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# ON DISCRETE APPROXIMATIONS FOR A CLASS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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## 1. Introduction

One of the main problems in the theory of semiconductors is the determination of the distribution of electrons over admissible quantum states. Deviations from the known equilibrium distribution function can occur e.g. under the influence of electrical fields and have to be taken into consideration in many physical and technical problems. For several purposes it is sufficient to describe the electron transport by linear Boltzmann equations (for physical theory, cf. [1]). The following simple one-dimensional example serves as an illustration of the structure of the linear Boltzmann equations occurring in the theory of electron transport:

$$aDu(x)+c(x)u(x)$$
$$= \int_{-l}^{l} \big[K_1(x,t)\,\delta\big(w(x)-w(t)\big)+K_2(x,t)\,\delta\big(w(x)-w(t)+w_0\big)\big]u(t)\,dt$$

with

$$c(x) = \int_{-l}^{l} \big[K_1(t,x)\,\delta\big(w(t)-w(x)\big)+K_2(t,x)\,\delta\big(w(t)-w(x)+w_0\big)\big]\,dt$$

and

$$K_j(x,t)\in C[-l,l]\times[-l,l], \qquad K_j(x,t)\geqslant 0, \qquad j=1,2,$$
$$w(x)\in C^1[-l,l],\ \text{symmetric}, \qquad Dw(x)>0 \quad \text{in} \quad (0,l),$$
$$a, w_0\in \boldsymbol{R}, \qquad a>0.$$

The boundary condition for the distribution function $u(x)$ is

$$u(-l) = u(l).$$

If $\int_x^{x_1} u(t)dt$ is interpreted as the number of electrons which are in the state interval $[x, x_1]$, then the right-hand side of the equation represents the number of electrons which, owing to interactions with the crystal, were scattered from a possible state $t$ into a unit state interval near $x$ in the given example two such processes of interactions were taken nto consideration). The term

$$K_j(x, t)\,\delta\big(w(x)-w(t)+w_0^{(j)}\big)$$

represents an interaction process in which transitions from a state $t$ are admitted only to states $x$ with energy $w(x)$ satisfying the equality

$$w(x) = w(t)-w_0^{(j)}.$$

$c(x)u(x)$ represents the number of electrons which, owing to the same scattering processes, were scattered from a unit state interval near $x$ into a possible state $t$. $aDu(x)$ is the so-called drift term describing the influence of a constant electrical field onto the distribution function.

In this paper we consider approximations for a class of linear functional differential equations which are generalizations of the one-dimensional linear Boltzmann equations from the theory of electron transport. The convergence of the solutions of the approximating equations to the exact solution of a functional differential equation under examination is investigated by means of convergence results from the theory of discrete convergence of linear operators developed by F. Stummel [4], [5]. The definitions which we need and statements of this theory are briefly presented in Section 3. In particular, the applicability of the projection method and of the finite difference method to the functional differential equations is demonstrated.

## 2. Definition of functional differential equations

Let $[a, b]$ be a finite interval of the real line $\boldsymbol{R}^1$ and let $C^k[a, b]$ ($C := C^0$), $L^2(a, b)$, $W^{m,2}(a, b)$ be the familiar spaces of real-valued functions defined on $[a, b]$. In these spaces the norms are defined as usual. We shall use double brackets to denote scalar products and single brackets to denote the values of bounded linear functionals.

For given $r-1$ distinct points $\hat{x}_1, \ldots, \hat{x}_{r-1} \in (a, b)$ and $\hat{x}_0 = a$, $\hat{x}_r = b$, write $X_j = (\hat{x}_{j-1}, \hat{x}_j)$, $j = 1, \ldots, r$, and $X = [a, b]\setminus\{\hat{x}_0, \ldots, \hat{x}_r\}$, and denote by $PC = PC(X)$ the space of real-valued functions defined on $X$

which are continuous on each $X_j$ and can be continuously extended to $\bar{X}_j$. The norm in $PC$ is defined by

$$\|u\|_{PC} := \sup_{x\in X} |u(x)|.$$

By $K_C, K_{PC}, K_L, K_W$ we denote the cones of non-negative functions from the corresponding $C, PC, L^2, W^{m,2}$.

Given $m \geqslant 1$ linear independent linear forms

$$R_s(u) := \sum_{k=0}^{m-1} [\alpha_{s,k} D^k u(a) + \beta_{s,k} D^k u(b)], \quad s = 1, \dots, m, \tag{2.1}$$

define

$$V := \{u \in W^{m,2}(a, b) \mid R_s(u) = 0, \ s = 1, \dots, m\}. \tag{2.2}$$

We assume that the cone $K_V = K_C \cap V$ has the property

$$\operatorname{int}(K_V) = \operatorname{int}(K_W) \cap V \neq \emptyset \tag{2.3}$$

($\operatorname{int}(K_V)$ denotes the interior of $K_V$).

Further, let there be given families of non-negative functionals $l^{(j)}(x) \in C'[a, b]$, $x \in \bar{X}_j$, $j = 1, \dots, r$, which are continuous on $\bar{X}_j$ in the following sense:

$$\lim_{\delta\to 0} \sup_{\substack{|x-x'|<\delta \\ x,x'\in\bar{X}_j}} \left|\left(u, l^{(j)}(x) - l^{(j)}(x')\right)\right| = 0, \quad \forall u \in C[a, b]. \tag{2.4}$$

We set

$$l(x) := l^{(j)}(x) \quad \text{for} \quad x \in X \tag{2.5}$$

and deduce from Banach–Steinhaus theorem the existence of a $\varkappa > 0$ such that

$$\|l(x)\| \leqslant \varkappa$$

holds for all $x \in X$. By means of this family we define the operator

$$B\colon C \to PC, \quad (Bu)(x) := \left(u, l(x)\right). \tag{2.6}$$

This operator has the properties

$$B \in B(C, PC), \quad B\colon K_C \to K_{PC}, \quad B \in B_0(V, PC). \tag{2.7}$$

Here $B(E, F)$ and $B_0(E, F)$ denote the set of all continuous, resp. completely continuous linear operators from $E$ to $F$. The operator $B$ is supposed to have, moreover, the following properties:

$$\begin{aligned} &\forall u \in \operatorname{int}(K_V) \Rightarrow Bu \neq 0, \\ &\exists c \in PC\colon \forall u \in V \Rightarrow \left((Bu, 1)\right) = \left((u, c)\right) \end{aligned} \tag{2.8}$$

(0 and 1 denote the null function and the unit function in $C[a, b]$, respectively, and $((\cdot, \cdot))$ denotes the scalar product in $L^2$). Hence we have $c \neq 0$ and $c \in K_{PC}$.

