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ALTERNATING DIRECTION GALERKIN METHOD WITH CAPACITANCE MATRIX FOR THE PARABOLIC PROBLEM WITH NATURAL BOUNDARY CONDITION

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We consider a parabolic equation with an initial-boundary condition of the third kind in $Q_T = \bar{\Omega} \times [0, T]$, where Ω is an arbitrary bounded domain in $\boldsymbol{R}^2$. To solve this problem we construct an alternating direction Galerkin (ADG) method with the bilinear approximation in the space of independent variable x_i, $i = 1, 2$, and the difference approximation in the time variable t. We prove that this method is unconditionally stable and convergent. The error of the method is $O(\tau + h)$ in the norm $L_\tau^2(0, T; H^1)$, where τ and h_i are the steps of the time and space grid, $h = \max\{h_1, h_2\}$.

The corresponding system of linear equations is solved by the capacitance matrix method, see [1], [2]. If τ is of order h then the cost of the method is proportional to the total number of unknows.

In [3] and [4], the ADG method is considered for the parabolic problem in $Q_T = [0, T] \times \bar{\Omega}$ where Ω is a rectangle or rectangular polygon. In [5] this method is generalized for an arbitrary domain Ω for the parabolic problem with a boundary Dirichlet condition.

1. Differential problem

We consider the initial-boundary value problem for a parabolic equation with a boundary condition of the third kind,

$$\frac{\partial u}{\partial t} + Lu = f(x, t), \quad (x, t) \in Q_T = \Omega \times (0, T], \tag{1.1}$$

$$u(x, 0) = u_0, \quad x \in \Omega, \qquad \frac{\partial u}{\partial \nu} + \sigma u = 0, \quad x \in \Gamma, \quad t \in (0, T], \tag{1.2}$$

where Ω is a bounded domain in $\boldsymbol{R}^2$ with the boundary $\Gamma = \partial\Omega$, which is piecewise smooth. The operators L and $\partial/\partial\nu$ in (1.1) and (1.2) are of the form

$$Lu = -\sum_{i,j=1}^{2} D_i\big(a_{ij}(x,t)D_j u\big) + \sum_{i=1}^{2} b_i(x,t)D_i u + c(x,t)u,$$

$$\frac{\partial u}{\partial\nu} = -\sum_{i,j=1}^{2} a_{ij}(x,t)D_j u\cos(x_i,\nu),$$

where ν denotes the outward normal to Γ and (x_i,ν) is the angle between ν and the x_i axis.

The weak form of the problem (1.1) and (1.2) is as follows:

For $f\in L^2\big(0,T;L^2(\Omega)\big)$ and $u_0\in H^1(\Omega)$ find a function u: $[0,T]\to H^1$ such that

$$u\in L^2\big(0,T;H^1(\Omega)\big),\qquad \frac{du}{dt}\in L^2\big(0,T;L^2(\Omega)\big), \tag{1.3}$$

$$\frac{d}{dt}\big(u(t),v\big)+a\big(t;u(t),v\big) = \big(f(t),v\big),\qquad \forall\, v\in H^1(\Omega),\quad t\in(0,T], \tag{1.4}$$

$$u(0) = u_0 \tag{1.5}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$ and

$$a(t;u,v) = \int_\Omega\Big\{\sum_{i,j=1}^{2} a_{ij}D_i uD_j v + \sum_{i=1}^{2} b_i D_i uv + cuv\Big\}\,d\Omega + \int_\Gamma \sigma uv\,d\Gamma.$$

Here $L^2\big(0,T;H(\Omega)\big)$ denotes the space of functions u: $[0,T]\to H(\Omega)$ such that

$$\int_0^T \|u(t)\|_H^2\,dt < +\infty$$

where $H(\Omega)$ is a Hilbert space.

