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AN INVERSE FUNCTION THEOREM FOR FRÉCHET SPACES

MIECZYSŁAW ALTMAN

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803, U.S.A.

Introduction

The method of contractor directions presented in monograph [1] and subsequently developed in a number of investigations (see [4]–[7]) has proved an extremely useful and powerful tool in the study of nonlinear operator equations in Banach spaces. However, for many problems in analysis the class of Banach spaces is insufficient and a broader class is needed. Having an inverse function theorem for such spaces is extremely important. The theorems of Nash [11] and Moser [10] are very significant steps in this direction. Since then many versions and applications have appeared, for example, Rabinowitz [12], Schwartz [13], Jacobovitz [9], Zehnder [14], to mention a few. In locally convex vector spaces, the solvability problem for nonlinear operators with closed range is discussed by Browder [7]. For Banach spaces, the problem is investigated in [1], where more references are given. A generalization of Moser's result is contained in [2]. Recently, Hamilton [8] in his very interesting paper, introduced a very important class of operators in Fréchet spaces, called tame maps, and proved an inverse function theorem. He uses smoothing operators as a basic tool for the convergence proof.

The diagonal method of contractor directions presented here is a further development of the method of contractor directions for Banach spaces. Its extension to locally convex spaces endowed with an increasing sequence of seminorms does not require the existence of smoothing operators. Instead, different kind of growth conditions are imposed on the operators in question. The class of nonlinear operators under investigation is rather general. In case of Fréchet spaces inverse function theorems are given which result from local existence and convergence theorems proved for algorithms constructed for the diagonal method of contractor directions. A further development along with applications to tame operators will appear separately. However, Hamilton's result for tame maps is not covered.

1. The diagonal method of contractor directions

Let X be a vector space endowed with an increasing sequence of seminorms

$$\|x\|_0 \leqslant \|x\|_1 \leqslant \ldots, \qquad \text{for all } x \in X.$$

Let Y be another vector space of the same type, i.e., with the sequence of seminorms

$$\|y\|_0 \leqslant \|y\|_1 \leqslant \ldots, \qquad \text{for all } y \in Y.$$

Let $P\colon D(P) \subset X \to Y$ be a nonlinear mapping, X and Y being complete in the usual sense.

We define for P at $x \in D(P)$ a family of sets $\Gamma_x(P, n)$ for $n = 0, 1, \ldots,$ as follows.

DEFINITION 1.1. $\Gamma_x(P, n) \subset Y$ is a *set of contractor directions at* $x \in D(P)$ *for* P *of order* n if there exists a constant $0 < q < 1$ with the following property. If $y \in \Gamma_x(P, n)$, then there exist a positive $0 < \varepsilon = \varepsilon(x, y) \leqslant 1$ and an element $h \in X$ such that

$$\|P(x+\varepsilon h) - Px - \varepsilon y\|_i \leqslant q\varepsilon \|y\|_i \quad \text{for all } 0 \leqslant i \leqslant n, \text{ where } x+\varepsilon h \in D(P). \tag{1.1}$$

It follows from this definition that $\Gamma_x(P, n, q) = \Gamma_x(P, n)$ depends on q. Since q does not depend on $x \in D(P)$, we use the notation $\Gamma_x(P, n)$ instead of $\Gamma_x(P, n, q)$.

LEMMA 1.1. *The closure in the seminorm* $\| \;\|_n$ *of the set* $\Gamma_x(P, n, q)$ *is contained in* $\Gamma_x(P, n, \bar{q})$ *if* $q < \bar{q} < 1$. *If* $\Gamma_x(P, n)$ *is dense (in* $\| \;\|_n$*) in some neighborhood* $V_0 = [y\colon \|y\|_n < b]$ *of* 0 *in* Y *then* $\Gamma_x(P, n) = Y$.

Proof. Let $y \in \Gamma_x(P, n)$ and $q < \bar{q} < 1$. If $v \in X$ is such that $\|v - y\|_i \leqslant \eta$, then

$$\begin{aligned}\|P(x+\varepsilon h) - Px - \varepsilon v\|_i &\leqslant \|P(x+\varepsilon h) - Px - \varepsilon y\|_i + \varepsilon \|y - v\|_i \leqslant q\varepsilon \|y\|_i + \varepsilon\eta \\ &\leqslant q\varepsilon(\|v\|_i + \eta) + \varepsilon\eta \leqslant \bar{q}\varepsilon \|v\|_i \quad \text{for } 0 \leqslant i \leqslant n\end{aligned}$$

if

$$\eta \leqslant \min_{0 \leqslant i \leqslant n} (\bar{q} - q)\|v\|_i/(1+q).$$

Since $\Gamma_x(P, n)$ is dense in V_0, by assumption, it follows from the above that $\Gamma_x(P, n) = V_0$. If $y \in Y$ is arbitrary, then $\beta y \in V_0$ for some positive $\beta \leqslant 1$ and consequently, $\Gamma_x(P, n) = Y$.

Denote by $\boldsymbol{B}$ the class of increasing continuous functions B such that

$$B(0) = 0 \quad \text{and} \quad B(s) > 0 \quad \text{for } s > 0;$$

$$\int_0^a s^{-1}B(s)\,ds < \infty \quad \text{for some positive } a.$$

LEMMA 1.2 [1]. *Let the positive sequence $\{a_n\}$ be defined as follows:*

$$a_{n+1} = (1 - q\varepsilon_n)a_n \quad \textit{for } n = 0, 1, \ldots,$$

where $0 < q < 1$ and $0 < \varepsilon_n \leqslant 1$ for $n = 0, 1, \ldots$

Let B be some function from class $\boldsymbol{B}$. Then the series $\sum_{n=0}^{\infty} \varepsilon_n B(a_n)$ is convergent, and we have

$$\sum_{n=0}^{\infty} \varepsilon_n B(a_n) \leqslant q^{-1} \int_0^a s^{-1}B(s)\,ds \quad \textit{with } a = e^q a_0, \tag{1.2}$$

and the remainder then satisfies

$$\sum_{i=n}^{\infty} \varepsilon_i B(a_i) < q^{-1} \int_b^{b_n} s^{-1}B(s)\,ds, \tag{1.3}$$

where

$$b_n = a_0 \exp\big(q(1-t_n)\big), \quad b = a_0 \exp\big(q(1-T)\big),$$

$$t_0 = 0, \quad t_n = \sum_{i=0}^{n-1} \varepsilon_i, \quad \textit{and} \quad T = \sum_{i=0}^{\infty} \varepsilon_i.$$

Moreover, $a_n \to 0$ as $n \to \infty$ if and only if $T = \infty$.