Along with $B$ we consider the operator

$$(2.9)\qquad A_0 = \sum_{k=0}^{m} a_k(x) D^k,\ a_k \in C^k[a, b],\ k = 1, \ldots, m,\ a_0 \in PC(X),\ a_m > 0$$

and the embedding operator $J^L$: $V \to L^2$ and we define

$$(2.10)\qquad A\colon V \to L^2, \quad A := A_0 + cJ^L.$$

We impose the condition

$$(2.11)\qquad \begin{aligned} &A \text{ bijective},\ \forall w \in K_L,\ w \neq 0 \Rightarrow A^{-1} w \in \operatorname{int}(K_V),\\ &((A_0 u, 1)) = 0,\ \forall u \in V. \end{aligned}$$

Consider the problem

$$(\mathrm{P1})\qquad (A - B) u = 0;$$

we wish to find a non-trivial solution $u \in V$.

In addition to this problem, given another linear differential operator $A_a$ of order $m_a < m$, we will consider the following inhomogeneous problem

$$(\mathrm{P}_2)\qquad (A + A_a - B) u = f;$$

we wish to find a solution $u \in V$ for a given $f \in L^2$, in particular $f \in K_{PC}$, under the assumptions

$$(2.12)\qquad \begin{aligned} &A + A_a \text{ bijective},\ \forall w \in K_L,\ w \neq 0 \Rightarrow (A + A_a)^{-1} w \in \operatorname{int}(K_V),\\ &((A_0 u, 1)) = 0,\ \forall u \in V,\\ &((A_a u, 1)) > 0,\ \forall u \in \operatorname{int}(K_V). \end{aligned}$$

Equations of the form $(\mathrm{P}_2)$ describe electron transport models with absorption and generation of particles.

Notice that $A$ and $A + A_a$ are Fredholm operators with index zero and thus $A$ and $A + A_a$ are bijective, provided that

$$((A_0 u, u)) \geqslant 0, \quad (((A_0 + A_a) u, u)) \geqslant 0, \quad \forall u \in V.$$

Obviously, $\lambda = 0$ belongs to the resolvent sets $\varrho(A, B)$, $\varrho(A + A_a, B)$ [1] and the spectra $\sigma(A, B)$, $\sigma(A + A_a, B)$ are countable sets with no accumulation points different from $\infty$. Each point of these spectra is an eigenvalue of finite (algebraic) multiplicity. The question we are interested in is whether $\lambda = 1$ is a point of the spectrum or of the resolvent set.

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[1] $\varrho(A, B) := \{\lambda \in \boldsymbol{C} \mid A - \lambda B \text{ bijective}\}$, $\sigma(A, B) := \boldsymbol{C} \setminus \varrho(A, B)$.

It can be answered by use of the following theorem (cf. [7]), which in its turn can be proved by means of theorems on positive operators in spaces with cones:

THEOREM 1. (i) *The closed disk* $|\lambda| \leqslant 1$ *is contained in the resolvent set* $\varrho(A+A_a, B)$ *and thus* $(P_2)$ *has exactly one solution* $u \in V$ *for any* $f \in L^2$. *If* $f \in K_L$ *(in particular, if* $f \in K_{PC}$*) then* $u \in \operatorname{int}(K_V)$.

(ii) $\lambda = 1$ *belongs to* $\sigma(A, B)$ *and has algebraic multiplicity* 1. *This point is the smallest (in absolute value) eigenvalue in* $\sigma(A, B)$ *and the corresponding eigenvector can be chosen from* $\operatorname{int}(K_V)$. *Accordingly,* $(P_1)$ *has exactly one normed solution* $u(x)$ *with* $u(a) > 0$ *and this solution is in* $\operatorname{int}(K_V)$.

## 3. Basic notions and statements from the theory of discrete convergence of linear operators

To get approximate solutions of the problems $(P_1)$, $(P_2)$, we examine suitable problems in finite-dimensional spaces. In order to define approximations for the problems $(P_1)$ and $(P_2)$ we need adequate approximations of spaces and of operators. The notions and results which can be used in the investigation of convergence properties of the approximate problems are to be found e.g. in the theory of discrete convergence of linear operators due to F. Stummel (see [4], [5]). In the sequel we give all the definitions and statements of this theory which we need.

Let $I_0$ be a denumerable infinite sequence of pairwise distinct elements and let $E$, $(E_i)_{i\in I_0}$ be normed spaces over the same scalar field $\boldsymbol{K}$ ($\boldsymbol{K} = \boldsymbol{R}$ or $\boldsymbol{K} = \boldsymbol{C}$). The sequence $(E_i)_{i\in I_0}$ is said to *form a discrete approximation of* $E$ if there is given a sequence of restriction operators $R_i^E\colon E \to E_i$, $i \in I_0$, with the following properties:

$$\begin{aligned} (3.1) \qquad & \|R_i^E u\|_{E_i} \to \|u\|_E \quad (i \in I_0), \\ & \|R_i^E(\alpha u + \beta v) - \alpha R_i^E u - \beta R_i^E v\|_{E_i} \to 0 \quad (i \in I_0), \\ & \forall\, u, v \in E,\ \forall \alpha, \beta \in \boldsymbol{K}. \end{aligned}$$

A discrete approximation is denoted by $\big(E, (E_i)_{i\in I_0}, (R_i^E)_{i\in I_0}\big)$. In $\big(E, (E_i)_{i\in I_0}, (R_i^E)_{i\in I_0}\big)$ discrete convergence for sequences $(u_i) \in \prod_{i\in I\subset I_0} E_i$ (to a limit $u \in E$) is defined by

$$(3.2) \qquad u_i \to u \ \ (i \in I) \Leftrightarrow \|u_i - R_i^E u\|_{E_i} \to 0 \ \ (i \in I).$$

We shall also write

$$u_i \to u \ \ (E_i \to E) \ \ (i \in I)$$

to denote discrete convergence in $\big(E, (E_i)_{i\in I_0}, (R_i^E)_{i\in I_0}\big)$. In defining concrete discrete approximations the following extension theorem for restriction operators may be helpful.

