COMPUTATIONAL MATHEMATICS
BANACH CENTER PUBLICATIONS, VOLUME 13
PWN – POLISH SCIENTIFIC PUBLISHERS
WARSAW 1984

APPROXIMATE SOLUTIONS OF A FREE-BOUNDARY PROBLEM IN THE THEORY OF JOURNAL BEARINGS

WILFRIED WEINELT

Technische Hochschule, Sektion Mathematik, Reichenhainer Str. 41, DDR-9022, Karl-Marx-Stadt, DDR

1. Introduction

In journal bearings the rotating shaft swims in the oil because of the force generated by the hydrodynamical pressure in the oil film, see e.g. [1]. This paper deals with the approximate solution of a free-boundary problem for this oil pressure

$$u = u(x, z) \tag{1}$$

for a cylindrical bearing of finite length B and the radius R of the shaft, which rotates with the given frequency ν. A motion of the shaft perpendicular to its axis is also allowed for. In (1) x and z are the coordinates of the points in the rectangle

$$\Omega = (0, 2\pi R) \times (0, B) \ni (x, z) \tag{2}$$

representing the mantle surface of the shaft removed into the xz-plane.

There are many contributions to this problem. Some technical encyclopaedias, for instance [2], take as basic results those from [3]. But paper [3] and also its further developments (see e.g. [4]) make use of the superposition principle for solutions of the nonlinear problem for the pressure in the oil film under the influence of cavitation effects, which gives rise to some doubts.(1) Many calculations of the path of the shaft are also based on the idea of superposition (see e.g. [5] and the literature there), but even in the case where the bearings are not of cylindrical form it does not seem clear how to proceed.

(1) Because of the nonlinearity of the problem, such a superposition is not correct and may only be an approximation, for which the error estimation is absent.

Further developments of the hydrodynamical theory of journal bearings are made in [6], [7], where [6] is essentially based on the assumption of the existence of a gas-oil mixture throughout the bearing. This leads to a nonlinear boundary value problem to which a discretization by finite differences is formally applied. A numerical analysis for this model would be of interest. Further [6] contains the idea of [8], [9]; but a numerical realization is given only in a special case. Mathematical models of larger complexity (including temperature) are considered in [7], but even the method for the determination of the pressure as an essential part of the model is not free from inaccuracies. Firstly [10] — in the case where pure rotation of the shaft is considered — uses the calculus of variational inequalities to show that the problem of hydrostatic pressure in journal bearings is uniquely solvable and that a further problem consists in the approximate solution of a variational inequality.

In this paper we use the calculus of variational inequalities to formulate our problem (in the next section). In Section 3 we consider a difference method to find approximate solutions. Questions of solving discrete problems are discussed in Section 4.

2. Statement of the problem

Let ν be the frequency of the rotating shaft and

$$d = d(x, z; t) \geqslant d_0 > 0 \tag{3}$$

the oil film thickness at the time t. The function d (and the time derivative $\dot{d}$) are of course uniquely determined by the position and by the velocity, of the shaft, respectively. If the viscosity of the oil is characterized by a constant parameter η, then from the Navier–Stokes equations we derive (see e.g. [1] or any other basic literature in this field) the Reynolds equation

$$Lu = -\frac{\partial}{\partial x}\left(d^3\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial z}\left(d^3\frac{\partial u}{\partial z}\right) = -12\eta\left(\nu\pi R\frac{\partial d}{\partial x} + \dot{d}\right) = f(x, z),$$

for the pressure u in that subdomain $\Omega^+ \subset \Omega$ where this pressure is greater than the cavitation pressure $u_c = \text{const}$.

In the whole domain Ω one has $u \geqslant u_c$, whereas in the coincidence set $\Omega^- = \{x \mid x \in \Omega,\ u(x) = u_c\}$ the Reynolds equation is replaced by the inequality $Lu - f \geqslant 0$. This, together with the condition of continuous normal derivative $\frac{\partial u}{\partial n}$ along the free boundary $\Gamma_0 = \Omega^- \cap \overline{\Omega^+}$, means physically, roughly speaking, that enough oil is in the bearing. (²)

(²) So that in each subdomain of Ω in which $f > 0$ we have $u \not\equiv u_c$.

At the boundary Γ of Ω let the values $\mu(x, z) \geqslant u_c$ of the pressure (Dirichlet conditions) be given. These values mean the outside pressure at $z = 0$ and $z = B$ and the given pressure along an axial oil groove (³) at $x = 0$ (and $x = 2\pi R$).