We assume that $a(t;u,v)$ satisfies the following condition: there exist constants $\gamma_0,\gamma_1>0$ such that $\forall\, t\in[0,T]$ $\forall\, v\in H^1(\Omega)$

$$\gamma_0\|v\|_{H^1}^2 \leqslant a(t;v,v)\leqslant \gamma_1\|v\|_{H^1}^2. \tag{1.6}$$

The problem (1.3)–(1.5) has a unique solution under some assumptions on the coefficients of the problem (see for example [6]).

2. Discrete problem

Our aim is to construct an alternating direction Galerkin (ADG) method for the problem (1.3)–(1.5).

First we construct a finite element space. Let R_h^2 be a rectangular grid in $\boldsymbol{R}^2$ with the steps h_1 and h_2. Let e_{ij} denote the rectangular element of the form $e_{ij} = [ih_1, (i+1)h_1] \times [jh_2, (j+1)h_2]$. For the domain Ω we define Ω_1 as the union of all elements e_{ij} such that $e_{ij} \subset \bar{\Omega}$ and Ω_2 as the union of all elements e_{ij} such that $e_{ij} \cap \bar{\Omega} \neq \{\emptyset\}$ (see Fig. 1).

The finite element space is defined as

$$V_h(\Omega_2) = \{v:\ v \in C(\bar{\Omega}),\ v|e_{ij} \in P_{1,1}\}$$

where $P_{1,1}$ is the set of the bilinear polynomial in x_1 and x_2. The function v on the element e_{ij} is uniquely defined by the values of v in the vertices of e_{ij}.

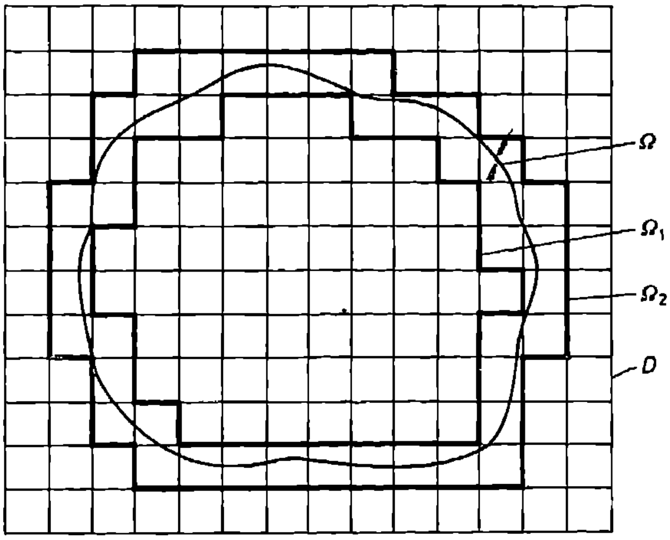

Fig. 1

Finally, let ω_τ be a time grid,

$$\omega_\tau = \{t = n\tau,\ n = 0, \ldots, N,\ N\tau = T\}.$$

The discrete AD Galerkin problem for the problem (1.1) and (1.2) is defined as follows.

Find a function U: $\omega_\tau \to V_h(\Omega_2)$ such that

(2.1)
$$(U_t^n, v)_{L^2(\Omega)} + \theta\tau(U_t^n, v)_{H^1(\Omega_2)} + \frac{\theta^2\tau^2}{1+\theta\tau}(D_1D_2U_t^n, D_1D_2v)_{L^2(\Omega_2)} + a^n(U^n, v)$$
$$= (f^n, v)_{L^2(\Omega)}, \quad \forall\, v \in V_h(\Omega_2),\ n = 0, \ldots, N-1,$$

(2.2)
$$U^0 = u_{0h}$$

where θ is a positive real parameter which will be chosen later on and

$$U^n = U(n\tau), \qquad U_t^n = (U^{n+1} - U^n)/\tau, \qquad a^n(U^n, v) = a(n\tau;\, U^n, v).$$

The function $u_{0h} \in V_h(\Omega_2)$ interpolates the extension of the function u_0 to Ω_2. By f^n we mean $f(n\tau)$. For simplicity we assume that $f \in C([0, T]; L^2(\Omega))$.