THEOREM 2. *To any sequence* $(r_i^E)$ *of densely defined restriction operators* $r_i^E$: $E \to E_i$, $i \in I_0$, *i.e. such that the domain* $D(r_i^E) = \Phi$ *is linear and dense in* $E$ *and* (3.1) *is satisfied* $\forall u$, $v \in \Phi$, *there exists a sequence of restriction operators* $R_i^E$: $E \to E_i$, $i \in I_0$, *with*

$$\|r_i^E u - R_i^E u\|_{E_i} \to 0 \quad (i \in I_0), \qquad \forall u \in \Phi;$$

*if* $(R_i^E)$ *and* $(\hat{R}_i^E)$ *are two sequences of such extensions, then*

$$\|R_i^E u - \hat{R}_i^E u\|_{E_i} \to 0 \quad (i \in I_0), \qquad \forall u \in E.$$

Therefore, to define a discrete approximation of a space, one only needs a sequence of densely defined restriction operators. Given a sequence $(r_i^E)$ of densely defined restriction operators, we can define by means of Theorem 2 several discrete approximations of $E$; the discrete convergence which results is independent of the particular extensions of $r_i^E$ and can be characterized as follows:

$$(3.3) \qquad u_i \rightharpoonup u \ (i \in I) \Leftrightarrow \forall \varepsilon > 0 \Rightarrow \exists \varphi_\varepsilon \in \Phi: \quad \|u - \varphi_\varepsilon\|_E < \varepsilon, \quad \limsup_{i \in I} \|u_i - r_i^E \varphi_\varepsilon\|_{E_i} < \varepsilon.$$

A sequence $(l_i)_{i \in I \subset I_0}$ of bounded linear functionals $l_i \in E_i'$ is said to be *discretely weakly convergent to a limit* $l \in E'$,

$$l_i \rightharpoonup l \qquad (i \in I),$$

if

$$(3.4) \qquad \forall u_i \to u \ (i \in I) \Rightarrow (u_i, l_i) \to (u, l) \ (i \in I).$$

We suppose that all spaces $E_i$ of a discrete approximation for a pre-Hilbert space $E$ are also pre-Hilbert spaces. In discrete approximations for pre-Hilbert spaces we also define discrete weak convergence for sequences of their elements:

$$(3.5) \qquad u_i \rightharpoonup u \ (i \in I) \Leftrightarrow \forall v_i \to v \ (i \in I) \Rightarrow \big((v_i, u_i)\big)_{E_i} \to \big((v, u)\big)_E.$$

A discrete approximation of a pre-Hilbert space is called *discretely weakly compact* if

$$(3.6) \qquad \forall (u_i)_{i \in I}, \text{ bounded} \Rightarrow \exists I_1 \subset I, \ \exists u \in E: \ u_i \rightharpoonup u \ (i \in I_1).$$

If $E$ is a separable Hilbert space then all discrete approximations of $E$ are discretely weakly compact, and obviously, a discrete approximation of a reflexive Banach space with $E_i \subset E$ and with discrete convergence equivalent to norm convergence is discretely weakly compact.

A discrete approximation is called *discretely compact* if

$$(3.7) \qquad \forall (u_i)_{i \in I}, \text{ bounded} \Rightarrow \exists I_1 \subset I, \ \exists u \in E: \ u_i \to u \ (i \in I_1).$$

Now let $(E, (E_i)_{i\in I_0}, (R_i^E)_{i\in I_0})$, $(F, (F_i)_{i\in I_0}, (R_i^F)_{i\in I_0})$ be two discrete approximations. For operators $A \in B(E, F)$, $A_i \in B(E_i, F_i)$, $i \in I \subset I_0$, we then introduce the following types of convergence:

(3.8) $$A_i \to A \ (i \in I) \Leftrightarrow \forall u_i \to u \ (i \in I) \Rightarrow A_i u_i \to Au \ (i \in I),$$

(3.9) $$A_i \rightharpoonup A \ (i \in I) \Leftrightarrow \forall u_i \rightharpoonup u \ (i \in I) \Rightarrow A_i u_i \rightharpoonup Au \ (i \in I),$$

(3.10) $$A_i \xrightarrow{\cdot} A \ (i \in I) \Leftrightarrow \forall u_i \rightharpoonup u \ (i \in I) \Rightarrow A_i u_i \to Au \ (i \in I).$$

Of course, $E$ and $F$ in (3.9) and $E$ in (3.10) are assumed to be pre-Hilbert spaces.

A sequence $(A_i)_{i\in I}$ is called *discretely compact* if

(3.11) $$\forall (u_i)_{i\in I_1 \subset I}, \text{ bounded} \Rightarrow \exists I_2 \subset I_1, \ \exists w \in F: \ A_i u_i \to w \ (i \in I_2)$$

and is called *discretely weakly compact* if

(3.12) $$\forall u_i \rightharpoonup 0 \ (i \in I) \Rightarrow A_i u_i \to 0 \ (i \in I).$$

The discrete convergence (3.10) can be characterized (see [6]) by

(3.13) $$A_i \xrightarrow{\cdot} A \ (i \in I) \Leftrightarrow A_i \to A \ (i \in I), \ (A_i) \text{ discretely weakly compact.}$$

If $E$ is a separable Hilbert space and $F$ a pre-Hilbert space, then

(3.14) $$A_i \xrightarrow{\cdot} A \ (i \in I) \Leftrightarrow A_i \rightharpoonup A \ (i \in I), \quad (A_i) \text{ discretely compact.}$$

Finally, $(A, (A_i)_{i\in I})$ is *consistent* if

(3.15) $$\forall u \in \Phi, \Phi \text{ dense in } E \Rightarrow \exists (u_i) \in \prod_{i\in I} E_i:$$
$$u_i \to u \ (i \in I), \quad A_i u_i \to Au \ (i \in I)$$

and a sequence $(A_i)_{i\in I}$ is said to be *inversely stable* if

(3.16) $$\exists \gamma > 0, \ \exists I^e \subset I \ (I^e \text{ denoting a final piece of } I) \text{ such that}$$
$$\gamma \|u_i\|_{E_i} \leqslant \|A_i u_i\|_{F_i}, \quad u_i \in E_i, \ i \in I^e.$$

Stummel ([5], Section 2.2) proved the following convergence theorem for the class of approximations

(3.17) $$A_i(z) u_i := (A_i - zB_i) u_i = w_i, \quad i \in I, \ z \in \boldsymbol{K},$$

for inhomogenous problems

(3.18) $$A(z) u := (A - zB) u = w, \quad z \in \boldsymbol{K},$$

of type

$$(3.19)\qquad A \in B(E, F), \quad B \in B_0(E, F), \quad \varrho(A, B) \neq \emptyset\ (^2):$$

THEOREM 3. *Let there be given two discrete approximations* $\big(E, (E_i)_{i\in I_0}, (R_i^E)_{i\in I_0}\big)$ *and* $\big(F, (F_i)_{i\in I_0}, (R_i^F)_{i\in I_0}\big)$. *If*

(i) $A \in B(E, F)$, $B \in B_0(E, F)$, $\varrho(A, B) \neq \emptyset$,

(ii) $A_i \in B(E_i, F_i)$, $B_i \in B_0(E_i, F_i)$, $\big(A, (A_i)\big)$ *consistent*, $\big(B, (B_i)\big)$ *consistent*, $i \in I \subset I_0$,

(iii) $(B_i)$ *discretely compact*,

(iv) $\Delta_b\big((A_i), (B_i)\big) := \{z \in K \mid \exists I^e = I^e(z) \subset I,\ \exists \gamma = \gamma(z) > 0:$
$\gamma \|u_i\|_{E_i} \leqslant \|(A_i - zB_i)u_i\|_{F_i},\ u_i \in E_i;\ R(A_i - zB_i) = F_i,\ i \in I^e\} \neq \emptyset$
($\Delta_b$ *is called the region of boundedness*),

*then*

$$\Delta_b\big((A_i), (B_i)\big) = \varrho(A, B),$$

$$A_i(z)^{-1} \to A(z)^{-1} \quad (i \in I^e(z)), \quad z \in \varrho(A, B).$$

In particular, the assertion of this theorem is valid under the following assumptions:

($V_1$) $E, F$ complete, $A$ bijective, $B \in B_0(E, F)$,

($V_2$) $\dim E_i = \dim F_i < \infty$, $i \in I_0$,

($V_3$) $A_i$, $B_i \in B(E_i, F_i)$, $\big(A, (A_i)\big)$ and $\big(B, (B_i)\big)$ consistent, $i \in I \subset I_0$,

($V_4$) $(B_i)$ discretely compact,

($V_5$) $(A_i)$ inversely stable.