So the problem of determining the pressure can be formulated in the following "classical" way:

(4) Find $u \in C(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^+)$ so that

$$Lu - f \geqslant 0, \quad u \geqslant u_c, \quad (u - u_c)\cdot(Lu - f) = 0 \text{ in } \Omega,$$

$$u = \mu \text{ on } \Gamma.$$

As shown e.g. in [11], problem (4) can be reformulated in a generalized sense as a variational inequality:

(5) Find $u \in K = \{v \mid v \in W_2^1(\Omega),\ v \geqslant u_c \text{ in } \Omega,\ v = \mu \text{ on } \Gamma\}$ so that

$$a(u, v-u) \geqslant (f, v-u) \quad \forall\, v \in K,$$

where

$$a(v, w) = \int_\Omega d^3 \nabla v \nabla w\, dx, \qquad (f, v) = \int_\Omega f v\, dx.$$

It is known (see e.g. [12]) that (5) has a unique solution.

3. A difference approximation

We apply here a difference method which we have investigated (see the literature quoted below) and realized also in somewhat more general situations than in the problem described above.

We use the notation and results of the general theory of the difference method [13].

We construct a uniform network

$$\mathfrak{R}_h = \{(x_i, z_j) \mid x_i = ih_x,\ z_j = jh_z;\ i, j = 0, \pm 1, \pm 2, \ldots;$$
$$N_x h_x = 2\pi R;\ N_z h_z = B;\ N_x, N_z > 0 \text{ whole}\}$$

and define

$$\omega = \Omega \cap \mathfrak{R}_h, \quad \gamma = \Gamma \cap \mathfrak{R}_h, \quad \bar{\omega} = \omega \cup \gamma, \quad h = \max\{h_x, h_z\}$$

and the difference expression

$$\Lambda y = -(a y_{\bar{x}})_x - (b y_{\bar{z}})_z \quad \text{in } \omega,$$

(³) Other cases are not considered here.

where

$$a(x_i, z_j) = d^3(x_i - h_x/2, z_j; t), \qquad b(x_i, z_j) = d^3(x_i, z_j - h_z/2; t)$$

and $y_{\bar{x}}$, y_x are, respectively, the back and the foreward difference quotients of the grid function y.

Problem (5) is then approximated by

$$(6) \qquad \begin{gathered} \Lambda y - f \geqslant 0, \quad y \geqslant u_c, \quad (y - u_c)\cdot(\Lambda y - f) = 0 \quad \text{in } \omega, \\ y = \mu \quad \text{on } \gamma. \end{gathered}$$

The following holds (see [14], [15], [18], [19]):

(i) The difference scheme (6) has a unique solution y.

(ii) The problem (6) is stable with respect to f, i.e., a perturbation Δf of f leads to such a change Δy of y that

$$\|\Delta y\|_{(1)} \leqslant C\|\Delta f\|_{(-1)}.$$

Here $\|\cdot\|_{(1)}$ and $\|\cdot\|_{(-1)}$ denote the discrete W_2^1-norm and the dual norm, respectively.

(iii) For sufficiently regular solutions of (5) the error function $Z(\bar{x}) = y(\bar{x}) - u(\bar{x})$ $(\bar{x} \in \overline{\omega})$ has the estimation

$$\|Z\|_{(1)} = O(h^{3/2}).$$

(iv) The error estimation in (iii) is optimal.

Remark. A partial result concerning the convergence of the free boundaries is given in [17].

4. Numerical solution of the discrete problem (6)

To solve the discrete problem (6) of inequalities we have applied the following mixed penalty iteration method:

$$(7) \qquad \begin{cases} \Lambda y + r(y)(y - u_c) = f \quad \text{in } \omega, \quad y = \mu \quad \text{on } \gamma, \\ r(y) = \begin{cases} 0, & y > u_c, \\ \varrho, & y \leqslant u_c, \end{cases} \quad \varrho = \text{const} > 0; \end{cases}$$

$$(8) \qquad \begin{cases} \Lambda y^0 = f \quad \text{in } \omega; \\ \big(\Lambda + r(y^{m-1})I\big)y^m = f + r(y^{m-1})u_c \quad \text{in } \omega, \quad m \geqslant 1; \\ \qquad y^m = \mu \quad \text{on } \gamma, \quad m \geqslant 0. \end{cases}$$

Let us denote by $y(\varrho)$, $y^m(\varrho)$ the solutions of (7), (8), respectively.

