 \varphi_0^3 \rangle \neq 0. \tag{3.18}$$

Furthermore we assume

$$C_2 > 0. \tag{3.19}$$

The following ansatz is applicable to the determination of nonisolated solutions in dependence of the perturbation parameter τ:

$$\begin{aligned} y(z) &= \sqrt{C_2}\varphi_0 z + [C_2 \tilde{u}_1 + K(z)\varphi_0] z^2 + [u_2 + 2\sqrt{C_2}\, K(z) \tilde{u}_1] z^3 + \\ &\qquad + [v(z) + K(z)^2 \tilde{u}_1] z^4, \\ \lambda(z) &= \lambda_0 + z^2 + \lambda_1(z) z^3, \\ \varphi(z) &= \varphi_0 + 2\sqrt{C_2}\tilde{u}_1 z + [\varphi_2 + 2K(z)\tilde{u}_1] z^2 + \psi(z) z^3, \\ z^3 &= -\frac{3}{2}\frac{b}{\sqrt{C_2}\, a_1}\tau; \quad v(z) \in \mathscr{N}(L_0)^{\perp}, \quad \psi(z) \in \mathscr{N}(L_0)^{\perp}. \end{aligned} \tag{3.20}$$

u_2 is defined in (3.20) as (unique) solution of the linear boundary value problem

$$\begin{aligned} L_0 u_2 &= -\frac{2}{3}\frac{\sqrt{C_2}\, a_1}{b} f_\tau^0 + \sqrt{C_2} f_{y\lambda}^0 \varphi_0 + C_2\sqrt{C_2} f_{yy}^0 \tilde{u}_1 \varphi_0 + C_2\sqrt{C_2}\tfrac{1}{6} f_{yyy}^0 \varphi_0^3, \\ &B[u_2] = 0, \quad \langle \varphi_0, u_2 \rangle = 0 \end{aligned} \tag{3.21}$$

and φ_2 is defined as (unique) solution of the linear boundary value problem

$$
\begin{aligned}
L_0\varphi_2 &= f^0_{y\lambda}\varphi_0+3C_2(f^0_{yy}\tilde{u}_1\varphi_0+\tfrac{1}{6}f^0_{yyy}\varphi_0^3),\\
B[\varphi_2] &= 0, \qquad \langle\varphi_0,\varphi_2\rangle = 0.
\end{aligned}
\tag{3.22}
$$

Using Taylor's theorem and the properties (2.2), we obtain for $f(\lambda,\tau,y)$ and $f_y(\lambda,\tau,y)$ expansions of the form

$$
\begin{aligned}
f(\lambda,\tau,y) &= f^0_y y+f^0_\tau\tau+f^0_{y\lambda}y(\lambda-\lambda_0)+f^0_{y\tau}y\tau+\tfrac{1}{2}f^0_{yy}y^2+\\
&\quad+\tfrac{1}{6}f^0_{yyy}y^3+\tfrac{1}{2}f^0_{yy\lambda}y^2(\lambda-\lambda_0)+\tfrac{1}{24}f^0_{yyyy}y^4+R_3(\lambda,\tau,y)
\end{aligned}
\tag{3.23}
$$

and

$$
\begin{aligned}
f_y(\lambda,\tau,y) &= f^0_y+f^0_{y\lambda}(\lambda-\lambda_0)+f^0_{y\tau}\tau+f^0_{yy}y+f^0_{yy\lambda}y(\lambda-\lambda_0)+\\
&\quad+\tfrac{1}{2}f^0_{yyy}y^2+\tfrac{1}{6}f^0_{yyyy}y^3+R_4(\lambda,\tau,y).
\end{aligned}
\tag{3.24}
$$

The remainders R_3 and R_4 are of the form

$$
R_3 = O(z^5) \quad\text{and}\quad R_4 = O(z^4).
\tag{3.25}
$$

Inserting ansatz (3.20) into (3.1) and employing the abbreviations

$$
\begin{aligned}
w_1(z) &\equiv \sqrt{C_2}\varphi_0+[C_2\tilde{u}_1+K(z)\varphi_0]z+[u_2+2\sqrt{C_2}K(z)\tilde{u}_1]z^2+\\
&\quad+[v(z)+K(z)^2\tilde{u}_1]z^3,\\
w_2(z) &\equiv C_2\tilde{u}_1+[u_2+2\sqrt{C_2}K(z)\tilde{u}_1]z+[v(z)+K(z)^2\tilde{u}_1]z^2,\\
w_3(z) &\equiv \varphi_0+2\sqrt{C_2}\tilde{u}_1z+[\varphi_2+2K(z)\tilde{u}_1]z^2+\psi(z)z^3,\\
w_4(z) &\equiv 2\sqrt{C_2}\tilde{u}_1+[\varphi_2+2K(z)\tilde{u}_1]z+\psi(z)z^2
\end{aligned}
\tag{3.26}
$$