If $K = C$, then under the assumptions of Theorem 3 the following convergence theorem for eigenvalues and eigenelements holds:

THEOREM 4 (see [5], Section 3.2). (i) *If $\lambda \in \sigma(A, B)$ is an eigenvalue of (algebraic) multiplicity $m$ and $w^{(1)}, \ldots, w^{(m)}$ is a basis of the corresponding algebraic eigenspace $E(\lambda)$, and if $U$ is a bounded closed neighbourhood of $\lambda$ with*

$$\sigma(A, B) \cap U = \{\lambda\},$$

*then there are a final piece $I^e \subset I$ and $\forall i \in I^e$ exactly $m$ eigenvalues $\lambda_i^{(1)}, \ldots, \lambda_i^{(m)}$ (counted according to multiplicity) and $m$ linearly independent vectors $w_i^{(1)}, \ldots, w_i^{(m)}$ from the sum $E(\lambda_i^{(1)}, \ldots, \lambda_i^{(m)})$ of the algebraic eigenspaces of $\lambda_i^{(1)}, \ldots, \lambda_i^{(m)}$ such that*

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($^2$) Here $\varrho(A, B)$ denotes the set $\{z \in K \mid A - zB \text{ bijective}\}$.

$$\sigma(A_i, B_i)\cap U = \{\lambda_i^{(1)}, \ldots, \lambda_i^{(m)}\}, \qquad i \in I^e,$$

*and*

$$\lambda_i^{(k)} \to \lambda, \qquad w_i^{(k)} \to w^{(k)} \quad (i \in I^e), \qquad k = 1, \ldots, m.$$

(ii) *If $P_i$ denotes the eigenprojection onto $E(\lambda_i^{(1)}, \ldots, \lambda_i^{(m)})$, then*

$$\big(E(\lambda),\ \big(E(\lambda_i^{(1)}, \ldots, \lambda_i^{(m)})\big)_{i\in I^e},\ (P_i R_i^E)_{i\in I^e}\big)$$

*is a discretely compact approximation, and the corresponding discrete convergence is equivalent to the discrete convergence in $\big(E, (E_i)_{i\in I_0}, (R_i^E)_{i\in I_0}\big)$.*

## 4. Sufficient conditions for inverse stability and discrete compactness

In the following sections two classes of discrete approximations for the problems $(P_1)$ and $(P_2)$ are investigated by means of Theorems 3 and 4. These theorems are applicable since the assumption $(V_1)$ is valid for $(P_1)$ and $(P_2)$. In this section we give certain simple testable conditions for approximations of the operators from $(P_1)$ and $(P_2)$, which imply $(V_4)$ and $(V_5)$, respectively. For this, let

$$\big(V, (V_i)_{i\in I_0}, (R_i^V)_{i\in I_0}\big), \qquad \big(V_C, (C_i)_{i\in I_0}, (R_i^C)_{i\in I_0}\big),$$
$$\big(L^2, (L_i^2)_{i\in I_0}, (R_i^L)_{i\in I_0}\big)$$

be discrete approximations of $V$, $V_C$, $L^2$, respectively. Here $V_C$ denotes the subspace $J^C V$ in $C$ ($J^C$ denotes the embedding operator from $V$ into $C$). Suppose there are defined cones $K_{V_i}$ and $K_{C_i}$ in $V_i$ and $C_i$ such that the following conditions are fulfilled:

$$K_{V_i} = K_{C_i}\cap V_i, \quad \operatorname{int}(K_{V_i}) \neq \emptyset, \quad i \in I_0, \tag{4.1}$$
$$\forall u_i \to u \ (i \in I),\ u_i \in V_i,\ u \in \operatorname{int}(K_v) \Rightarrow$$
$$\exists I^e \subset I\colon u_i \in \operatorname{int}(K_{V_i}),\ i \in I^e.$$

Furthermore, let $l_i^{(j)}(x) \in C_i'$, $x \in \bar{X}_j$, $j = 1, \ldots, r$, $i \in I \subset I_0$ be families of non-negative functionals which are continuous on $\bar{X}_j$ in the sense of (2.4), and let $l_i(x)$ be the family of functionals (2.5) formed by $l_i^{(j)}(x)$, $j = 1, \ldots, r$. We then define

$$\hat{B}_i\colon C_i \to PC, \quad (\hat{B}_i u_i)(x) := \big(u_i, l_i(x)\big), \quad x \in X,\ i \in I. \tag{4.2}$$

Obviously, $\hat{B}_i \in B(C_i, PC)$. Moreover, the following holds:

THEOREM 5. *Suppose that*

(i) $V_i$ *is continuously embedded in* $C_i$ *with the embedding operator* $J_i^C$, $i \in I_0$, *and*

$$J_i^C \rightharpoonup J^C \qquad (i \in I_0),$$

(ii) $\sup\limits_{i\in I}\sup\limits_{x\in X} \|l_i(x)\| < a$, *with* $a > 0$,

$$\lim_{i\in I}\sup_{x\in X} \left|\left(R_i^C\varphi, l_i(x)\right) - \left(\varphi, l(x)\right)\right| = 0, \ \forall \varphi \in C^\infty \cap V,$$

(iii) $(r_i)_{i\in I}$ *is a sequence of operators* $r_i$: $PC \to L_i^2$ *with the property*

$$u_i \to u(PC) \ (i \in I) \Rightarrow \|r_i u_i - R_i^L u\|_{L_i^2} \to 0 \ (i \in I),$$

*and consider the operators*

$$B_i := r_i \hat{B}_{i|V_i}, \qquad i \in I. \tag{4.3}$$

*Then the sequence* $(B_i)$ *is discretely compact and* $B_i \to B$ $(i \in I)$.