Let $\{\Gamma_x(P, n)\}$ be a family of sets of contractor directions at $x \in D(P)$ for $P\colon D(P) \subset X \to Y$ such that $-Px \in \Gamma_x(P, n)$ for all $n \geqslant p$, where $p \geqslant 0$ is an arbitrary fixed integer, and for all $x \in U_0$, $U_0 = D(P) \cap S(x_0, r)$, where $S(x_0, r) = [x\colon \|x - x_0\| < r]$ is a neighborhood of the given $x_0 \in D(P)$. Then we can construct a sequence of elements $x_n \in U_0$ for $n = 0, 1, \ldots$, and a sequence $\{\varepsilon_n\}$ with $0 < \varepsilon_n \leqslant 1$ for $n = 0, 1, \ldots$ as follows. $x_1 = x_0 + \varepsilon_0 h_0$, where ε_0 and h_0 satisfy relation (1.1) with $x = x_0$, $\varepsilon = \varepsilon_0$, $y = -Px_0$, and $i = p$. Suppose that x_i has been defined for $0 \leqslant i \leqslant n$. Then we find $0 < \varepsilon_n \leqslant 1$ and $h_n \in X$ such that

$$\|P(x_n + \varepsilon_n h_n) - (1 - \varepsilon_n)Px_n\|_i \leqslant q\, \varepsilon_n \|Px_n\|_i \quad \text{for all } p \leqslant i \leqslant n, \tag{1.4}$$

where $x_n + \varepsilon_n h_n \in U_0$, $0 < q < 1$ being a global constant. Then we put

$$x_{n+1} = x_n + \varepsilon_n h_n. \tag{1.5}$$

LEMMA 1.3. *Given $k \geqslant p$, consider the sequence $\{a_n\}$ such that*

$$a_k = \|Px_k\|_{k+p}, \quad a_{n+1} = \big(1-(1-q)\varepsilon_n\big)a_n \quad \text{for } n \geqslant k. \tag{1.6}$$

Then we have

$$\|Px_n\|_{k+p} \leqslant a_n \quad \text{for all } n \geqslant k+p. \tag{1.7}$$

Proof. It results from (1.4) that

$$\|Px_{n+1}\|_{k+p} \leqslant \big(1-(1-q)\varepsilon_n\big)\|Px_n\|_{k+p} = a_{n+1}$$

for all $n \geqslant k+p$, by virtue of (1.6). Hence, by induction, relation (1.7) follows.

LEMMA 1.4. *Suppose, in addition to the hypotheses of Lemma 1.3, that the element h_n defined in relation (1.4) satisfies the following inequalities*

$$\|h_n\|_k \leqslant B_k(\|Px_n\|_{k+p(k)}) \quad \text{with } B_k \in \boldsymbol{B}, \tag{1.8}$$

for all $n \geqslant k+p(k)$, $k = 0, 1, \ldots$, where $p(k) \geqslant 0$ are arbitrary integers, $p(0) = p$.

Then with $\beta_k = \|Px_k\|_{k+p(k)}\exp\big((1-q)(1-t_{k+p(k)})\big)$, we have

$$\sum_{n=k+p(k)}^{\infty} \varepsilon_n \|h_n\|_k < (1-q)^{-1}\int_b^{\beta_k} s^{-1}B_k(s)\,ds \quad \text{for all } k \geqslant 0, 1, \ldots, \tag{1.9}$$

where

$$t_0 = 0, \quad t_n = \sum_{i=0}^{n-1}\varepsilon_i, \quad T = \sum_{i=0}^{\infty}\varepsilon_i, \quad b = \|Px_k\|_{k+p(k)}\exp\big((1-q)(1-T)\big)$$

and in particular,

$$\sum_{n=0}^{\infty}\varepsilon_n\|h_n\|_0 \leqslant (1-q)^{-1}\int_0^a s^{-1}B_0(s)\,ds, \quad a = e^{1-q}\|Px_0\|_p. \tag{1.10}$$

Proof. By virtue of (1.8), the estimate of the series in (1.9) follows from Lemma 1.2 with B, a_0, q replaced by B_k, $\|Px_k\|_{k+p(k)}$, $(1-q)$, respectively, and from Lemma 1.3, where p should be replaced by $p(k)$.

LEMMA 1.5. *Under the hypotheses of Lemma 1.4, the sequence $\{x_n\}$ $(n = 0, 1, \ldots)$ defined by (1.5) lies in $U_0 = D(P)\cap S(x_0, r)$, where*

$$r = (1-q)^{-1}\int_0^a s^{-1}B(s)\,ds, \quad a = e^{1-q}\|Px_0\|_p, \tag{1.11}$$

and

$$\|x_n\|_k \leqslant \|x_k\|_k + (1-q)^{-1}\int_0^{\beta_k} s^{-1}B(s)\,ds, \tag{1.12}$$

where

$$\beta_k = \|Px_k\|_{k+p(k)} \exp\left((1-q)(1-t_{k+p(k)})\right), \quad t_0 = 0,$$

and

$$t_n = \sum_{i=0}^{n-1} \varepsilon_i,$$

or

$$\|x_n\|_k \leqslant \|x_k\|_k + (1-q)^{-1} \int_0^{\beta_k} s^{-1} B(s)\, ds, \tag{1.13}$$

where

$$\beta_k = \|Px_k\|_{k+p(k)} e^{1-q}, \quad n \geqslant k, \quad \text{for } k = 0, 1, \ldots$$

Moreover, $\{x_n\}$ *is a Cauchy sequence.*

Proof. The estimate (1.12) follows from the estimate (1.9) of Lemma 1.4, and therefore relation (1.13) holds. Relations (1.11) and (1.10) yield that the sequence $\{x_n\}$ lies in U_0. Relation (1.9) implies that $\{x_n\}$ is a Cauchy sequence.