yield

$$
\begin{aligned}
L_0v(z) &= K(z)\{f^0_{y\lambda}\varphi_0+3C_2f^0_{yy}\varphi_0\tilde{u}_1+\tfrac{1}{2}C_2f^0_{yyy}\varphi_0^3\}+\lambda_1(z)\sqrt{C_2}f^0_{y\lambda}\varphi_0+\\
&\quad+f^0_{y\lambda}\{w_2(z)+\lambda_1(z)[w_2(z)+K(z)\varphi_0]z\}+\\
&\quad+f^0_{yy}\big\{\tfrac{1}{2}w_2(z)^2+\sqrt{C_2}\varphi_0\big(u_2+[v(z)+K(z)^2\tilde{u}_1]z\big)+\\
&\qquad+K(z)\varphi_0\big(w_2(z)-C_2\tilde{u}_1\big)\big\}+\\
&\quad+\tfrac{1}{6}f^0_{yyy}\big\{3C_2\varphi_0^2w_2(z)+3\sqrt{C_2}\varphi_0\big(w_2(z)+K(z)\varphi_0\big)^2z+\\
&\qquad+\big(w_2(z)+K(z)\varphi_0\big)^3z^2\big\}-\\
&\quad-\frac{2}{3}\frac{\sqrt{C_2}a_1}{b}f^0_{y\tau}w_1(z)+\frac{1}{2}f^0_{yy\lambda}\{1+\lambda_1(z)z\}w_1(z)^2+\\
&\quad+\frac{1}{24}f^0_{yyyy}w_1(z)^4+\frac{1}{z^4}R_3\\
&= P_2\big(v(z),\lambda_1(z),K(z);z\big)
\end{aligned}
\tag{3.27a}
$$

and

$$L_0\psi(z) = \lambda_1(z)f^0_{y\lambda}\varphi_0 + 6\sqrt{C_2}\,K(z)\{f^0_{yy}\varphi_0\tilde{u}_1 + \tfrac{1}{6}f^0_{yyy}\varphi_0^3\} + \tag{3.27b}$$
$$+\Big[-\frac{2\sqrt{C_2}a_1}{3b}f^0_{y\tau} + f^0_{yy}\{u_2 + [v(z) + K(z)^2\tilde{u}_1]z\} +$$
$$+ f^0_{yy\lambda}\{1 + \lambda_1(z)z\}w_1(z) +$$
$$+ \frac{1}{2}f^0_{yyy}\{2\sqrt{C_2}\varphi_0 w_2(z) + [w_2(z) + K(z)\varphi_0]^2 z\} +$$
$$+ \frac{1}{6}f^0_{yyyy}w_1(z)^3 + \frac{1}{z^3}R_4\Big]w_3(z) +$$
$$+ [f^0_{y\lambda}\{1 + \lambda_1(z)z\} + f^0_{yy}C_2\tilde{u}_1 + \tfrac{1}{2}f^0_{yyy}\{C_2\varphi_0^2 +$$
$$+ 2\sqrt{C_2}\,K(z)\varphi_0^2 z\}]w_4(z) +$$
$$+ f^0_{yy}\{\sqrt{C_2}\varphi_0[\varphi_2 + \psi(z)z] + K(z)\varphi_0[\varphi_2 +$$
$$+ 2K(z)\tilde{u}_1 + \psi(z)z]z\}$$
$$\equiv Q_2\big(v(z), \lambda_1(z), K(z), \psi(z);\ z\big).$$

Again we construct an equivalent boundary value problem to solve equations (3.27)

$$\begin{aligned}
L_0 v(z) &= P_2\big(v(z), \lambda_1(z), K(z); z\big), \qquad B[v(z)] = 0,\\
L_0\psi(z) &= Q_2\big(v(z), \lambda_1(z), K(z), \psi(z); z\big), \qquad B[\psi(z)] = 0,\\
&\lambda_1(z)' = 0, \qquad K(z)' = 0,\\
&\mu_1' - \varphi_0^T v(z) = 0, \qquad \mu_1(a) = \mu_1(b) = 0,\\
&\mu_2' - \varphi_0^T \psi(z) = 0, \qquad \mu_2(a) = \mu_2(b) = 0.
\end{aligned} \tag{3.28}$$

We have now the following

THEOREM 3.2. *Let L, f and λ_0 satisfy conditions* (2.2), (2.3), (2.6), (2.7) *and* (3.15). *Then there is a positive constant τ_0 such that for each τ with $|\tau| \leqslant \tau_0$ there exists an isolated solution $\big(v(z), \psi(z), K(z), \lambda_1(z), \mu_1(z), \mu_2(z)\big)$ of the boundary value problem* (3.28).