*Proof.* Since $V$ is a separable Hilbert space, to any bounded sequence $(u_i) \in \prod\limits_{i\in I_1} V_i^1$ there correspond $I_2 \subset I_1$ and $u \in V$ with $u_i \rightharpoonup u$ $(V_i \to V)$ $(i \in I_2)$. Thus according to (i), $u_i \to u$ $(C_i \to V_C)$ $(i \in I_2)$. For an arbitrary $\varepsilon > 0$ et $\varphi \in C^\infty \cap V$ be chosen in such a way that $\|u - \varphi\|_C \leqslant \varepsilon$. From (ii) it follows that

$$\begin{aligned}\|\hat{B}_i u_i - Bu\|_{PC} &= \sup_{x\in X}\left|\left(u_i, l_i(x)\right) - \left(u, l(x)\right)\right| \\ &\leqslant \sup_{x\in X}\left|\left(u_i - R_i^C\varphi, l_i(x)\right)\right| + \sup_{x\in X}\left|\left(R_i^C\varphi, l_i(x)\right) - \left(\varphi, l(x)\right)\right| + \\ &\qquad + \sup_{x\in X}\left|\left(\varphi - u, l(x)\right)\right| \\ &\leqslant \|u_i - R_i^C\varphi\|_{C_i}\cdot a + \sup_{x\in X}\left|\left(R_i^C\varphi, l_i(x)\right) - \left(\varphi, l(x)\right)\right| + \|\varphi - u\|_C\cdot\varkappa\end{aligned}$$

and therefore

$$\lim_{i\in I_2}\|\hat{B}_i u_i - Bu\|_{PC} \leqslant \varepsilon(a + \varkappa).$$

Consequently $\hat{B}_i u_i \to Bu(PC)$ $(i \in I_2)$. Assumption (iii) leads to $B_i u_i \to Bu$ $(i \in I_2)$. So $(B_i)_{i\in I}$ is discretely compact. The discrete convergence $B_i \to B$ $(i \in I)$ is obvious.

Using the following theorem we shall investigate the inverse stability of approximations for the operators $A$ and $A + A_a$, respectively.

THEOREM 6. *If* $\left(E, (E_i)_{i\in I_0}, (R_i^E)_{i\in I_0}\right)$ *is a discretely weakly compact approximation of a Hilbert space* $E$, *if* $\left(F, (F_i)_{i\in I_0}, (R_i^F)_{i\in I_0}\right)$ *is a discrete approximation of a pre-Hilbert space* $F$ *and* $\left(\tilde{F}, (\tilde{F}_i)_{i\in I_0}, (R_i^{\tilde{F}})_{i\in I_0}\right)$ *is a dis-*

*crete approximation of a normed space* $\tilde{F}$, *and if* $A \in B(E, F)$ *is injective, then a sequence* $(A_i)_{i\in I}$, $A_i \in B(E_i, F_i)$, *is inversely stable if* $A_i$ *can be split into the sum of two operators* $A_i^{(1)}, A_i^{(2)}$

$$A_i = A_i^{(1)} + A_i^{(2)}, \quad i \in I,$$

*with the following properties:*

(i) $(A_i^{(2)})$ *is discretely weakly compact,*

(ii) $\exists \alpha > 0$, $\exists (K_i)_{i\in I}$, $K_i \in B(E_i, \tilde{F}_i)$, $(K_i)$ *discretely weakly compact and*

$$\alpha \|u_i\|_{E_i} \leqslant \|A_i^{(1)} u_i\|_{F_i} + \|K_i u_i\|_{\tilde{F}_i}, \quad u_i \in E_i, \; i \in I,$$

(iii) $A_i \rightharpoonup A$ $(i \in I)$.

*Proof.* If $(A_i)$ were not inversely stable, then a subsequence $I_1 \subset I$ and a sequence $(u_i)$, $u_i \in E_i$, $\|u_i\|_{E_i} = 1$, $i \in I_1$, with $\|A_i u_i\|_{F_1} \to 0$ $(i \in I_1)$ would exist. Since $\big(E, (E_i)_{i\in I_0}, (R_i^E)_{i\in I_0}\big)$ was assumed to be discretely weakly compact, there exist a subsequence $I_2 \subset I_1$ and a $u \in E$ with $u_i \rightharpoonup u$ $(i \in I_2)$. From (iii), $A_i u_i \rightharpoonup Au$ $(i \in I_2)$ results. Thus $Au = 0$ and consequently $u = 0$. Then according to (i) $A_i^{(2)} u_i \to 0$ $(i \in I_2)$ holds and thus $A_i^{(1)} u_i \to 0$ $(i \in I_2)$, too. The inequality in (ii) leads to $u_i \to 0$ $(i \in I_2)$, in contradiction to $\|u_i\|_{E_i} = 1$.

## 5. Discrete approximations of functional differential equations in subspaces of $V$, $L^2$

Let $I_0 := N$ be the sequence of natural numbers and let $V_i \subset V$, $i \in N$, be a sequence of finite-dimensional subspaces of $V$ with

$$\inf_{v\in V_i} \|u - v\|_m \to 0 \quad (i \in N), \quad \forall u \in V.$$

Let $\Pi_i$ be the orthogonal projection onto $V_i$ and

$$R_i^V := \Pi_i.$$

Obviously, the sequence $(\Pi_i)$ satisfies condition (3.1) and thus

$$\big(V, (V_i)_{i\in N}, (\Pi_i)_{i\in N}\big) \tag{5.1}$$

is a discretely weakly compact approximation of $V$.

Since

$$u_i \to u \;(V_i \to V) \Leftrightarrow \|u_i - \Pi_i u\|_m \to 0 \Leftrightarrow \|u_i - u\|_m \to 0 \Leftrightarrow u_i \to u(V),$$

the discrete convergence corresponding to (5.1) is equivalent to the norm convergence in $V$. It is also easily seen that the discrete weak convergence corresponding to (5.1) is equivalent to the weak convergence in $V$.

Furthermore, let $C_i := J^C V_i$ and

$$R_i^C u := \Pi_i u, \quad \forall u \in V.$$

From $u_i \to u(V)$ follows $u_i \to u(C)$, and therefore

$$\big(V_C, (C_i)_{i\in N}, (\Pi_i)_{i\in N}\big) \tag{5.2}$$

is a discrete approximation of $V_C$, and the discrete convergence in (5.2) is equivalent to the norm convergence in $C$.

Let

$$K_{V_i} = K_{C_i} := K_C \cap C_i, \tag{5.3}$$

by which (4.1) is obviously fulfilled.

Let $(P_i)$ be a sequence of projections with

$$P_i \in B(PC, L^2) \quad \text{and} \quad P_i u \to u(L^2), \quad \forall u \in PC$$

(e.g. the orthogonal projections in $L^2$ which project onto finite-dimensional subspaces of $L^2$ spanned by the elements of complete systems in $L^2$, or interpolation operators interpolating the functions of $PC$ at the zeros of orthogonal polynomials). We then put

$$L^2 \supset L_i^2 := R(P_i) \quad \text{and} \quad r_i^L u := P_i u, \quad \forall u \in PC,$$

and assume $\dim L_i^2 = \dim V_i$, $i \in N$.