For given $x_n, h_n = h_n(x_n)$, $0 < q < 1$, put

$$\Phi_i(\varepsilon, x_n, h_n) = \|P(x_n + \varepsilon h_n) - (1-\varepsilon) Px_n\|_i / \varepsilon, \tag{1.14}$$

$$\Phi_i(1, x_n, h_n) \leqslant q \|Px_n\|_i \quad \text{for all } p \leqslant i \leqslant n, \tag{1.15}$$

where $p = p(0)$ is the same as above. We assume that there exist a positive function f and a constant $0 < q < 1$ which have the following properties:

$$f(s_n) \to 0 \quad \text{implies} \quad s_n \to 0 \text{ as } n \to 0, \tag{1.16}$$

where $\{s_n\}$ is a positive sequence. If condition (1.15) is not satisfied, then there exists a positive $\varepsilon_n < 1$ such that

$$f(q\|Px_n\|_i) \leqslant \Phi(\varepsilon_n, x_n, h_n) \leqslant q\|Px_n\|_i \quad \text{for all } p \leqslant i \leqslant n. \tag{1.17}$$

By using this assumption, the following iterative procedure can be defined. If condition (1.15) is satisfied then we put

$$x_{n+1} = x_n + h_n. \tag{1.18}$$

If condition (1.15) is not satisfied, then we put

$$x_{n+1} = x_n + \varepsilon_n h_n. \tag{1.19}$$

THEOREM 1.1. *Suppose that the following conditions are fulfilled for* $P\colon D(p) \subset X \to Y$. *Relations* (1.16) *and* (1.7) *hold for the sequence* $\{x_n\}$ $(n = 0, 1, \ldots)$ *defined by* (1.18) *and* (1.19). *In addition, assume that for* *each $i \geqslant p$, we have*

$$\Phi_i(\varepsilon_n, x_n, h_n) \to 0 \quad \textit{whenever } \varepsilon_n \to 0, \tag{1.20}$$

where the sequence $\{h_n\}$ satisfies relation (1.8) *for all $n = 0, 1, \ldots$*

Then the sequence $\{x_n\}$ lies in $U_0 = D(P) \cap S(x_0, r)$ with r defined by (1.11) *and converges to a solution x of equation $Px = 0$.*

Proof. By virtue of (1.17) relation (1.4) is always satisfied for $\{x_n\}$. By Lemma 1.5, $\{x_n\} \subset U_0$. First suppose that the sequence $\{\varepsilon_n\}$ converges to 0. Then the convergence of $\{\|Px_n\|_i\}$ to 0 as $n \to \infty$ holds for each $i \geqslant p$, by virtue of (1.20), (1.17) and (1.16). If $\{\varepsilon_n\}$ is not convergent to 0, then $\sum_{n=0}^{\infty} \varepsilon_n = \infty$, and by Lemma 1.2, $\{a_n\}$ converges to 0. It results from relation (1.7) of Lemma 1.3 that $\{\|Px_n\|_i\}$ converges to 0 for each $i \geqslant p$. This completes the proof.

Let us observe that if Y is a Banach space, then we can put $f(s) = \beta s$, where $0 < \beta < 1$ is arbitrary.

2. The first algorithm for the diagonal method of contractor directions

Let $P\colon D(P) \subset X \to Y$ be a continuous nonlinear mapping and $U_0 = D(P) \cap \cap S$, where $S = S(x_0, r) = [x\colon \|x - x_0\|_0 < r]$. Suppose that the following hypotheses are satisfied.

The first and second Fréchet derivatives $P'(x)$ and $P''(x)$ exist and are continuous at all $x \in U_0$ and have the following properties:

(i) For each $x \in U_0$, there exists an element $h(x)$ such that $P'(x)h(x) = -Px$.

(ii) For each $k = 0, 1, \ldots$, there exist a function $B_k \in \boldsymbol{B}$ and an integer $p(k) \geqslant 0$, $p(0) = p \geqslant 0$ such that

$$\|h(x)\|_k \leqslant B_k(\|Px\|_{k+p(k)}) \quad \text{for } k = 0, 1, \ldots \tag{2.1}$$

We are now in a position to define the algorithm. Given $x_0 \in D(P)$ and $h_0 = h_0(x_0)$ satisfying relation (i) and (2.1) with $p(0) = p \geqslant 0$, and a global constant $0 < q < 1$, we define ε_0 as follows:

$$\varepsilon_0 = q\|Px_0\|_p / \max_{0 \leqslant t \leqslant 1} \|P''(x_0 + th_0)(h_0, h_0)\|_p,$$

provided $\varepsilon_0 < 1$. If this is not the case, then we put $\varepsilon_0 = 1$. In both cases, we put $x_1 = x_0 + \varepsilon_0 h_0$. Suppose now that $x_0, x_1, \ldots, x_n \in D(P)$ have been already chosen. Then we put

$$\varepsilon_n = \min_{p \leqslant i \leqslant m} \{q\|Px_n\|_i / \max_{0 \leqslant t \leqslant 1} \|P''(x_n + th_n)(h_n, h_n)\|_i\}, \tag{2.2}$$

provided $\varepsilon_n < 1$. If this is not the case, then we put $\varepsilon_n = 1$. In both cases, we put

$$x_{n+1} = x_n + \varepsilon_n h_n, \tag{2.3}$$

where $h_n = h_n(x_n)$ satisfies relations (i) and (2.1) with $x = x_n$. In this way, the sequence (2.3) is well-defined for $n = 0, 1, \ldots$ It is clear that the sequence $\{x_n\}$ satisfies relation (1.1) with $x = x_n$ and $y = -Px_n$. In fact, we have, by virtue of (i) with $x = x_n$,

$$\begin{aligned}\|P(x_n+\varepsilon_n h_n)-(1-\varepsilon_n)Px_n\|_i &= \|P(x_n+\varepsilon_n h_n)-Px_n-\varepsilon_n P'(x_n)h_n\|_i \\ &\leqslant \varepsilon_n^2 \int_0^1 \|P''(x_n+t\varepsilon_n h_n)(h_n, h_n)\|_i\,dt \\ &\leqslant \varepsilon_0^2 \max_{0\leqslant t\leqslant 1} \|P''(x_n+th_n)(h_n, h_n)\|_i.\end{aligned}$$

Hence, by virtue of (2.2), we have

$$\|P(x_n+\varepsilon_n h_n)-(1-\varepsilon_n)Px_n\|_i \leqslant q\|Px_n\|_i, \quad \text{for all } p\leqslant i\leqslant n. \tag{2.4}$$

Regularity Hypotheses

RH_1: It is assumed that the sequence $\{\varepsilon_n\}$ defined by relation (2.2) has the following property:

$$\varepsilon_n = q\|Px_n\|_{i_n}/\max_{0\leqslant t\leqslant 1}\|P''(x_n+th_n)(h_n, h_n)\|_{i_n},$$

where $i_n \leqslant n$ and $i_n\to\infty$ as $n\to\infty$.