It is possible to construct a continuous family of nonisolated solutions of (2.1) *with ansatz* (3.20) *containing the special singular solution* $(\lambda_0, 0, 0, \varphi_0)$.

Proof. For $z \to 0$ problem (3.28) is reduced to:

$$L_0 v(0) - K(0)\{f^0_{y\lambda}\varphi_0 + 3C_2 f^0_{yy}\varphi_0\tilde{u}_1 + \tfrac{1}{2}C_2 f^0_{yyy}\varphi_0^3\} - \lambda_1(0)\sqrt{C_2}f^0_{y\lambda}\varphi_0$$
$$= C_2 f^0_{y\lambda}\tilde{u}_1 + f^0_{yy}\left\{\sqrt{C_2}\,u_2\varphi_0 + \frac{1}{2}C_2^2\tilde{u}_1^2\right\} +$$

$$+\frac{1}{2}\, C_2^2 f^0_{yyy}\tilde{u}_1\varphi_0^2-\frac{2C_2a_1}{3b}\, f^0_{y\tau}\varphi_0+\frac{1}{2}\, C_2 f^0_{yy\lambda}\varphi_0^2+\frac{1}{24}\, C_2^2 f^0_{yyyy}\varphi_0^4,$$

$$L_0\psi(0)-K(0)6\sqrt{C_2}\{f^0_{yy}\varphi_0\tilde{u}_1+\tfrac{1}{6}f^0_{yyy}\varphi_0^3\}-\lambda_1(0)f^0_{y\lambda}\varphi_0$$

$$= -\frac{2\sqrt{C_2}a_1}{3b}f^0_{y\tau}\varphi_0+f^0_{yy}\{u_2\varphi_0+2C_2\sqrt{C_2}\tilde{u}_1^2+\sqrt{C_2}\varphi_0\varphi_2\}+$$

$$+2\sqrt{C_2}f^0_{y\lambda}\tilde{u}_1+\sqrt{C_2}f^0_{yy\lambda}\varphi_0^2+2C_2\sqrt{C_2}f^0_{yyy}\tilde{u}_1\varphi_0^2+\tfrac{1}{6}C_2\sqrt{C_2}f^0_{yyyy}\varphi_0^4, \tag{3.29}$$

$$\lambda_1(0)'=0,\qquad K(0)'=0,$$

$$\mu_1'-\varphi_0^T v(0)=0,\qquad \mu_2'-\varphi_0^T\psi(0)=0,$$

$$B[v(0)]=0,\qquad B[\psi(0)]=0,$$

$$\mu_1(a)=\mu_1(b)=0,\qquad \mu_2(a)=\mu_2(b)=0.$$

Let $(v^0, \psi^0, K^0, \lambda_1^0, \mu_1^0, \mu_2^0)$ be a solution of the homogeneous problem (3.29).

Then

$$L_0v^0-K^0\{f^0_{y\lambda}\varphi_0+3C_2f^0_{yy}\varphi_0\tilde{u}_1+\tfrac{1}{2}C_2f^0_{yyy}\varphi_0^3\}-\lambda_1^0\sqrt{C_2}f^0_{y\lambda}\varphi_0=0,$$

$$L_0\psi^0-K^0 6\sqrt{C_2}\{f^0_{yy}\varphi_0\tilde{u}_1+\tfrac{1}{6}f^0_{yyy}\varphi_0^3\}-\lambda_1^0f^0_{y\lambda}\varphi_0=0,$$

$$\lambda_1^{0\prime}=0,\qquad K^{0\prime}=0,\qquad \mu_1^{0\prime}-\varphi_0^T v^0=0,\qquad \mu_2^{0\prime}-\varphi_0^T\psi^0=0,$$

$$B[v^0]=0,\qquad B[\psi^0]=0,\qquad \mu_1^0(a)=\mu_1^0(b)=0,\qquad \mu_2^0(a)=\mu_2^0(b)=0.$$