The sequence $(P_i)$ satisfies (3.1) for all $u, v \in PC$ and thus we get a discrete approximation

$$\big(L^2, (L_i^2)_{i\in N}, (R_i^L)_{i\in N}\big) \tag{5.4}$$

defined by the extensions $R_i^L$ of $P_i$. The discrete convergence in (5.4) is equivalent to the norm convergence in $L^2$ as is seen from the following:

$$u_i \to u\ (L_i^2 \to L^2)\ (i \in N_1) \Leftrightarrow \forall \varepsilon > 0 \Rightarrow \exists \varphi_\varepsilon \in PC:\ \|u - \varphi_\varepsilon\| < \varepsilon,$$

$$\limsup_{i\in N_1} \|u_i - P_i\varphi_\varepsilon\| < \varepsilon$$

$$\Rightarrow \limsup_{i\in N_1} \|u_i - u\| \leqslant \limsup_{i\in N_1} \|u_i - P_i\varphi_\varepsilon\| + \limsup_{i\in N_1} \|P_i\varphi_\varepsilon - \varphi_\varepsilon\| + \|\varphi_\varepsilon - u\| < 2\varepsilon$$

$$\Rightarrow u_i \to u(L^2)\ (i \in N_1)$$

and, on the other hand,

$$u_i \to u\ (L^2)\ (i \in N_1) \Rightarrow \forall \varepsilon > 0 \Rightarrow \exists \varphi_\varepsilon \in PC:\ \|u - \varphi_\varepsilon\| < \varepsilon$$

$$\Rightarrow \limsup_{i\in N_1} \|u_i - P_i\varphi_\varepsilon\| \leqslant \limsup_{i\in N_1} \|u_i - u\| + \|u - \varphi_\varepsilon\| + \limsup_{i\in N_1} \|\varphi_\varepsilon - P_i\varphi_\varepsilon\| < \varepsilon$$

$$\Rightarrow u_i \to u\ (L_i^2 \to L^2)\ (i \in N_1).$$

Then again the discrete weak convergence in (5.4) is equivalent to the weak convergence in $L^2$.

For approximations of the operator $B$ we choose

$$l_i(x) := l(x), \quad r_i := P_i, \quad i \in \boldsymbol{N}. \tag{5.5}$$

Then all assumptions of Theorem 5 are satisfied, and for

$$B_i = P_i B|_{V_i}, \quad i \in \boldsymbol{N}, \tag{5.6}$$

we get $B_i \to B$ ($i \in \boldsymbol{N}$) and the sequence $(B_i)$ is discretely compact. The operators $A$ and $A+A_a$ are decomposed into

$$A^{(1)} := a_m D^m \quad \text{and} \quad A^{(2)} := \sum_{k=0}^{m-1} a_k D^k + cJ^L$$

(for convenience we have denoted the coefficients of $A_0+A_a$ also by $a_k$). Obviously, $A^{(2)} \in B_0(V, PC)$.

We assume

$$A^{(1)}(V_i) \subset PC, \quad i \in \boldsymbol{N}, \tag{5.7}$$

and put

$$A_i^{(j)} := P_i A^{(j)}|_{V_i}, \quad j = 1, 2, \quad A_i = A_i^{(1)} + A_i^{(2)}, \quad i \in \boldsymbol{N}. \tag{5.8}$$

The operators $A_i$ are consistent with $A$ (with $A+A_a$, respectively). To apply Theorem 6 to $(A_i)$ we choose

$$\tilde{F} = \tilde{F}_i := W^{m-1}(a, b), \; R_i^{\tilde{F}} := \mathrm{Id}_{\tilde{F}}, \; K_i := J : V \to W^{m-1}, \quad i \in \boldsymbol{N}, \tag{5.9}$$

($J$ denotes the embedding operator) and suppose

$$\exists \beta > 0 \colon \beta \|w\| \leqslant \|P_i w\|, \; \forall w \in A^{(1)}(V_i), \tag{5.10}$$
$$w_i \rightharpoonup w(L^2) \Rightarrow P_i w_i \rightharpoonup w(L^2), \quad w_i \in A^{(1)}(V_i).$$

The assumptions (5.10) are fulfilled, e.g., if

$$A^{(1)}(V_i) \subset R(P_i), \quad i \in \boldsymbol{N}. \tag{5.11}$$

$(A_i^{(2)})$ is discretely weakly compact, and from (5.9) and (5.10) it is easily seen that also the conditions (ii) and (iii) of Theorem 6 are fulfilled. Therefore we get:

For almost all $i \in \boldsymbol{N}$ the solutions $u_i \in V_i$ of the approximate problems

$$P_i(A - \lambda_i^{\min} B) u_i = 0, \quad \|u_i\|_m = 1, \quad u_i(a) > 0, \quad i \in \boldsymbol{N}, \tag{$\mathrm{P}_{1i}$}$$

and

$$P_i(A + A_a - B) u_i = P_i f, \quad f \in K_{PC}, \quad i \in \boldsymbol{N}, \tag{$\mathrm{P}_{2i}$}$$

exist, are uniquely determined and are in $\mathrm{int}(K_{V_i})$. They converge in the norm of $W^{m,2}$ to the corresponding solutions of $(\mathrm{P}_1)$ and $(\mathrm{P}_2)$, respectively. Here $\lambda_i^{\min}$ denotes the smallest (in absolute value) eigenvalue from $\sigma(A_i, B_i)$ which for almost all $i \in \boldsymbol{N}$ has (algebraic) multiplicity 1. Moreover, $\lambda_i^{\min} \to 1$. If, e.g., $V_i = L_i^2$ and $1 \in V_i$, then for almost all $i \in \boldsymbol{N}$ we have $\lambda_i^{\min} = 1$. Finally,

$$\{\lambda \in \boldsymbol{C} \mid |\lambda| \leqslant 1\} \subset \varrho(A_i + A_{ai}, B_i)$$

for almost all $i \in \boldsymbol{N}$.

We notice that both problems $(\mathrm{P}_{1i})$ and $(\mathrm{P}_{2i})$ can be solved also iteratively by essentially the same procedure.