RH_2: It is assumed that $\sum_{n=0}^{\infty}\varepsilon_n = \infty$.

RH_3: It is assumed that 0 is a cluster point of the sequence $\{M_n - \varepsilon_n\}$, where

$$M_n = \max_{p\leqslant i\leqslant n}\{q\|Px_n\|_i/\max_{0\leqslant t\leqslant 1}\|P''(x_n+th_n)(h_n, h_n)\|_i\}. \tag{2.5}$$

RH_4: It is assumed that $\varepsilon_n\to 0$ as $n\to\infty$ implies that $Px_n\to 0$ as $n\to\infty$.

RH_5: It is assumed that $\varepsilon_n\to 0$ as $n\to\infty$ implies that

$$\big(P(x_n+\varepsilon_n h)-Px_n\big)/\varepsilon_n\to 0 \quad \text{as } n\to\infty,$$

where $\{\varepsilon_n\}$ is the same as in (2.2) in all the hypotheses quoted above.

Remark 2.1. In all cases the sequence $\{\varepsilon_n\}$ defined by (2.2) can be replaced by the following

$$\varepsilon_n = \min_{p\leqslant i\leqslant n}\{q\|Px_n\|_i/\max_{0\leqslant t\leqslant 1}\|P''(x_n+th_n)(h_n, h_n)\|_i\cdot C_i\} \tag{2.6}$$

for some sequence of $C_i \geqslant 1$ independent of n.

3. Local existence and convergence theorems

The following lemma holds for the algorithm defined in Section 2 by relation (2.3).

LEMMA 3.1. *Suppose that the mapping* $P\colon D(P)\subset C\to Y$ *satisfies the hypotheses* (i), (ii) *of Section* 2, *and in addition,*

$$(3.1)\qquad r\geqslant(1-q)^{-1}\int\limits_0^a s^{-1}B(s)\,ds \quad \textit{with} \quad a=e^{1-q}\|Px_0\|_p.$$

Then there exists a family $\{\Gamma_x(P,n)\}$ *of sets of contractor directions such that* $-Px\in\Gamma_x(P,n)$ *for all* $x=x_n$ *and* $n\geqslant p$, *where* $\{x_n\}$, $n=0,1,\ldots$ *is the sequence generated by relation* (2.3) *of the algorithm in question. Moreover,* $\{x_n\}$ *is a Cauchy sequence and lies in* $U_0=D(P)\cap S(x_0,r)$.

Proof. The existence of a family of sets of contractor directions results from relation (2.4). It follows from Lemma 1.5 that $\{x_n\}$ is a Cauchy sequence which lies in U_0.

THEOREM 3.1. *Under the hypotheses of Lemma* 3.1, *suppose in addition, that one of the regularity hypotheses* RH_2 *or* RH_4 *is satisfied. Then the sequence* $\{x_n\}$ *defined by relation* (2.3) *lies in* U_0 *and converges to a solution of equation* $Px=0$.

Proof. By virtue of Lemma 3.1, it remains to be seen that the sequence $\{Px_n\}$ converges to 0 as $n\to\infty$. Suppose first that hypothesis RH_2 is satisfied. Then by Lemma 1.2, $a_n\to 0$ as $n\to\infty$. It follows from relation (1.7) of Lemma 1.3 that

$$\|Px_n\|_i\to 0 \quad \text{as } n\to\infty \quad \text{for each } i=0,1,\ldots$$

Suppose now that hypothesis RH_4 is satisfied. If $\varepsilon_n\to 0$ as $n\to\infty$, then by assumption, $Px_n\to 0$ as $n\to\infty$. If the sequence $\{\varepsilon_n\}$ does not converge to 0, then obviously $\sum\limits_{n=0}^{\infty}\varepsilon_n=\infty$, and the hypothesis RH_2 is satisfied.

THEOREM 3.2. *Under the hypotheses of Lemma* 3.1, *suppose in addition, that the hypotheses* RH_1 *and* RH_3 *are satisfied, and that the sequence*

$$\text{(iii)}\qquad \{\max_{0\leqslant t\leqslant 1}\|P''(x_n+th_n)(h_n,h_n)\|_i\}$$

is bounded for each $i=p,p+1,\ldots$, *where* $\{x_n\}$ *and* $\{h_n\}$ *are defined by the algorithm. Then the sequence* $\{x_n\}$ *lies in* U_0 *and converges to a solution of equation* $Px=0$.

Proof. Suppose that the sequence $\{M_n\}$ defined by relation (2.5) contains a subsequence convergent to 0. Then relation (iii) yields the existence of a subsequence of $\{Px_n\}$ convergent to 0. But $\{Px_n\}$ is a Cauchy sequence, since so is $\{x_n\}$, by virtue of Lemma 1.3. Therefore, $\{Px_n\}$ converges to 0 as $n\to\infty$. If the sequence $\{M_n\}$ does not contain a subsequence convergent to 0, then the sequence $\{\varepsilon_n\}$ is not convergent to 0 and, consequently, $\sum_{n=0}^{\infty}\varepsilon_n = \infty$, i.e., the hypothesis RH_2 is satisfied and the proof follows from Theorem 3.1.

