COMPUTATIONAL MATHEMATICS
BANACH CENTER PUBLICATIONS, VOLUME 13
PWN – POLISH SCIENTIFIC PUBLISHERS
WARSAW 1984

A NOTE ON THE SOLUTION OF INTEGER LINEAR PROGRAMMING PROBLEMS USING DUALITY PROPERTIES

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1. Introduction

In general, numerical solution of integer linear programming problems is connected with considerable difficulties, e.g. in many cases one has to examine a large number of feasible (or not feasible) solutions to calculate an optimum, the rounding errors often lead to instabilities during the solution process. Effective algorithms exist only for special classes of integer linear programming problems. A fact opposes these phenomena, namely, the modelling of many practical questions gives rise to integer linear programming problems. Therefore it is of great interest to develop methods for effective solution of special integer linear programming problems. The word "effective" means here the demands for an exact, quick, not failing algorithm, requiring not too much numerical expense.

In the following we present a technique for the solution of integer linear programming problems which is based on duality investigations.

Duality theorems are applied especially in linear programming with high effect. However in the past there were not many authors who dealt with duality in integer programming. First Balas [1], [2] described a duality principle in integer programming but his results have not been used to solve integer linear programming problems. There are other duality principles, e.g. the constructive duality – see Fisher and Shapiro [3], the surrogate constraint duality – see Glover [4]. Meyer and Fleisher [11] investigated a very special class of integer programs. A full discussion of these questions may be found in [10].

2. Some duality properties in integer linear programming

Continuing the investigations of Balas, we consider the integer linear programming problem (all or mixed)

$$(1)\qquad \begin{cases} \max\limits_{x} c^{1T}x^1 + c^{2T}x^2, \\ \qquad A^1x^1 + A^2x^2 \leqslant b, \\ \qquad\qquad x^2 \geqslant 0, \\ \qquad\qquad x^1 \geqslant 0 \text{ integer}. \end{cases}$$

We denote by A^1 an $m\times n_1$ matrix, by A^2 an $m\times(n-n_1)$ matrix, and by $x = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$ an n-dimensional column vector with $x^1 = (x_1, x_2, \ldots, x_{n_1})^T$ ($x^1 \geqslant 0$ integer means as usual that x_j is a nonnegative integer for all $j \in \mathfrak{N}_1 = \{1, 2, \ldots, n_1\}$). Let x^2, c^1, c^2, b, and the null vectors be of dimensions as needed. All components of the vectors and elements of the matrices are real numbers and not necessarily integers.

Now, in general, it is well known that primal-dual formulations for integer linear programming problems (and also for integer nonlinear programming problems) exhibit a duality gap. Therefore, our aim is to formulate a dual problem in a way admitting a proof of a strong duality theorem. We consider

$$(2)\qquad \begin{cases} \max\limits_{x^1}\min\limits_{u}\ c^{1T}x^1 + u^T(b - A^1x^1), \\ \qquad A^{2T}u \geqslant c^2, \\ \qquad\quad u \geqslant 0, \\ \qquad\quad x^1 \geqslant 0 \text{ integer} \end{cases}$$

with $u = (u_1, u_2, \ldots, u_m)^T$ and we introduce

DEFINITION 2.1. The dual of problem (1) is problem (2). ■

It should be remarked that $A^{2T}u \geqslant c^2$ contains only continuous variables. Before we establish a strong duality theorem let us state some elementary properties of problem (1) and problem (2). First, the following lemma is easily seen.

LEMMA 2.1. *The dual of problem* (2) *is problem* (1). ■

Furthermore, the following can be shown:

LEMMA 2.2. *The dual problem* (2) *can alternatively be formulated as*

$$(3)\qquad \begin{cases} \max\limits_{x^1}\min\limits_{u} u^Tb - v^{1T}x^1, \\ \qquad A^Tu - v = c, \\ \qquad\qquad u \geqslant 0, \\ \qquad\qquad x^1 \geqslant 0 \ \textit{integer}, \\ \qquad\qquad v^1 \ \textit{unconstrained} \end{cases}$$

where $v = \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}$, $v^1 = (v_1, v_2, \ldots, v_{n_1})^T$. ■

We now turn to the formulation of

THEOREM 2.1 (strong duality theorem). *If the primal problem* (1) *has an optimal solution* $x_0 = \begin{bmatrix} x_0^1 \\ x_0^2 \end{bmatrix}$, *then there exists* u_0 $(\geqslant 0)$ *such that* (x_0^1, u_0) *is an optimal solution to the dual problem* (2), *and*

$$c^{1T}x_0^1 + c^{2T}x_0^2 = c^{1T}x_0^1 + u_0^T(b - A^1x_0^1)$$

holds. ■

COROLLARY 2.1. *If the dual problem has an optimal solution, then the primal problem does so, and the optimal values of both objective functions are equal.* ■

In the present note we will not give any proof, for proofs and for more details see papers [7] and [10]. We only mention that one can prove simple complementary relations.