3. Algorithm

In this section we define an algorithm for computing the solution of the problem (2.1) and (2.2).

First we reduce the problem (2.1) and (2.2) to a system of linear equations. Let $\bar{\Omega}_{i,h}$, $i = 1, 2$, denote the sets of the grid points (nodes) belonging to $\bar{\Omega}_i$. For every point $x = (ih_1, jh_2) \in \bar{\Omega}_{2,h}$ we define a basis function φ_{ij} of the form

$$\varphi_{ij}(x_1, x_2) = \varphi_i(x_1)\varphi_j(x_2) = \varphi\left(\frac{x_1}{h_1} - i\right)\varphi\left(\frac{x_2}{h_2} - j\right)$$

where $\varphi(s)$, $s \in \boldsymbol{R}$, is the roof function.

The space $V_h(\Omega_2)$ is spanned by the functions φ_{ij}, i.e.,

$$V_h = \operatorname{span}\{\varphi_{ij}\}, \qquad (i, j) \in \bar{\Omega}_{2,h},$$

where $(i, j) \in \bar{\Omega}_{2,h}$ means that $(ih_1, jh_2) \in \bar{\Omega}_{2,h}$.

We seek the solution U^n of the form

$$U^n(x_1, x_2) = \sum_{(i,j)\in\bar{\Omega}_{2,h}} a_{ij}^n \varphi_{ij}(x_1, x_2).$$

Using this form in (2.1) and (2.2), we get

$$\begin{aligned} (3.1) \qquad & \sum_{(i,j)\in\bar{\Omega}_{2,h}} a_{ij,t}^n \Big\{(\varphi_{ij}, \varphi_{pq})_{L^2(\Omega)} + \theta\tau(\varphi_{ij}, \varphi_{pq})_{H^1(\Omega_2)} + \\ & \qquad + \frac{\theta^2\tau^2}{1+\theta\tau}(D_1D_2\varphi_{ij}, D_1D_2\varphi_{pq})_{L^2(\Omega_2)}\Big\} = F_{pq}^n, \\ & \qquad\qquad (p, q) \in \bar{\Omega}_{2,h}, \quad n = 0, \ldots, N-1, \end{aligned}$$

$$(3.2) \qquad a_{pq}^0 = u_{0h}(ph_1, qh_2), \qquad (p, q) \in \bar{\Omega}_{2,h},$$

where

$$F_{pq}^n = (f^n, \varphi_{pq})_{L^2(\Omega)} - a^n(U^n, \varphi_{pq}).$$

We rewrite system (3.1) as a difference scheme. Let us introduce the following difference operators:

$$\tilde{\Lambda}_i = \frac{h_i}{6}(T_i^+ + 4T^0 + T_i^-), \qquad \Lambda_i = -\frac{1}{h_i}(T_i^+ - 2T^0 + T_i^-)$$

where $T_i^{\pm} x = x \pm h_i e_i$, $e_1 = (1, 0)$, $e_2 = (0, 1)$.

We decompose the grid (node) sets $\bar{\Omega}_{i,h}$, $i = 1, 2$, in the form $\bar{\Omega}_{i,h} = \Omega_{i,h} \cup \partial\Omega_{i,h}$ where $\partial\Omega_{i,h}$ are the sets of the grid points belonging to $\partial\Omega_i$. Define $\Gamma_h = \partial\Omega_{1,h} \cup \partial\Omega_{2,h}$. Let N_{pq} denote a neighbourhood of the node $(p, q) \in \Gamma_h$ containing the nodes of the elements e_{ij} for which the node (p, q) is a vertex. We can now rewrite the system (3.1) as the following difference scheme: for $n = 0, 1, \ldots, N-1$

$$(3.3) \qquad (1+\theta\tau)\left(\tilde{\Lambda}_1 + \frac{\theta\tau}{1+\theta\tau}\Lambda_1\right)\left(\tilde{\Lambda}_2 + \frac{\theta\tau}{1+\theta\tau}\Lambda_2\right) a^n_{pq,t} = F^n_{pq}, \qquad (p, q) \in \Omega_{1,h},$$

$$(3.4) \qquad \sum_{(i,j)\in\Gamma_h} g_{ij}\, a^n_{ij,t} = F^n_{pq}, \qquad (p, q) \in \Gamma_h.$$