THEOREM 3.3. *In addition to the hypotheses of Lemma* 3.1, *suppose that relation* (iii) *is satisfied. Then the sequence* $\{x_n\}$ *defined by relation* (2.3) *converges to a solution of equation* $Px = 0$ *if and only if the hypothesis* RH_5 *is satisfied.*

Proof. Suppose that the sequence $\{\varepsilon_n\}$ converges to 0. We have

$$\|P(x_n+\varepsilon_n h_n)-(1-\varepsilon_n)Px_n\|_i \leqslant \varepsilon_n^2 \max_{0\leqslant t\leqslant 1}\|P''(x_n+th_n)(h_n, h_n)\|_i \tag{3.2}$$

for all $i = 0, 1, \ldots$ It follows from relations (3.2) and (iii) that the hypotheses RH_5 is satisfied if and only if the sequence $\{Px_n\}$ converges to 0. If the sequence $\{\varepsilon_n\}$ does not converge to 0, then obviously, $\sum_{n=0}^{\infty}\varepsilon_n = \infty$, and the proof follows from Theorem 3.1.

4. The second algorithm for the diagonal method of contractor directions

Let $P\colon D(P)\subset X\to Y$ be a continuous nonlinear mapping which satisfies the hypotheses (i) and (ii) of Section 2. We assume, in addition, that the following relation is satisfied. There exist positive constants $C_0, C_1, \ldots$ such that

$$\max_{0\leqslant t\leqslant 1}\|P''(x+th)(h, h)\|_i \leqslant C_i\|Px\|_i \quad \text{for all } i = 0, 1, \ldots, \text{ and} \tag{4.1}$$
$$x\in U_0 = D(P)\cap S(x_0, r), \text{ where } P'(x)h = -Px.$$

We are now in a position to define the algorithm. Given $x_0\in D(P)$ and $h_0 = h_0(x_0)$ satisfying relation (i) and (2.1) with $p(0) = p\geqslant 0$, and a global constant $0<q<1$, we define ε_0 as follows:

$$\varepsilon_0 = q/C_0 \quad \text{provided} \quad \varepsilon_0<1.$$

If this is not the case, then we put $\varepsilon_0 = 1$. In both cases, we put $x_1 = x_0 + \varepsilon_0 h_0$. Suppose now that $x_0, x_1, \ldots, x_n\in D(P)$ have been already chosen. Then we put

$$\varepsilon_n = q/\max_{0\leqslant i\leqslant n} C_i \quad \text{provided} \quad \varepsilon_n<1. \tag{4.2}$$

If this is not the case, then we put $\varepsilon_n = 1$. In both cases, we put

$$x_{n+1} = x_n+\varepsilon_n h_n, \tag{4.3}$$

where $h_n = h_n(x_n)$ satisfies relation (i) and (2.1) with $x = x_n$. In this way, the sequence (4.3) is well-defined for $n = 0, 1, \ldots$ It is clear that the sequence $\{x_n\}$ satisfies relation (1.1) with $x = x_n$ and $y = -Px_n$. In fact, we have, by virtue of (4.2), (4.3) and (i) with $x = x_n$,

$$\begin{aligned}\|P(x_n+\varepsilon_n h_n)-(1-\varepsilon_n)Px_n\|_i &= \|P(x_n+\varepsilon_n h_n)-Px_n-\varepsilon_n P'(x_n)h_n\|_i \\ &\leqslant \varepsilon_n^2\int_0^1 \|P''(x_n+t\varepsilon_n h_n)(h_n, h_n)\|_i\,dt \\ &\leqslant \varepsilon_n^2 \max_{0\leqslant t\leqslant 1}\|P''(x_n+th_n)(h_n, h_n)\|_i .\end{aligned}$$

Hence, by virtue of (4.2), we obtain

$$\|P(x_n+\varepsilon_n h_n)-(1-\varepsilon_n)Px_n\|_i \leqslant q\varepsilon_n\|Px_n\|_i \quad \text{for all } 0\leqslant i\leqslant n. \tag{4.4}$$

THEOREM 4.1. *In addition to the hypotheses* (i), (ii), (3.1) *of Lemma* 3.1, *suppose that the following growth condition is satisfied for relation* (4.1)

$$\sum_{n=0}^{\infty} 1/\max_{0\leqslant i\leqslant n} C_i = \infty. \tag{4.5}$$

Then the sequence $\{x_n\}$ *defined by* (4.3) *lies in* U_0 *and converges to a solution of equation* $Px = 0$.

Proof. The proof follows immediately from Theorem 3.1, since $\sum_{n=0}^{\infty} \varepsilon_n = \infty$, by virtue of (4.5) and (4.2).

Remark 4.1. It follows from assumption (3.1) of Theorem 4.1 that if

$$r_c > (1-q)^{-1}\int_0^a s^{-1}B(s)\,ds \quad \text{with} \quad a = e^{1-q}c,$$

then for all y satisfying the condition

$$\|Px_0 - y\|_p < c < \|Px_0\|_p, \tag{4.6}$$

the equation $Px - y = 0$ has a solution x such that $\|x - x_0\|_0 < r_c$. Moreover, if $P\bar{x} = \bar{y}$, where $\bar{y}$ satisfies relation (4.6) and

$$\bar{r} > (1-q)^{-1}\int_0^a s^{-1}B(s)\,ds \quad \text{with} \quad a = e^{1-q}\bar{c},$$

where $\bar{c}$ is such that $[y\colon \|\bar{y}-y\|_p < \bar{c}] \subset [y\colon \|Px_0 - y\|_p < c]$, then for each y with $\|\bar{y}-y\|_p < \bar{c}$, there exists a solution x of equation $Px - y = 0$ such that $\|\bar{x}-x\|_0 < \bar{r}$. Hence, it follows that if P is one-to-one, then the inverse of P is continuous.

Proof. We replace the operator P by the operator with values $Px - y$ and apply the corresponding theorem. This remark also holds for Theorems 3.1–3.4.