## 6. Discrete approximations of functional differential equations in the linear vector spaces $\boldsymbol{R}^s$

Again let $I_0 := \boldsymbol{N}$ be the sequence of natural numbers and let $-\infty < \bar{a} \leqslant a < b \leqslant \bar{b} < \infty$. Suppose that for each $i \in \boldsymbol{N}$, there are given $m+1$ sets of net points in $[\bar{a}, \bar{b}]$,

(6.1) $$\bar{a} \leqslant x_i^{k,1} < \ldots < x_i^{k,s} \leqslant \bar{b}, \quad k = 0, 1, \ldots, m, \quad s = s(i, k) \in \boldsymbol{N},$$

with

$$h_i^k := \max_{2 \leqslant j \leqslant s} |x_i^{k,j} - x_i^{k,j-1}| \to 0 \quad (i \in \boldsymbol{N}),$$

and

$$x_i^{k,1} \to a \quad (i \in \boldsymbol{N}), \quad x_i^{k,s} \to b \quad (i \in \boldsymbol{N}), \quad k = 0, 1, \ldots, m.$$

We assume for convenience

$$\{x_i^{m,j} \mid j = 1, \ldots, s\} \subset [a, b].$$

Consider $m+1$ convergent quadrature formulas

$$\sum_{j=1}^{s} w_i^{k,j} \bar{\varphi}(x_i^{k,j}) \to \int_a^b \varphi(x)\,dx \quad (i \in \boldsymbol{N}), \ \varphi \in C[a, b]$$

with positive weights $(w_i^{k,j})$, $j = 1, \ldots, s$ ($\bar{\varphi}$ denoting a continuous continuation of $\varphi$ to $[\bar{a}, \bar{b}]$) and write

(6.2) $$\big((u_i, v_i)\big)_i^{(k)} := \sum_{j=1}^{s(i,k)} w_i^{k,j} u_i^j v_i^j \quad \text{for } u_i, v_i \in \boldsymbol{R}^s, \quad k = 0, 1, \ldots, m.$$

We then define the spaces

(6.3) $$L_i^{2,k} := \{\boldsymbol{R}^{s(i,k)}, \big((\cdot, \cdot)\big)_i^{(k)}\}, \quad i \in \boldsymbol{N}, \ k = 0, 1, \ldots, m,$$

and use the densely defined restriction operators

(6.4) $$r_i^{(k)}: L^2 \to L_i^{2,k}, \quad D(r_i^{(k)}) = C,$$
$$(r_i^{(k)}\varphi)^j = (r_i^{(k)}\bar{\varphi})^j := \bar{\varphi}(x_i^{k,j}), \quad j = 1, \ldots, s,$$

with an arbitrarily fixed continuation $\bar{\varphi}$ of $\varphi$ to produce the $m+1$ discrete approximations

$$(6.5)\qquad \left(L^2,\ (L_i^{2,k})_{i\in N},\ (R_i^{(k)})_{i\in N}\right)$$

of $L^2$ by extensions $R_i^{(k)}$ of $r_i^{(k)}$.

If we take the spaces

$$(6.6)\qquad C_{k,i} := \{\boldsymbol{R}^{s(i,k)},\ \|\cdot\|_{\infty,i}^{(k)}\},\qquad i\in \boldsymbol{N},\ k = 0, 1, \ldots, m,$$

with

$$(6.7)\qquad \|u_i\|_{\infty,i}^{(k)} := \max_{1\leqslant j\leqslant s} |u_i^j|,\qquad u_i\in \boldsymbol{R}^{s(i,k)},$$

and the restriction operators

$$(6.8)\qquad R_{k,i}^C u := r_i^{(k)} u,\qquad u\in C,$$

then we get $m+1$ discrete approximations

$$(6.9)\qquad \left(C,\ (C_{k,i})_{i\in N},\ (R_{k,i}^C)_{i\in N}\right)$$

for the space $C$.

In order to define a discrete approximation for $W^{m,2}$ we take operators

$$(6.10)\qquad \delta_i^{(k)}\colon \boldsymbol{R}^{s(i,k-1)}\to \boldsymbol{R}^{s(i,k)},\qquad k = 1, \ldots, m,\ i\in \boldsymbol{N},$$

with the following properties:

(6.11) (i) $D_i^k r_i^{(0)}\varphi\to D^k\varphi\,(L_i^{2,k}\to L^2)\ (i\in \boldsymbol{N})$,

$\forall\varphi\in C^\infty[a, b]$ and $\bar{\varphi}\in C^{m1}[\bar{a}, \bar{b}]$, $m_1 > m$, with

$$D_i^k := \prod_{r=1}^{k} \delta_i^{(r)},$$

(ii) $\hat{\delta}_i^{(k)} r_i^{(k)}\psi\to D\psi\,(L_i^{2,k-1}\to L^2)\ (i\in \boldsymbol{N})$,

$\forall\psi\in C_0^\infty[a, b]$ and $\bar{\psi}\in C^{m2}[\bar{a}, \bar{b}]$, $m_2 > 1$, with

$\hat{\delta}_i^{(k)}\colon\ \boldsymbol{R}^{s(i,k)}\to \boldsymbol{R}^{s(i,k-1)}$,

$\left((u_i,\ \delta_i^{(k)} v_i)\right)_i^{(k)} = -\left((\hat{\delta}_i^{(k)} u_i,\ v_i)\right)_i^{(k-1)}$,

for $u_i\in \boldsymbol{R}^{s(i,k)}$, $v_i\in \boldsymbol{R}^{s(i,k-1)}$,

(iii) $$\left.\begin{array}{l} u_i\rightharpoonup u\,(L_i^{2,k-1}\to L^2)\ \ (i\in \boldsymbol{N}_1\subset \boldsymbol{N}) \\ \delta_i^{(k)} u_i\rightharpoonup Du\,(L_i^{2,k}\to L^2)(i\in \boldsymbol{N}_1\subset \boldsymbol{N})\end{array}\right\}\Rightarrow u_i\to u\,(C_{k-1,i}\to C)(i\in \boldsymbol{N}_1)$$

for $u\in W^{1,2}$, $(u_i)\in \prod_{i\in N_1} \boldsymbol{R}^{s(i,k-1)}$, $k = 1, \ldots, m$, and we set

$$((u_i, v_i))_{m,i} := \sum_{k=0}^{m} ((D_i^k u_i, D_i^k v_i))_i^{(k)}, \qquad u_i, v_i \in \boldsymbol{R}^{s(i,0)}. \tag{6.12}$$

With this scalar product, let

$$W_i^{m,2} := \{\boldsymbol{R}^{s(i,0)}, ((\cdot, \cdot))_{m,i}\}, \qquad i \in \boldsymbol{N}. \tag{6.13}$$

We then use the discrete approximation

$$(W^{m,2}, (W_i^{m,2})_{i\in \boldsymbol{N}}, (R_i^W)_{i\in\boldsymbol{N}}), \tag{6.14}$$

which is defined by extensions of the following densely defined restriction operators:

$$r_i^W\colon W^{m,2} \to W_i^{m,2}, \quad D(r_i^W) = C^\infty, \quad r_i^W u := r_i^{(0)} u, \quad i \in \boldsymbol{N}. \tag{6.15}$$

For discrete convergence and discrete weak convergence in (6.14) the following holds (see [7]):

$$\begin{aligned} &u_i \overset{(\rightharpoonup)}{\rightarrow} u\,(W_i^{m,2} \to W^{m,2})\ (i \in \boldsymbol{N}_1 \subset \boldsymbol{N}) \\ &\Leftrightarrow D_i^k u_i \overset{(\rightharpoonup)}{\rightarrow} D^k u\,(L_i^{2,k} \to L^2)\ (i \in \boldsymbol{N}_1 \subset \boldsymbol{N}) \quad \text{for } k = 0, 1, \ldots, m, \end{aligned} \tag{6.16}$$

($D_i^0$ and $D^0$ denote the identities of $L_i^{2,0}$ and $L^2$, respectively) and

$$\begin{aligned} &u_i \rightharpoonup u\,(W_i^{m,2} \to W^{m,2})\ (i \in \boldsymbol{N}_1) \\ &\Rightarrow D_i^k u_i \to D^k u\,(C_{k,i} \to C)\ (i \in \boldsymbol{N}_1) \quad \text{for } k = 0, 1, \ldots, m-1, \end{aligned} \tag{6.17}$$

(cf. [2]).