Note that the coefficients g_{ij} do not depend on n.

To solve the scheme (3.3) and (3.4) we apply the capacitance matrix method. First we describe this technique for an arbitrary difference scheme defined on the grid $\bar{D}_h$,

$$(3.5) \qquad Av(x) = b(x), \qquad x \in \bar{D}_h$$

where A is a difference linear operator. We assume that the problem (3.5) has a unique solution. Let S_h denote some subset of $\bar{D}_h$, $S_h \subset \bar{D}_h$, and let B be a difference linear operator which may differ from the operator A only at the grid points of S_h, i.e.

$$(Av)(x) = (Bv)(x), \qquad x \in (\bar{D}_h \setminus S_h)$$

For $y \in S_h$ we introduce the functions e_y defined on $\bar{D}_h$ as

$$e_y(x) = \begin{cases} 1 & \text{for } x = y, \\ 0 & \text{for } x \neq y. \end{cases}$$

The capacitance matrix algorithm for the solution of (3.5) is defined by the following steps.

1. For each $y \in S_h$ compute the vector v_y,

$$Bv_y(x) = e_y(x), \qquad x \in \bar{D}_h.$$

2. Compute the capacitance matrix C,

$$C = \{(Av_y)(x)\}, \qquad y,\ x \in S_h.$$

3. Compute the vector $\tilde{v}$,

$$B\tilde{v}(x) = \tilde{b}(x), \qquad x \in \bar{D}_h,$$

where

$$\tilde{b} \begin{cases} = b & \text{for } x \in (\bar{D}_h \setminus S_h), \\ \text{arbitrary} & \text{for } x \in S_h. \end{cases}$$

4. Compute the vector d,

$$d(x) = b(x) - (A\tilde{v})(x), \qquad x \in S_h.$$

5. Compute the vector z,

$$Cz = d.$$

6. Compute the vector v,

$$Bv(x) = \tilde{b}(x) + \sum_{y \in S_h} z(y) e_y(x), \qquad x \in \bar{D}_h.$$

It can be proven that the matrix C is nonsingular if the difference schemes with the operators A and B have unique solutions.

We are now in a position to use this algorithm for the solution of (3.3) and (3.4). Let D denote a rectangle $(a_1, b_1) \times (a_2, b_2)$ containing $\bar{\Omega}$ with a_i and b_i independent of h_1 and h_2. We assume that $\bar{D}$ is the union of the elements e_{ij} and the distance between ∂D and Ω_2 is greater than $h = \max\{h_1, h_2\}$. The set of the grid points (nodes) belonging to $\bar{D}$ is denoted by $\bar{D}_h = D_h \cup \partial D_h$ where ∂D_h is the set of nodes lying on ∂D. Let $\tilde{\Gamma}_h$ be the set of grid points $x \in (\bar{D}_h \setminus \bar{\Omega}_{2,h})$ such that $T_i^{\pm} x \in \bar{\Omega}_{2,h}$ or $T_j^{+} T_i^{-} x \in \bar{\Omega}_{2,h}$ for some $i \neq j$ where $T_i^{\pm} x = x \pm h_i e_i$. We augment the difference scheme (3.3) and (3.4) by the equations

$$a_{pq,t}^{n} = 0, \qquad (p, q) \in (\tilde{\Gamma}_h \cup \partial D_h) \tag{3.6}$$

$$(1+\theta\tau)\left(\tilde{\Lambda}_1 + \frac{\theta\tau}{1+\theta\tau}\Lambda_1\right)\left(\tilde{\Lambda}_2 + \frac{\theta\tau}{1+\theta\tau}\Lambda_2\right) a_{pq,t}^{n} = 0, \qquad (p, q) \in \big(D_h \setminus (\bar{\Omega}_{2,h} \cup \tilde{\Gamma}_h)\big). \tag{3.7}$$

Obviously, the solution of this system is zero.