Remark 4.2. Suppose that X and Y are Banach spaces and that for some $C_i > 0$,

$$\max_{0\leqslant t\leqslant 1} \|P''(x+th)(j, h)\|_i \leqslant C_i \|h\|_i^2 \quad \text{for all } x \in U_0, \tag{4.7}$$

where

$$P'(x)h = -Px.$$

If

$$\|h\|_i \leqslant B_i(\|Px\|_i) \quad \text{with} \quad B_i(s) = C_i^{-1/2} s^{1/2}, \tag{4.8}$$

then we obviously obtain that

$$\max_{0\leqslant t\leqslant 1} \|P''(x+th)(h, h)\|_i \leqslant C_i \|Px\|_i. \tag{4.9}$$

Hence, assumption (4.9) follows from (4.8) provided that condition (4.7) is satisfied. Condition (4.8) appears in a different form in the Kantorovich hypothesis used in his convergence proof for Newton's method (see [5]). This argument shows that the hypothesis (4.1) seems to be a natural extension of the one made in case of Banach spaces. If the seminorms in X are norms, then we obtain

COROLLARY 4.1. *Suppose that in addition to the hypotheses of Theorem* 4.1, *the following condition is satisfied*

$$\big\|P'(x)^{-1}\big[P'\big(x+t(\bar{x}-x)\big)-P'(x)\big]\,(\bar{x}-x)\big\|_0 < \|\bar{x}-x\|_0^2 \tag{4.10}$$

$$\text{for all} \quad x, \bar{x} \in [u\colon \|u-x_0\|_0 < r_c] \text{ and } 0 \leqslant t \leqslant 1,$$

where r_c is the same as in Remark 4.1. *Then the mapping P has a continuous inverse defined for all y satisfying relation* (4.6).

Proof. We have from Taylor's formula

$$P\bar{x}-Px-P'(x)(\bar{x}-x) = \int_0^1 \big[P'\big(x+t(\bar{x}-x)\big)-P'(x)\big](\bar{x}-x)\,dt.$$

If $P\bar{x} = Px$, then

$$\bar{x}-x = \int_0^1 P'(x)\big[P'\big(x+t(\bar{x}-x)\big)-P'(x)\big](\bar{x}-x)\,dt.$$

Hence, by virtue of (4.10), $\bar{x} = x$. The continuity follows from Remark 4.1.

Consider now a particular case of relation (4.1). As in the case of tame operators (see Hamilton [8]), suppose that the following assumption is satisfied. There exist constants C_i', $i = 0, 1, \ldots$, such that

$$\|P''(x)(h, h)\|_i \leqslant C_i'(\|h\|_i\|h\|_0 + \|x\|_i\|h\|_0^2). \tag{4.11}$$

Suppose, in addition, that

$$\|h\|_i \leqslant C_i'' \|Px\|_{i+p(i)}^{1/2} \quad \text{for } i = 0, 1, \ldots, \tag{4.12}$$

where $P'(x)h = -Px$, i.e., the function $B_i \in \boldsymbol{B}$ in relation (1.8) is $B_i(s) = C_i'' s^{1/2}$. Then we obtain from (4.11) and (4.12)

$$(4.13)\quad \max_{0\leqslant t\leqslant 1} \|P''(x+th)(h,h)\|_i \leqslant C_i'\left(\|Px\|_{i+p(i)}^{1/2}\|Px\|_p^{1/2} + (\|x\|_i + \|Px\|_{i+p(i)}^{1/2})\|Px\|_p\right),$$

where $p = p(0)$. Finally, assume that there exist constants $\bar{C}_i$, $i = 0, 1, \ldots$ such that

$$(4.14)\quad \|Px\|_{i+p(i)}^{1/2}\|Px\|_p^{1/2} + (\|x\|_i + \|Px\|_{i+p(i)}^{1/2})\|Px\|_p \leqslant \bar{C}_i\|Px\|_i$$

for all $x \in U_0$. Combining relations (4.13) and (4.14) yields

$$(4.15)\quad \max_{0\leqslant t\leqslant 1} \|P''(x+th)(h,h)\|_i \leqslant C_i\|Px\|_i \quad \text{for } i = 0, 1, \ldots,$$

where C_i is the constant resulting from those in (4.13) and (4.14). Hence, we obtain the following

THEOREM 4.2. *In addition to the hypotheses* (3.1), (i) *and* (ii) *with* (2.1) *replaced by* (4.12), *suppose that relation* (4.14) *is satisfied. If the growth condition* (4.5) *is satisfied for relation* (4.15), *then the sequence* $\{x_n\}$ *defined by* (4.3) *lies in* U_0 *and converges to a solution of equation* $Px = 0$.

Proof. The proof follows immediately from Theorem 4.1.

It is evident that Corollary 4.1 also applies to Theorem 4.2.

5. The third algorithm for the diagonal method of contractor directions

Let $P\colon D(P) \subset X \to Y$ be a nonlinear map which satisfies the hypotheses (i) and (ii) of Section 2. We assume, in addition, that there exists a sequence $\{C_i(s,t)\}$ of positive functions such that

$$(5.1)\quad \max_{0\leqslant t\leqslant 1} \|P''(x+th)(h,h)\|_i \leqslant C_i(\|x\|_{i+r(i)}, \|Px\|_{i+q(i)})\|Px\|_i$$

for $i \geqslant p, \ldots$, where $r(i) \geqslant 0$, $q(i) \geqslant 0$ are arbitrary integers, $P'(x)h = -Px$ for all $x \in U_0 = D(P)\cap S(x_0, r)$ with $S(x_0, r) = [x\colon \|x-x_0\| < r]$, r being defined below. We can now define the following algorithm. Given $x_0 \in D(P)$ and $0 < q < 1$, put

$$\varepsilon_0 = \min\{1, q/C_0(\|x_0\|_{r(0)}, \|Px_0\|_{q(0)})\} \quad \text{and} \quad x_1 = x_0 + \varepsilon_0 h_0,$$

where $P'(x_0)h_0 = -Px_0$.