With regard to weak duality we will introduce the notation:

$$\begin{aligned}
X &= \{x \mid Ax \leqslant b,\ x^2 \geqslant 0,\ x^1 \geqslant 0 \text{ integer}\},\\
U &= \{u \mid A^{2T}u \geqslant c^2, u \geqslant 0\},\\
\tilde{X} &= \{x \in X \mid A^1x^1 \leqslant b\},\\
\tilde{U} &= \{u \in U \mid A^{1T}u \geqslant c^1\},\\
z_P &= \max_x c^T x,\\
z_D &= \max_{x^1} \min_u c^{1T}x^1 + u^T(b - A^1x^1).
\end{aligned}$$

We assume the convention that

$$X = \emptyset,\ U = \emptyset \quad \text{lead to} \quad z_P = -\infty,\ z_D = \infty,$$

and analoguosly

$$\begin{aligned}
\tilde{X} &= \emptyset\colon\ z_P = -\infty,\ z_D = -\infty,\\
\tilde{U} &= \emptyset\colon\ z_P = \infty,\ z_D = \infty.
\end{aligned}$$

Turning now to problem (1) and problem (3) it is consistent with the above convention to formulate

THEOREM 2.2 (weak duality theorem). *If* $\tilde{u} \in \tilde{U}$ *exists, then* $\tilde{u}^T b - v^{1T}x^1 \leqslant \tilde{u}^T b$, *and if* $\tilde{x}^1 \in \tilde{X}$ *exists, then* $u^T b - v^{T1}\tilde{x}^1 \geqslant c^{1T}\tilde{x}^1$. ■

There are additional weak duality statements but they will not be pursued in this paper (see [10]). With all these weak duality relations bounds for the objective functions are given. This is important for the following algorithm.

3. A two-step-algorithm for solving integer linear programming problems

Looking at (2) and applying (3) we find, writing $t := x^1$ and $t = (t_1, t_2, \ldots, t_{n_1})^T$, that

$$(4)\qquad \begin{cases} \max\limits_{t_j} \min\limits_{u} u^T(b - a_j t_j) + c_j t_j \\ \qquad A_j^T u - w = d, \\ \qquad\qquad u \geqslant 0, \\ \qquad\qquad w \geqslant 0, \\ \qquad\qquad t_j \geqslant 0 \text{ integer}, \end{cases}$$

where $j \in \mathfrak{N}_1$, $A_j = ((a_{ik}))$, $i = 1, \ldots, m$, $k = 1, \ldots, j-1, j+1, \ldots, n$, $d = (c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_n)^T$, $w = (w_1, w_2, \ldots, w_{n-1})^T$ and a_j denotes the jth column of A.

One can observe that every linear programming problem (4) contains only one integer parameter t_j. This will be essential in developing the method. Before doing this we remark what follows. Facing the problem (4) for all $j \in \mathfrak{N}_1$, we are given the intervals I_j. Integer values $t_j^0 \in I_j$ for all $j \in \mathfrak{N}_1$ are determined by those intervals and by Theorem 2.2. From problem (1) we get x^2, and (t^0, x^2) is a feasible solution to the problem (1) with $t^0 = (t_1^0, t_2^0, \ldots, t_{n_1}^0)^T$, but not necessarily an optimum.

In this case, where a feasible but not optimal solution is found, we add a search-step for the determination of an optimal solution to the problem (1) (in principle, the calculation of all optimal solutions is possible if there exists more than one optimal solution).

Now let us outline a two-step-algorithm for solving the integer linear programming problem (1) as follows. A detailed description of this algorithm may be found in [7] and [10].

Step 1 ("optimal" intervals):

(i) Solve the n_1 linear programming problems (4) (each one with a single integer parameter t_j).

(ii) Find for all $j \in \mathfrak{N}_1$ the intervals $I_j := [g_j^l, g_j^u]$ where g_j^l, g_j^u are integers.

Remark. Let $x_0 = \begin{bmatrix} x_0^1 \\ x_0^2 \end{bmatrix}$ be an optimal solution to the problem (1), then $x_0^1 \in \bigtimes_{j \in \mathfrak{N}_1} [g_j^l, g_j^u]$.

(iii) Calculate $t^0 = (t_1^0, t_2^0, \ldots, t_{n_1}^0)^T$ and x^2.

Remark. The components of t^0 and x^2 form a feasible solution to the problem (1).

Now one can ask: is $x = \begin{bmatrix} t^0 \\ x^2 \end{bmatrix}$ optimal?

If x is not optimal, then go to step 2 (otherwise stop). In this way feasible solutions with increasing values of the objective function and an optimum to the problem (1) will be constructed.

Step 2 (optimal solution):

(i) $h:=1$ (h is an iteration index).

(ii) By Theorem 2.2 and according to the definition of I_j it follows that one can determine new ("better") values $t_j^{0(h)}$ for all $j \in \mathfrak{N}_1$:

$$t^{0(h)} = (t_1^{0(h)}, t_2^{0(h)}, \ldots, t_{n_1}^{0(h)});$$

for more details see [7] and [10].
Calculate $x^{2(h)}$.