The system (3.3), (3.4), (3.6) and (3.7) for a fixed n is taken as the initial difference scheme with the operator A (see (3.5)).

We now formulate the auxiliary difference scheme with the operator B. For a fixed n it has the form

$$(1+\theta\tau)\left(\tilde{\Lambda}_1 + \frac{\theta\tau}{1+\theta\tau}\Lambda_1\right)\left(\tilde{\Lambda}_2 + \frac{\theta\tau}{1+\theta\tau}\Lambda_2\right) \tilde{a}_{pq,t}^{n} = \tilde{F}_{pq}^{n}, \qquad (p, q) \in D_h, \tag{3.8}$$

$$\tilde{a}_{pq,t}^{n} = 0, \qquad (p, q) \in \partial D_h. \tag{3.9}$$

Note that for this case, the operators A and B differ at the points $x \in (\Gamma_h \cup \tilde{\Gamma}_h)$. Thus we now have $S_h = \Gamma_h \cup \tilde{\Gamma}_h$. Let $\bar{D}_h$ be given as

$$\bar{D}_h = \{(a_1 + ph_1, a_2 + qh_2) : p = 0, \ldots, N_1+1,\ q = 0, \ldots, N_2+1,\ h_s(N_s+1) = b_s - a_s,\ s = 1, 2\}.$$

The algorithm for solving (3.8) and (3.9) has three phases. For $q = 1, \ldots, N_2$

$$\left(\tilde{\Lambda}_1 + \frac{\theta\tau}{1+\theta\tau}\Lambda_2\right) v_{pq}^n = \tilde{F}_{pq}^n/(1+\theta\tau), \quad p = 1, \ldots, N_1,$$

(3.10)

$$v_{pq}^n \equiv \left(\tilde{\Lambda}_2 + \frac{\theta\tau}{1+\theta\tau}\Lambda_2\right) \tilde{a}_{pq,t}^n = 0, \quad p = 0;\ N_1+1;$$

for $p = 1, \ldots, N_1$

$$\left(\tilde{\Lambda}_2 + \frac{\theta\tau}{1+\theta\tau}\Lambda_2\right) \tilde{a}_{pq,t}^n = v_{pq}^n, \quad q = 1, \ldots, N_2, \tag{3.11}$$

$$\tilde{a}_{pq,t}^n = 0, \quad q = 0;\ N_2+1;$$

$$\tilde{a}_{pq}^{n+1} = \tilde{a}_{pq}^n + \tau\tilde{a}_{pq,t}^n, \quad p = 1, \ldots, N_1;\ q = 1, \ldots, N_2. \tag{3.12}$$

The matrices of the corresponding systems are tridiagonal.

The difference scheme (3.3) and (3.7) (without (3.5)) is solved by the capacitance matrix technique as follows. We label points of S_h. For $n = 0$ and successively chosen functions e_y, $y \in S_h$, we solve the following problems (step 1)

$$(1+\theta\tau)\left(\tilde{\Lambda}_1 + \frac{\theta\tau}{1+\theta\tau}\Lambda_1\right)\left(\tilde{\Lambda}_2 + \frac{\theta\tau}{1+\theta\tau}\Lambda_2\right) v_y(ph_1, qh_2) = e_y(ph_1, qh_2), \tag{3.13}$$

$$(p, q) \in D_h,$$

$$v_y = 0, \quad (p, q) \in \partial D_h \tag{3.14}$$

by the algorithm (3.11) and (3.12).