Suppose now that $x_0, x_1, \ldots, x_n \in D(P)$ have been chosen. Then we put

$$(5.2)\quad \varepsilon_n = \min\{1, q/\max_{0\leqslant i\leqslant n} C_i(\|x_n\|_{i+r(i)}, \|Px_n\|_{i+q(i)})\}$$

and

$$x_{n+1} = x_n + \varepsilon_n h_n, \quad \text{where } P'(x_n) h_n = -Px_n. \tag{5.3}$$

Thus, the sequence in (5.3) is well-defined for $i = 0, 1, \ldots$ It is clear that the sequence $\{x_n\}$ satisfies relation (1.1) with $x = x_n$ and $y = -Px_n$. In fact, we have, by virtue of (5.2) and (5.3),

$$\begin{aligned}\|P(x_n+\varepsilon_n h_n)-(1-\varepsilon_n)Px_n\|_i &= \|P(x_n+\varepsilon_n h_n)-Px_n-\varepsilon_n P'(x_n)h_n\|_i \\ &\leqslant \varepsilon_n^2 \int_0^1 \|P''(x_n+t\varepsilon_n h_n)(h_n, h_n)\|_i dt \\ &\leqslant \varepsilon_n^2 \max_{0\leqslant t\leqslant 1} \|P''(x_n+th_n)(h_n, h_n)\|_i.\end{aligned}$$

Hence, by virtue of (5.2), we obtain

$$\|P(x_n+\varepsilon_n h_n)-(1-\varepsilon_n)Px_n\|_i \leqslant q\varepsilon_n \|Px_n\|_i, \quad \text{for all } 0 \leqslant i \leqslant n. \tag{5.4}$$

THEOREM 5.1. *In addition to the hypotheses* (i), (ii), (3.1) *of Lemma* 3.1, *suppose that the following growth condition is satisfied for relation* (5.1)

$$\sum_{n=0}^{\infty} 1/\max_{0\leqslant i\leqslant n} C_i(\|x_n\|_{i+r(i)}, \|Px_n\|_{i+q(i)}) = \infty, \tag{5.5}$$

whenever $\{x_n\}$ *is a Cauchy sequence which lies in* U_0. *Then the sequence* $\{x_n\}$ *defined by* (5.3) *and* (5.2) *lies in* U_0 *and converges to a solution of equation* $Px = 0$.

Proof. The proof follows immediately from Theorem 3.1, since $\sum_{n=0}^{\infty} \varepsilon_n = \infty$, by virtue of (5.5) and (5.2).

Remark 5.1. Corollary 4.1 also applies to Theorem 5.1.

EXAMPLE. Suppose that the following assumption is satisfied as in the case of tame operators (see Hamilton [8]):

$$\|P''(x)(h, h)\|_i \leqslant C_i(\|h\|_i\|h\|_0 + \|x\|_i\|h\|_0^2), \quad i = 0, 1, \ldots \tag{5.6}$$

Then we obtain

$$\max_{0\leqslant t\leqslant 1} \|P''(x+t\varepsilon h)(h, h)\|_i \leqslant C_i(\|h\|_i\|h\|_0 + \varepsilon\|h\|_i)\|h\|_0^2,$$

where $P'(x)h = -Px$. Now suppose that

$$\|h\|_i \leqslant C_i'\|Px\|_{i+p(i)}^{1/2} \quad \text{for } i = 0, 1, \ldots, \tag{5.7}$$

i.e., the function $B_i \in \boldsymbol{B}$ in relation (1.8) is $B_i(s) = C_i' s^{1/2}$. Then we obtain from the above

$$\begin{aligned}&\max_{0\leqslant t\leqslant 1} \|P''(x+t\varepsilon h)(h, h)\|_i \\ &\qquad \leqslant C_i(\|Px\|_{i+p(i)}^{1/2}/\|Px\|_p^{1/2} + \|x\|_i + \varepsilon\|Px\|_{i+p(i)}^{1/2})\|Px\|_p,\end{aligned} \tag{5.8}$$

where $p(0) = p$ and C_i is the constant resulting from those in (5.6) and (5.7). Thus, relation (5.8) is a particular case of the more general relation (5.1).

6. The fourth algorithm for the diagonal method of contractor directions

Assuming that relations (5.6) and (5.7) hold, we can define the following algorithm. Given $x_0 \in D(P)$ and $0 < q < 1$, solve for ε the following equation

$$\|Px_0\|_p^{1/2}\varepsilon^2 + (1+\|x\|_0)\varepsilon - q/C_0 = 0. \tag{6.1}$$

Put $\varepsilon_0 = \min(1, \varepsilon)$, where ε is the solution of equation (6.1), and $x_1 = x_0 + \varepsilon_0 h_0$, where $P'(x_0)h_0 = -Px_0$. Suppose now that $x_0, x_1, \ldots, x_n \in D(P)$ have already been defined. Then we solve the equation

$$\|Px_n\|_{n+p(n)}^{1/2}\varepsilon^2 + (\|Px_n\|_{n+p(n)}^{1/2}/\|Px_n\|_p^{1/2} + \|x_n\|_n)\varepsilon - q/C_n = 0, \tag{6.2}$$

and put $\varepsilon_n = \min(1, \varepsilon_n^*)$, where ε_n^* is the solution of equation (6.2). Put

$$x_{n+1} = x_n + \varepsilon_n h_n, \quad \text{where } P'(x_n)h_n = -Px_n. \tag{6.3}$$

Thus, the sequence in (6.3) is well-defined for $i = 0, 1, \ldots$ It is clear that the sequence $\{x_n\}$ satisfies relation (1.1) with $x = x_n$, $y = -Px_n$. In fact, we have, by virtue of (6.2) and (6.3),

$$\begin{aligned}\|P(x_n+\varepsilon_n h_n) - (1-\varepsilon)Px_n\|_i &= \|P(x_n+\varepsilon_n h_n) - Px_n - \varepsilon_n P'(x_n)h_n\|_i \\ &\leqslant \varepsilon_n^2 \max_{0\leqslant t\leqslant 1} \|P''(x_n + t\varepsilon_n h_n)\|_i.\end{aligned}$$

Hence, by virtue of (5.8) with $x = x_n$, $h = h_n$, and $i = n$, and (6.2), we obtain

$$\|P(x_n+\varepsilon_n h_n) - (1-\varepsilon_n)Px_n\|_i \leqslant q\varepsilon_n\|Px_n\|_i \quad \text{for } P \leqslant i \leqslant n. \tag{6.4}$$

THEOREM 6.1. *In addition to the hypotheses* (i), (ii), (3.1) *of Lemma* 3.1, *suppose that one of the following growth conditions is satisfied for relation* (5.8)

$$\sum_{n=0}^{\infty} \|Px_n\|_p^{1/2}/C_n(\|Px_n\|_{n+p(n)}^{1/2} + \|Px_n\|_p^{1/2}\|x_n\|_n) = \infty, \tag{6.5}$$

or

$$\sum_{n=0}^{\infty}\left\{\left[1+\frac{4q\|Px_n\|_{n+p(n)}^{1/2}\|Px_n\|_p}{C_n(\|Px_n\|_{n+p(n)}^{1/2}+\|Px_n\|_p^{1/2}\|x_n\|_n)^2}\right]^{1/2}-1\right\} = \infty, \tag{6.6}$$

whenever $\{x_n\}$ *is a Cauchy sequence which lies in* U_0. *Then the sequence* $\{x_n\}$ *defined by* (6.3) *lies in* U_0 *and converges to a solution of equation* $Px = 0$.