In concrete applications the conditions (6.11) (i) and (ii) can be easily verified. The validity of (6.11) (iii) can be shown by means of interpolation operators (see [7])

$$\mathrm{Int}^{(k)}\colon \boldsymbol{R}^{s(i,k-1)} \to W^{1,2}(\bar a, \bar b), \qquad k = 1, \ldots, m. \tag{6.18}$$

Now we present discrete approximations for the spaces $L^2$, $V_C$, $V$ which will be used to define approximations of the problems ($P_1$) and ($P_2$). We set

$$\begin{aligned} L_i^2 &:= L_i^{2,m}, \quad R_i^L := R_i^{(m)}, \\ C_i &:= C_{0,i}, \quad R_i^C := R_{0,i}^C, \quad i \in \boldsymbol{N}. \end{aligned} \tag{6.19}$$

Let $K_{C_i}$ be the cone of non-negative vectors of $\boldsymbol{R}^{s(i,0)}$. To approximate the space $V$, we choose subspaces $\hat{\boldsymbol{R}}^{s(i,0)} \subset \boldsymbol{R}^{s(i,0)}$ with $\dim \hat{\boldsymbol{R}}^{s(i,0)} = \dim L_i^2$ and scalar products $((\cdot, \cdot))_{V_i}$ such that in the resulting Hilbert spaces

$$V_i := \{\hat{\boldsymbol{R}}^{s(i,0)}, ((\cdot, \cdot))_{V_i}\}, \qquad i \in \boldsymbol{N}, \tag{6.20}$$

the following inequalities hold:

$$\alpha \|u_i\|_{m,i} \leqslant \|u_i\|_{V_i} \leqslant \beta \|u_i\|_{m,i}, \quad \forall u_i \in V_i, \; i \in \boldsymbol{N}, \tag{6.21}$$

with $\alpha, \beta > 0$ independent of $i$, and

$$\operatorname{int}(K_{V_i}) \neq \emptyset \quad \text{for} \quad K_{V_i} = K_{C_i} \cap V_i.$$

We then consider the discrete approximations

$$\big(V, (V_i)_{i\in\boldsymbol{N}}, (R_i^V)_{i\in\boldsymbol{N}}\big) \tag{6.22}$$

defined by the extensions of densely defined restriction operators

$$\hat{r}_i\colon V \to V_i, \quad D(\hat{r}_i) = C^\infty \cap V \tag{6.23}$$

with the property

$$\|r_i^{(0)}\varphi - \hat{r}_i\varphi\|_{m,i} \to 0 \quad (i \in \boldsymbol{N}), \; \forall \varphi \in C^\infty \cap V. \tag{6.24}$$

The condition (4.1) is then fulfilled (cf. [7]).

It is easily seen that then the discrete convergence in (6.22) is equivalent to the discrete convergence in (6.16). The discrete weak convergence in (6.22) is proved to be equivalent to the discrete weak convergence in (6.16) too. Therefore condition (i) in Theorem 5 is fulfilled and $\big(V, (V_i)_{i\in\boldsymbol{N}}, (R_i^V)_{i\in\boldsymbol{N}}\big)$ remains discretely weakly compact.

The operator $B$ is approximated as follows: For a sequence of prolongation operators

$$Q_i\colon C_i \to C, \quad \|Q_i\| \leqslant \gamma, \quad Q_i R_i^C \varphi \to \varphi(C) \; (i \in \boldsymbol{N}), \quad \varphi \in C^\infty \cap V \tag{6.25}$$

(e.g. linear interpolation operators) we define

$$(\hat{B}_i u_i)(x) = \big(u_i, l_i(x)\big) := \big(Q_i u_i, l(x)\big) \tag{6.26}$$

and we set

$$(r_i u)^j := \tfrac{1}{2}\big(u(x_i^{m,j}+0) + u(x_i^{m,j}-0)\big), \quad j = 1, \ldots, s(i,k), \; u \in PC. \tag{6.27}$$

By these settings, all assumptions of Theorem 5 are satisfied and we have again

$$B_i \to B \; (i \in \boldsymbol{N}) \quad \text{and} \quad (B_i) \text{ discretely compact}.$$

For simplicity we consider here only the following type of approximations for $A$ and $A + A_a$, respectively:

$$A_i = A_i^{(1)} + A_i^{(2)},$$

with

$$A_i^{(1)} := r_i a_m D_i^m, \quad A_i^{(2)} := \sum_{k=0}^{m-1} r_i a_k M_i^{(k)} D_i^k + r_i c M_i^{(0)} \tag{6.28}$$

and linear operators

$$M_i^{(k)}: L_i^{2,k} \to L_i^{2,m}, \quad k = 0, 1, \ldots, m-1, \tag{6.29}$$

which are discretely convergent to the identity of $L^2$ (with respect to generalized approximations; cf. [3]). To apply Theorem 6 to $(A_i)$ we set

$$\begin{aligned} &\tilde{F} := W^{m-1,2}(a, b), \quad \tilde{F}_i := W_i^{m-1,2} := \{\boldsymbol{R}^{s(i,0)}, ((\cdot, \cdot))_{m-1,i}\}, \\ &K_i := J_i\colon V_i \to W_i^{m-1,2} \quad (J_i \text{ the embedding operator}) \end{aligned} \tag{6.30}$$

and we define the discrete approximation of $\tilde{F}$ by the extensions of the following densely defined restriction operators:

$$r_i^{(0)}: W^{m-1,2} \to \tilde{F}_i, \quad D(r_i^{(0)}) = C^\infty.$$

Then all conditions of Theorem 6 are fulfilled, and therefore the sequence $(A_i)$ is inversely stable. Obviously, $(A_i)$ is consistent with $A$ and $A + A_a$, respectively, and thus we again obtain the existence, uniqueness, positivity, and convergence of the solutions of the approximate problems defined by the operators $(A_i)$, $(B_i)$.

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