Having found the function v_y, we successively compute rows of the matrix C (step 2). The matrix C is independent of n. Note that the solution v_{pq}^n of (3.10) with $\tilde{F}_{pq}^0 = e_y$ is equal to zero with the exception of the horizontal line passing through the point y. Note also that we compute $v_y(x)$ only at these points in which the values v_y needed to compute the coefficients of C.

In the third step we solve (3.8) and (3.9) with the right-hand side $\tilde{F}_{pq}^0$ which is equal to

$$\tilde{F}_{pq}^0 = \begin{cases} F_{pq}^0 & \text{for } (p, q) \in \bar{\Omega}_{1,h}, \\ 0 & \text{for } (p, q) \in (D_h \setminus \bar{\Omega}_{1,h}). \end{cases}$$

Here we assume that F_{pq}^0 is already given. The evaluation of F_{pq}^0 usually requires use of numerical integration. Having found $\tilde{v}(x)$, we compute the vector d (step 4).

Next we solve the linear system with the matrix C (step 5), for instance, by the Gauss elimination. It is convenient to make a factorization of the matrix C into the product LU where L and U are lower and upper triangular matrices. This factorization of C is stored and will be used for $n = 1, \ldots, N-1$. In the last step 6 we solve the scheme (3.8) and (3.9) with $\tilde{F}^0_{pq}$ of the form

$$\tilde{F}^0_{pq} = \begin{cases} F^0_{pq} & \text{for } (p, q) \in \Omega_{1,h}, \\ z & \text{for } (p, q) \in S_h, \\ 0 & \text{for } (p, q) \in \big(D_h \setminus (\Omega_{1,h} \cup S_h)\big). \end{cases}$$

That is now we compute $a^0_{pq,t}$. Applying the formula (3.12), we obtain a^1_{pq}. To find $a^n_{pq,t}$ for $n = 1, \ldots, N-1$ we proceed as above apart from the steps 1 and 2, and we use the known factorization C in the step 5.

We now estimate the cost of the described algorithm. Note that the number of points of $S_h = \Gamma_h \cup \tilde{\Gamma}_h$ at most is of order h^{-1}, so the cost of the step 1 is of order h^{-3} and the cost of the step 2 is of order h^{-2} of arithmetic operations. These steps and the factorization of C are done only once during the whole process. For n fixed the cost of the other steps is of order h^{-2}. Thus the total cost of the algorithm is

$$O(\tau^{-1} h^{-2} + h^{-3})$$

of arithmetic operations.

If we take τ proportional to h, as we usually do in practice, then the cost is of order h^{-3}. Note that the total number of unknowns is also of order h^{-3}. This shows that our algorithm is (asymptotically) optimal.

4. Stability and convergence

In this section we shall prove stability and estimate the error of the method (2.1) and (2.2). We introduce the following norms

$$\|v\|_{L^\infty_\tau(H)} = \max_n \|v^n\|_H, \qquad \|v\|_{L^2_\tau(H)} = \Big(\tau \sum_{n=0}^{N} \|v^n\|^2_H\Big)^{1/2}.$$

THEOREM 1. *If condition* (1.6) *holds,*

$$\frac{\partial a_{ij}}{\partial t}, \ \frac{\partial b_i}{\partial t}, \ \frac{\partial c}{\partial t} \in C(Q_T), \qquad \frac{\partial \sigma}{\partial t} \in C\big(\Gamma \times (0, T)\big) \quad \text{and} \quad \theta \geqslant 0.5\, \gamma_1,$$

then the solution U^n *of* (2.1) *and* (2.2) *satisfies the inequality*

$$\|U_t\|_{L^2_\tau(L^2(\Omega))} + \|U\|_{L^\infty_\tau(H^1(\Omega))} \leqslant M\{\|f\|_{L^2_\tau(L^2(\Omega))} + \|U^0\|_{H^1(\Omega)}\} \tag{4.1}$$

where M *is a positive constant independent of* τ *and* h_i.