Proof. Consider the larger root of equation (6.2)

$$\varepsilon_n^* = \{[(\|Px_n\|_{n+p(n)}^{1/2} + \|Px_n\|_p^{1/2}\|x_n\|_n)^2/\|Px_n\|_p + 4q\|Px_n\|_{n+p(n)}^{1/2}/C_n]^{1/2} - (\|Px_n\|_{n+p(n)}^{1/2} + \|Px_n\|^{1/2}\|x_n\|_p)/\|Px_n\|_p^{1/2}\}/2\|Px_n\|_{n+p(n)}^{1/2}.$$

Hence,

$$(6.7)\qquad \varepsilon_n^* = \frac{\|Px_n\|_{n+p(n)}^{1/2} + \|Px_n\|_p^{1/2}\|x_n\|_n}{2\|Px_n\|_p^{1/2}\|Px_n\|_{n+p(n)}^{1/2}} \times \left\{\left[1 + \frac{4q\|Px_n\|_{n+p(n)}^{1/2}\|Px_n\|_p}{C_n(\|Px_n\|_{n+p(n)}^{1/2} + \|Px_n\|_p^{1/2}\|x_n\|_n)^2}\right]^{1/2} - 1\right\}$$
$$\geqslant \frac{\|Px_n\|_{n+p(n)}^{1/2} + \|Px_n\|_p^{1/2}\|x_n\|_n}{2\|Px_n\|_p^{1/2}\|Px_n\|_{n+p(n)}^{1/2}} \times \frac{4q\|Px_n\|_{n+p(n)}^{1/2}\|Px_n\|_p}{2\cdot 2^{1/2}C_n(\|Px_n\|_{n+p(n)}^{1/2} + \|Px_n\|_p^{1/2}\|x_n\|_n)^2}$$
$$= q\|Px_n\|_p^{1/2}/2^{1/2}C_n(\|Px_n\|_{n+p(n)}^{1/2} + \|Px_n\|_p^{1/2}\|x_n\|_n),$$

provided that for some $0 < a < 1$ we have

$$4q\|Px_n\|_{n+p(n)}^{1/2}\|Px_n\|_p^{1/2}/C_n(\|Px_n\|_{n+p(n)}^{1/2} + \|Px_n\|_{n+p(n)}^{1/2}\|x_n\|_n)^2 \leqslant a < 1,$$

and since $0 < a < 1$ implies $(1+a)^{1/2} - 1 > a/2^{3/2}$. If this is not the case, then $\sum_{n=0}^{\infty} \varepsilon_n^* = \infty$, since the sequence $\{\|Px_n\|_p^{1/2}\}$ is bounded. Hence, we have in both cases that $\sum_{n=0}^{\infty} \varepsilon_n = \infty$, and the proof follows from Theorem 3.1. It is clear from (6.7) that (6.6) implies $\sum_{n=0}^{\infty} \varepsilon_n^* = \infty$.

Sufficient conditions can be given for relation (6.5) to be fulfilled. For instance, suppose that there exist positive constants M_n which have the following property.

Relation

$$(6.8)\qquad \|Px\|_n^{1/2} \leqslant M_n\|Px\|_p^{1/2}/(1 + \|x\|_n)$$

is satisfied for all $x \in U_0$ and for almost all $n > p$. Then we obtain the following

THEOREM 6.2. *Theorem* 6.1 *holds true if condition* (6.5) *is replaced by* (6.8) *and*

$$(6.9)\qquad \sum_{n=0}^{\infty} 1/C_n M_{n+p(n)} = \infty.$$

Proof. We have from (6.9) that

$$(6.10)\qquad \sum_{n=0}^{\infty} \|Px_n\|_p^{1/2}/C_n\|Px_n\|_{n+p(n)}(1 + \|x_n\|_{n+p(n)}) = \infty,$$

by virtue of (6.8) with $x = x_n$ and n replaced by $n+p(n)$ for almost all $n > p$. It is clear that relation (6.5) follows from (6.10).

With regard to relation (6.6) consider the following condition. Suppose that there exist positive constants d_n such that the relation

$$\|Px\|_{n+p(n)}^{1/2}\|Px\|_p \geqslant d_n C_n(\|Px\|_{n+p(n)}^{1/2} + \|Px\|_p^{1/2}\|x\|_n)^2 \tag{6.11}$$

is fulfilled for all $x \in U_0$ and almost all n. Then we obtain the following

THEOREM 6.3. *Theorem* 6.1 *holds true if condition* (6.6) *is replaced by* (6.11) *and*

$$\sum_{n=0}^{\infty}[(1+d_n)^{1/2}-1] = \infty. \tag{6.12}$$

Proof. It follows from (6.11) with $x = x_n$ and from (6.12) that condition (6.6) is satisfied.

Condition (6.5) can be replaced by the following coersiveness hypothesis. Suppose that there exists a positive function f such that $f(s_n)\to 0$ as $n\to\infty$ implies $s_n\to 0$ as $n\to\infty$, and relation

$$\|Px\|_p^{1/2}/C_n(\|Px\|_{n+p(n)}^{1/2} + \|Px\|_p^{1/2}\|x\|_n) \geqslant f(\|Px\|_n) \tag{6.13}$$

is satisfied for all $x \in U_0$ and all $n > p$. Then we obtain the following

THEOREM 6.4. *Theorem* 6.1 *holds true if condition* (6.5) *is replaced by* (6.13).

Proof. The proof follows from the argument used in the proof of Theorem 6.1 and from the fact that $\varepsilon_n^*\to 0$ as $n\to\infty$ implies $\|Px_n\|_n\to 0$ as $n\to\infty$, by virtue of relation (6.13) with $x = x_n$, since the sequence $\{x_n\}$ lies in U_0.

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