(iii) We have the new feasible solution

$$x^{(h)} = \begin{bmatrix} t^{0(h)} \\ x^{2(h)} \end{bmatrix},$$

and

$$c^{1T} t^{0(h)} + c^{2T} x^{2(h)} > c^{1T} t^0 + c^{2T} x^2$$

holds.
If $x^{(h)}$ is not optimal, take $h := h+1$ and go to (ii) (of step 2).
Otherwise stop.

THEOREM 3.1 (finiteness). *This two-step-algorithm is finite.* ■

To illustrate the method just described let us consider the following elementary mixed integer linear programming problems:

EXAMPLE 3.1.

$$(5) \qquad \begin{cases} \max x_1 - 3x_2, \\ -x_1 + 2x_2 \leqslant 3, \\ 2x_1 + 2x_2 \leqslant 8, \\ x_1 - 8x_2 \leqslant 0, \\ x_1, x_2 \geqslant 0, \\ x_1 \text{ integer}. \end{cases}$$

From (4) and (5) we derive

$$(6) \qquad \begin{cases} \max\limits_{t_1} \min\limits_{u} u_1(3+t_1) + u_2(8-2t_1) - u_3 t_1 + t_1, \\ 2u_1 + 2u_2 - 8u_3 - w_1 = -3, \\ t_1, u_1, u_2, u_3, w_1 \geqslant 0, \\ t_1 \text{ integer}. \end{cases}$$

Solving problem (6) we have $u = (0, 0, \frac{3}{8})^T$, $\tilde{I}_1 = [0, \frac{32}{9})$, and $g_1^l = 0$, $g_1^u = 3$; finally $t_1^0 = 3$.

For $x_2 = \frac{3}{8}$ it results from Theorem 2.1 that $x^0 = (3, \frac{3}{8})^T$ is an optimal solution to problem (5) (x^0 can be shown to be unique).

Problem (5) is solved in step 1 without use of step 2.

Example 3.2:

$$
(7) \qquad \begin{cases} \max 4x_1 + 5x_2 + x_3, \\ \quad 3x_1 + 4x_2 + 2x_3 \leqslant 11, \\ \quad\quad x_1, x_2, x_3 \geqslant 0, \\ \quad\quad\quad x_1, x_2 \text{ integer}. \end{cases}
$$

Applying step 1 one can get the feasible solution $t_1^0 = 3$, $t_2^0 = 0$, $x_3 = 1$, and the objective function has the value 13. Turning to step 2 and using $u = \frac{4}{3}$ (from step 1) one can see that $14 + \frac{2}{3}$ is an upper bound for the optimal value of the objective function. By use of Theorem 2.1 and with regard to feasibility we obtain $t_2^{0(1)} := t_2^0 + 2$ and finally $t_1^{0(1)} := t_1^0 - 2$.

Hence, the optimal solution to problem (7) is $x_0 = (1, 2, 0)^T$.

We add some remarks on the two-step-algorithm just defined:

1. In principle, every all integer or mixed integer linear programming problem can be solved by this method. The articles [6], [8], [10] contain some details concerning the solution of 0-1 linear problems.

2. The existence of any feasible solution to the problem (1) is stated in step 1.

3. Rounding errors influence the solution process only unessentially.

4. The numerical expense can be estimated before starting the algorithm. The method seems to be effective for problems with $n_1 \ll n$.

5. From a detailed discussion of the algorithm it follows that some special classes of integer linear programming problems are effectively solvable, e.g. problems with nonnegative coefficients only or problems with $m \gg n$ (see [7], [9], [10]).

6. In the articles [7], [10] one can find a discussion of relations between the two-step-algorithm and other methods (e.g. Lagrange-functions, Benders-technique).

7. There exists an interesting interpretation for industrial management for the mixed integer linear programming problem (1) and its dual problem (2) (see [5], [10]).

4. On some computational experiences

The algorithm described in Section 3 and its four modifications (compare Remark 5 at the end of Section 3) have been programmed. About 80 problems (examples taken from other authors, constructed test problems, and problems from industrial management) have been solved on different computers. The problems had no special structure (the greatest dimension of a problem was $(m, n) = (30, 45)$). A few known algorithms (e.g. Gomory-, Benders-, Balas-methods) have been also employed to solve these problems. In the papers [9], [10] a comparison of all the results is presented.

Summing up one can state:

1. The two-step-algorithm solved about 95 p.c. of all examples (this is in contrast to the other applied methods, e.g. the Gomory-methods solved about 45 p.c. of all examples). For each problem a feasible solution was found with bounds for the objective function.

2. The storage requirements can be estimated before starting the computer runs (it can be possible to solve problems of higher dimensions than those described above).

3. The computing time for step 1 depends essentially on the number fo integer variables. If this number is small then one can expect a small otmputing time. A linear growth of the computing time with respect co the number of integer variables can be established.

4. In general, the number of arithmetical operations in step 2 increases linearly with m and n.

For more details see the articles [9], [10].

It thus turns out that the two-step-algorithm is a useful method for solving the mentioned classes of integer linear programming problems.

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Presented to the Semester
Computational Mathematics
February 20 — May 30, 1980