Proof. Putting $v = \tau U_t^n$ in (2.1), summing up this equation from $n = 0$ to $k-1$ and using the identity $U^n = 0.5(U^{n+1}+U^n-\tau U_t^n)$, we get

$$(4.2)\qquad \tau\sum_{n=0}^{k-1}\Big\{\|U_t^n\|^2_{L^2(\Omega)}+a^n\big((U^{n+1}+U^n)/2,\ U_t^n\big)+\tau\theta\|U_t^n\|^2_{H^1(\Omega_2)}+ \\ +\frac{\theta^2\tau^2}{1+\theta\tau}\|D_1D_2U_t^n\|^2_{L^2(\Omega_2)}-\tau 0.5\, a^n(U_t^n,\ U_t^n)\Big\} \\ =\tau\sum_{n=0}^{k-1}(f^n,\ U_t^n)_{L^2(\Omega)}.$$

Using (1.6) and the ε-inequality $(ab\leqslant 0.5\varepsilon^{-1}a^2+0.5\varepsilon b^2,\ \varepsilon>0)$ it can be proved that

$$(4.3)\qquad \tau\sum_{n=0}^{k-1}a^n\big((U^{n+1}+U^n)/2,\ U_t^n\big) \\ \geqslant(\tfrac{1}{2}\gamma_0-\varepsilon_0)\|U^k\|^2_{H^1(\Omega)}-\tfrac{1}{2}\gamma_1\|U^0\|^2_{H^1(\Omega)}-M\sum_{n=0}^{k-1}\tau\|U^n\|^2_{H^1(\Omega)}.$$

We estimate the right-hand side of (4.2) as follows:

$$(4.4)\qquad (f^n,\ U_t^n)_{L^2(\Omega)}\leqslant\varepsilon_1\|U_t^n\|^2_{L^2(\Omega)}+(4\varepsilon_1)^{-1}\|f^n\|^2_{L^2(\Omega)}.$$

Using the estimates (4.3) and (4.4) in (4.2), we get

$$(4.5)\qquad (\tfrac{1}{2}\gamma_0-\varepsilon_0)\|U^k\|^2_{H^1(\Omega)}+\tau\sum_{n=0}^{k-1}\Big\{(1-\varepsilon_1)\|U_t^n\|^2_{L^2(\Omega)}+ \\ +\frac{\theta^2\tau^2}{1+\theta\tau}\|D_1D_2U_t^n\|^2_{L^2(\Omega_2)}+\tau\big(\theta\|U_t^n\|^2_{H^1(\Omega_2)}-\tfrac{1}{2}\gamma_1\|U_t^n\|^2_{H^1(\Omega)}\big)\Big\} \\ \leqslant M\tau\sum_{n=0}^{k-1}\{\|f^n\|^2_{L^2(\Omega)}+\|U^n\|^2_{H^1(\Omega)}\}+\tfrac{1}{2}\gamma_1\|U^0\|^2_{H^1(\Omega)}.$$

To obtain the required estimate (4.1) for $\theta\geqslant 0.5\gamma_1$ we choose respectively ε_i, $i=0,1$, and we use the Gronwall inequality.

We shall now prove an error estimate for the problem (2.1) and (2.2).

THEOREM 2. *Let the assumption of Theorem* 1 *hold and let u be a solution of* (1.3)–(1.5) *such that*

$$\frac{d^2u}{dt^2}\in L^2\big(0,T;H^1(\Omega)\big),\qquad \frac{du}{dt}\in L^2\big(0,T;H^2(\Omega)\big),\qquad f\in C\big(0,T;L^2(\Omega)\big).$$

Then

$$\|U-u\|_{L^\infty_\tau(H^1(\Omega))} \leqslant M(\tau+h) \tag{4.6}$$

where M is a positive constant independent of τ, $h = \max\{h_1, h_2\}$, and U^n is the solution of (2.1) *and* (2.2).