Since

$$L_0v^0\in\mathscr{R}(L_0),\qquad L_0\psi^0\in\mathscr{R}(L_0)\qquad\text{and}\qquad \mathscr{R}(L_0)=\mathscr{N}(L_0^*)^\perp$$

it follows that

$$\begin{aligned}0=\langle\psi_0, L_0v^0\rangle&=\langle\psi_0, K^0\{f^0_{y\lambda}\varphi_0+3C_2f^0_{yy}\varphi_0\tilde{u}_1+\tfrac{1}{2}C_2f^0_{yyy}\varphi_0^3\}+\\&\qquad+\lambda_1^0\sqrt{C_2}f^0_{y\lambda}\varphi_0\rangle,\\0=\langle\psi_0, L_0\psi^0\rangle&=\langle\psi_0, K^0 6\sqrt{C_2}\{f^0_{yy}\varphi_0\tilde{u}_1+\tfrac{1}{6}f^0_{yyy}\varphi_0^3\}+\lambda_1^0f^0_{y\lambda}\varphi_0\rangle.\end{aligned}\tag{3.30}$$

The first equation in (3.30) implies

$$\lambda_1^0=0.$$

Therefore we obtain

$$0=\langle\psi_0, L_0\psi^0\rangle=K^0 6\sqrt{C_2}\langle\psi_0, f^0_{yy}\varphi_0\tilde{u}_1+\tfrac{1}{6}f^0_{yyy}\varphi_0^3\rangle.$$

Supposition (3.18) yields

$$K^0=0.$$

We conclude in correspondence to the proof of Theorem 3.1 that

$$v^0=0,\qquad \psi^0=0,\qquad \mu_1^0=0,\qquad \mu_2^0=0.$$

Again, let $\mathscr{L}$ denote the differential operator defining the homogeneous equation (3.29). This equation has only the trivial solution. Hence, the nullspace of $\mathscr{L}^*$ is one dimensional, i.e. $\mathscr{R}(\mathscr{L}) = C_n[a, b]$ and $\mathscr{L}$ is invertible.

We obtain the statements of Theorem 3.2 by the implicit function theorem. ■

4. Solution branches through nonisolated solutions

We are now interested in the determination of solution branches of (2.1) passing through nonisolated solutions $(\lambda, \tau, y, \varphi)$. Let $\tau = \bar{\tau}$ fixed. The following boundary value problem is treated

$$(4.1a) \qquad Ly = f(\lambda, \bar{\tau}, y), \quad a \leqslant t \leqslant b,$$

$$(4.1b) \qquad B[y] = 0.$$

We denote by $\bar{y}, \bar{\lambda}, \bar{\varphi}$ the components of the singular solution at $\tau = \bar{\tau}$.

Let L_1 be the linear differential operator $L - f_y(\bar{\lambda}, \bar{\tau}, \bar{y})$ with domain $\mathscr{D}(L_1) \equiv \mathscr{D}(L)$.

Assume that

$$(4.2) \qquad \dim \mathscr{N}(L_1) = 1, \quad \mathscr{N}(L_1) = \operatorname{span}\{\bar{\varphi}\}.$$

Then, the nullspace of the adjoint operator is also one dimensional

$$\dim \mathscr{N}(L_1^*) = 1, \quad \mathscr{N}(L_1^*) = \operatorname{span}\{\bar{\psi}\}.$$

We seek for solutions λ, y of (4.1) in the form (cf. [8])

$$(4.3) \qquad y(\varepsilon) = \bar{y} + \varepsilon\bar{\varphi} + \varepsilon^2 w(\varepsilon), \quad w(\varepsilon) \in \mathscr{N}(L_1)^{\perp}, \quad \lambda(\varepsilon) = \bar{\lambda} + \varepsilon^2 \varrho(\varepsilon).$$

Using Taylor's theorem and properties (2.2), we obtain for the right-hand side of (4.1) the expansion

$$(4.4) \qquad f(\lambda, \bar{\tau}, y) = f(\bar{\lambda}, \bar{\tau}, \bar{y}) + f_y(\bar{\lambda}, \bar{\tau}, \bar{y})(y - \bar{y}) + f_\lambda(\bar{\lambda}, \bar{\tau}, \bar{y})(\lambda - \bar{\lambda}) + \\ + \tfrac{1}{2} f_{yy}(\bar{\lambda}, \bar{\tau}, \bar{y})(y - \bar{y})^2 + R_5(\lambda, \bar{\tau}, y).$$