Then (see [16]):

(v) The problems (7) and (8) are uniquely solvable.

(vi) In (7) one has convergence from below, i.e., $y(\varrho) \leqslant y$ for all $\bar{x} \in \bar{\omega}$, and the estimation

$$\max_{\bar{x} \in \bar{\omega}} \left(y(\bar{x}) - y(\varrho)(\bar{x})\right) \leqslant C_1 \varrho^{-1}, \qquad C_1 = |\min_{\bar{x} \in \bar{\Omega}} f(\bar{x})|.$$

(vii) The iteration process (8) converges after a finite number of steps:

$$y^m(\varrho) = y(\varrho) \quad \text{for} \quad m \geqslant m_0(h).$$

References

[1] O. Pinkus, B. Sternlicht, *Theory of hydrodynamic lubrication*, Mc Graw Hill, 1961.

[2] *Wissensspeicher Tribotechnik*, Autorenkollektiv, Fachbuchverlag, Leipzig 1978.

[3] H. Sassenfeld, A. Walther, *Gleitlagerberechnungen*, VDI-Forschungsheft 441, Band 20 (1954), VDI-Verlag Düsseldorf.

[4] H.-J. Butenschön, *Das hydrodynamische zylindrische Gleitlager endlicher Breite unter instationärer Belastung*, Diss., Karlsruhe 1976.

[5] H. Gläser, *Stand, Probleme und Perspektiven der Berechnung dynamisch beanspruchter Radialgleitlager*, Schmierungstechnik 7 (1979), 196–198.

[6] R. Wegmann, *Erweiterung der hydrodynamischen Schmiertheorie durch Einbeziehung des Unterdruckgebietes in die Berechnung instationär belasteter Gleitlager*, Diss., Rostock 1978.

[7] U. Kaftan, *Zur Berechnung des Druck-, Geschwindigkeits- und Temperaturfeldes in ölgeschmierten Mehrgleitflächenlagern unter Berücksichtigung der Temperaturabhängigkeit der Viskosität*, Diss., Magdeburg 1978.

[8] B. Jacobsson, L. Floberg, *The finite journal bearing, considering vaporization*, Transactions of Chalmers University of Technology, Gothenburg, 10 (1957), Nr. 190.

[9] K.-O. Olsson, *Cavitation in dynamically loaded bearings*, Transactions of Chalmers University of Technology, Gothenburg, 34 (1965), Nr. 308.

[10] G. Cimatti, *On a problem of the theory of lubrication governed by a variational inequality*, Appl. Math. and Optimization 3, 2/3 (1977), 227–242.

[11] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéares*, Paris 1969.

[12] J.-L. Lions, G. Stampacchia, *Variational inequalities*, Comm. Pure Appl. Math. 20 (1967), 493–519.

[13] A. A. Samarskiĭ, *Theory of difference schemes* (in Russian), Moscow 1977.

[14] W. Weinelt, *On the finite difference method to solve boundary value problems for differential inequalities*, Žurn. Vyč. Mat. i Mat. Fis. 18.3 (1978), 642–652.

[15] —, *Zur näherungsweisen Lösung von Variationsungleichungen. Tagungsberichte der 7. Tagung über Methoden und Probleme der Mathematischen Physik*, Karl-Marx-Stadt 1979, S. 73–80.

[16] W. Weinelt, G. Helmert, *Iterative Lösung spezieller nichtlinearer Differenzenschemata*, Beitr. z. Numer. Math. 7 (1979), 139–158.

[17] W. Weinelt, *Konvergenz des freien Randes bei der näherungsweisen Lösung von Variationsungleichungen mittels Differenzenverfahren,* Wiss. Zeitschr. d. TH Karl-Marx-Stadt 22.5 (1980), 433–442.
[18] St. Domschke, W. Weinelt, *Optimale Konvergenzaussagen für ein Differenzenverfahren zur Lösung einer Klasse von Variationsungleichungen,* Beitr. z. Numer. Math. (in print).
[19] W. Weinelt, *On the approximate solution of variational inequalities* (in Russian), Diff. Uravnenia 17.11 (1981), 2029–2040.
[20] W. Weinelt, K.-H. Hartwig, *ATM iteration solving elliptic difference schemes with a strongly variable coefficient* (in German), Beitr. z. Numer. Math. 11 (1983), 33–39.

Presented to the Semester
Computational Mathematics
February 20 – May 30, 1980