The remainder R_5 satisfies

$$R_5 = O(\varepsilon^3).$$

Inserting ansatz (4.3) into (4.1) yields

$$(4.5) \qquad L_1 w(\varepsilon) = f_\lambda(\bar{\lambda}, \bar{\tau}, \bar{y})\varrho(\varepsilon) + \tfrac{1}{2} f_{yy}(\bar{\lambda}, \bar{\tau}, \bar{y})\big(\bar{\varphi} + \varepsilon w(\varepsilon)\big)^2 + \frac{1}{\varepsilon^2} R_5$$

$$\equiv R\big(w(\varepsilon), \varrho(\varepsilon); \varepsilon\big)$$

$$B[w(\varepsilon)] = 0, \quad \langle \bar{\varphi}, w(\varepsilon) \rangle = 0.$$

Again, we construct an equivalent boundary value problem of dimension $N \equiv n+2$ to solve (4.5)

$$L_1 w(\varepsilon) = R\big(w(\varepsilon), \varrho(\varepsilon); \varepsilon\big),$$
$$\varrho(\varepsilon)' = 0, \tag{4.6}$$
$$\mu_3' - \bar{\varphi}^T w(\varepsilon) = 0, \quad \mu_3(a) = \mu_3(b) = 0.$$

We can formulate for problem (4.6) the following theorem:

THEOREM 4.1. *Let L, f and λ_0 satisfy conditions* (2.2), (4.2) *and*

$$c \equiv \langle \bar{\psi}, f_\lambda(\bar{\lambda}, \bar{\tau}, \bar{y}) \rangle \neq 0. \tag{4.7}$$

Then there is a positive constant ε_0 such that for each ε with $|\varepsilon| \leqslant \varepsilon_0$ there exists an isolated solution $\big(w(\varepsilon), \varrho(\varepsilon), \mu_3(\varepsilon)\big)$ of the boundary value problem (4.1).

It is possible to construct a continuous family of solutions of (2.1) *at $\tau = \bar{\tau}$ with ansatz* (4.3) *containing the singular solution $(\bar{\lambda}, \bar{\tau}, \bar{y}, \bar{\varphi})$.*

Proof. For $\varepsilon \to 0$ problem (4.6) is reduced to

$$L_1 w(0) - \varrho(0) f_\lambda(\bar{\lambda}, \bar{\tau}, \bar{y}) = \tfrac{1}{2} f_{yy}(\bar{\lambda}, \bar{\tau}, \bar{y}) \bar{\varphi}^2,$$
$$B[w(0)] = 0, \tag{4.8}$$
$$\mu_3' - \bar{\varphi}^T w(0) = 0, \quad \mu_3(a) = \mu_3(b) = 0.$$

Because of assumption (4.7) the homogeneous equation (4.8) has only the trivial solution.

Theorem 4.1 now follows, as in the proof of Theorem 3.1 [3.2], from the application of the implicit function theorem. ∎

There are two possibilities to realize the numerical implementation of problem (4.6). The first one consists in a separate treatment of (3.12) [(3.28)] and (4.6). Here, we have to work on the same set of net points in both problems, because the right-hand side of (4.6) is known only on these points. Therefore, initial value methods with automatic stepsize control are not applicable.

The other possibility permits the use of actual software for initial value problems, i.e. methods with error control per unit step. For this purpose an expanded boundary value problem is build up from (3.12) [(3.28)] and (4.6):

$$L_0 v(z) = P_{1[2]}\big(v(z), \lambda_1(z), K(z); z\big), \quad B[v(z)] = 0,$$
$$L_0 \psi(z) = Q_{1[2]}\big(v(z), \lambda_1(z), K(z), \psi(z); z\big), \quad B[\psi(z)] = 0,$$
$$L_1 w(\varepsilon) = R\big(w(\varepsilon), \varrho(\varepsilon); v(z), \lambda_1(z), K(z), \psi(z); \varepsilon, z\big), \quad B[w(\varepsilon)] = 0,$$
$$\lambda_1(z)' = 0, \quad K(z)' = 0, \quad \varrho(\varepsilon)' = 0,$$

$$\mu_1' - \varphi_0^T v(z) = 0, \quad \mu_1(a) = \mu_1(b) = 0,$$
$$\mu_2' - \varphi_0^T \psi(z) = 0, \quad \mu_2(a) = \mu_2(b) = 0,$$
$$\mu_3' - [\varphi_0 + \varphi_1 z + \psi(z) z^2]^T w(\varepsilon) = 0, \quad \mu_3(a) = \mu_3(b) = 0, \tag{4.9}$$
$$\big[\mu_3' - [\varphi_0 + 2\sqrt{C_2}\,\tilde{u}_1 z + \{\varphi_2 + 2K(z)\tilde{u}_1\} z^2 + \psi(z) z^3]^T w(\varepsilon) = 0,$$
$$\mu_3(a) = \mu_3(b) = 0\big].$$

We have solved system (4.9) with parallel shooting techniques in our sample calculations.

5. A numerical example

We report some calculations with the problem:

$$y'' + \lambda \sin y = \tau \cos \pi t, \qquad y'(0) = y'(1) = 0, \tag{5.1}$$

describing the buckling of a thin rod under compression. The numerical technique used is the multiple shooting method. The interval [0, 1] is divided into 4 segments. The resulting initial value problems are solved by a Runge–Kutta method of order 6 with automatic stepsize control.

1. Bifurcation point: $\lambda_0 = \pi^2$,

Eigenfunction: $\varphi_0(t) = \sqrt{2}\cos \pi t$,

2. Bifurcation coefficient: $a_2 = 0$,

Constant C_2 (3.17): $C_2 = \dfrac{4}{3\pi^2} > 0$,

Function $\tilde{u}_1$ (3.16): $\tilde{u}_1(t) \equiv 0$,

Function u_2 (3.21): $u_2(t) = -\dfrac{\sqrt{2}}{36\sqrt{3}\,\pi^3}\cos 3\pi t$,

Function φ_2 (3.22): $\varphi_2(t) = -\dfrac{\sqrt{2}}{24\pi^2}\cos 3\pi t$,

Computational Results:

Table 1. Nonisolated solution at $\tau = 0.001$

t	$y(t)$	$\varphi(t)$
0	7.39636 1068 E-2	1.41409 2505 E0
0.25	5.23031 5649 E-2	1.00008 5600 E0
0.50	3.27999 2534 E-17	1.26454 5065 E-15
0.75	−5.23031 5649 E-2	−1.00008 5600 E0
1	−7.39636 1068 E-2	−1.41409 2505 E0
$\lambda = 9.88988\ 5799$ E0		

Table 2. Nonisolated solution at $\tau = 0.01$

t	$y(t)$	$\varphi(t)$
0	1.59036 0029 E-1	1.41365 0965 E0
0.25	1.12485 3166 E-1	1.00039 7816 E0
0.50	4.33462 9350 E-16	1.70914 4806 E-14
0.75	−1.12485 3166 E-1	−1.00039 7816 E0
1	−1.59036 0029 E-1	−1.41365 0965 E0
$\lambda = 9.96394\ 7471$ E0		

Table 3. Nonisolated solution at $\tau = 0.1$

t	$y(t)$	$\varphi(t)$
0	3.39527 2348 E-1	1.41158 7155 E0
0.25	2.40381 8474 E-1	1.00185 7138 E0
0.50	1.72338 0703 E-14	3.52714 5346 E-13
0.75	−2.40381 8474 E-1	−1.00185 7138 E0
1	−3.39527 2348 E-1	−1.41158 7155 E0
$\lambda = 1.03119\ 2106$ E1		

Table 4. Solution branch through the nonisolated solution given in Table 1: $\varepsilon = 0.001$

t	$y(t)$	
0	7.53777 0086 E-2	
0.25	5.33032 4374 E-2	
0.50	1.07886 6256 E-15	$\lambda = 9.88989\ 3123$ E0
0.75	−5.33032 4374 E-2	
1	−7.53777 0086 E-2	

Table 5. Solution branch through the nonisolated solution given in Table 1: $\varepsilon = 0.01$

t	$y(t)$	
0	8.81042 8953 E-2	
0.25	6.23041 8656 E-2	
0.50	−1.58074 7573 E-15	$\lambda = 9.89054\ 8168$ E0
0.75	−6.23041 8656 E-2	
1	−8.81042 8953 E-2	

Table 6. Solution branch through the nonisolated solution given in Table 1: $\varepsilon = 0.1$

t	$y(t)$	
0	2.15334 9142 E-1	
0.25	1.52338 5445 E-1	
0.50	1.17965 2369 E-12	$\lambda = 9.93171\ 5618$ E0
0.75	−1.52338 5445 E-1	
1	−2.15334 9142 E-1	

All computations were performed on an EC 1040 computer in double precision arithmetic carrying 16 significant digits.

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