Proof. We rewrite equation (1.4) for $t = n\tau$ as

$$(u_t^n, v)_{L^2(\Omega)} + a^n(u^n, v) = (f^n, v)_{L^2(\Omega)} + (\delta^n, v)_{L^2(\Omega)} \tag{4.7}$$

where

$$\delta^n = u_t^n - (du/dt)^n \quad \text{and} \quad \|\delta^n\|_{L^2(L^2)} = O(\tau).$$

Let $Z^n = U^n - W^n$ in Ω_2 where W^n is for a time being an arbitrary function from $V_h(\Omega_2)$. First we prove the estimate (4.6) for Z^n. The estimate for $U^n - u^n$ will follow from the triangle inequality.

Subtracting from the both sides of (2.1) the terms of the left-hand side of (2.1), in which the function U^n is replaced by W^n, using equation (4.7) and summing up the corresponding expressions for $0 \leqslant n \leqslant k-1$, we get

$$\begin{aligned} \tau \sum_{n=0}^{k-1} \Big\{ &(Z_t^n, v)_{L^2(\Omega)} + \theta\tau (Z_t^n, v)_{H^1(\Omega_2)} + \\ &+ \frac{\theta^2\tau^2}{1+\theta\tau} (D_1 D_2 Z_t^n, D_1 D_2 v)_{L^2(\Omega_2)} + a^n(Z^n, v)\Big\} \\ = \tau \sum_{n=0}^{k-1} \Big\{ &-(\delta^n, v)_{L^2(\Omega)} - \theta\tau (W_t^n, v)_{H^1(\Omega_2)} - \\ &- \frac{\theta^2\tau^2}{1+\theta\tau} (D_1 D_2 W_t^n, D_1 D_2 v)_{L^2(\Omega_2)} + a^n(u^n - W^n, v) + (u_t^n - W_t^n, v)_{L^2(\Omega)}\Big\}. \end{aligned} \tag{4.8}$$

We estimate the second term of the right-hand side of (4.8). Putting $v = \tau Z_t^n$ and applying the formula of summation by parts in the variable t, we obtain

$$\begin{aligned} &\theta\tau^2 \sum_{n=0}^{k-1} (W_t^n, Z_t^n)_{H^1(\Omega_2)} \\ &= -\theta\tau^2 \sum_{n=1}^{k-1} (W_{tt}^{n-1}, Z^n)_{H^1(\Omega_2)} + \theta\tau (W_t^{k-1}, Z^k)_{H^1(\Omega_2)} - \theta\tau (W_t^0, Z^0)_{H^1(\Omega_2)}. \end{aligned}$$

Using the ε-inequality, we have

(4.9) $$\theta\tau^2 \sum_{n=0}^{k-1} (W_t^n, Z_t^n)_{H^1(\Omega_2)}$$
$$\leqslant \theta\tau \sum_{n=1}^{k-1} \left\{\varepsilon_3 \|Z^n\|^2_{H^1(\Omega_2)} + \frac{\tau^2}{4\varepsilon_3} \|W_{tt}^{n-1}\|^2_{H^1(\Omega_2)}\right\} + \varepsilon_4 (\|Z^k\|^2_{H^1(\Omega_2)} + \|Z^0\|^2_{H^1(\Omega_2)}) + \\ + \frac{\theta^2\tau^2}{4\varepsilon_4} (\|W_t^{k-1}\|^2_{H^1(\Omega_2)} + \|W_t^0\|^2_{H^1(\Omega_2)}).$$

In (4.8) we put $v = \tau Z_t^n$. The terms of the left-hand side of this equation we transform and estimate as in the proof of Theorem 1. The terms of the right-hand side (except the second one) are estimated by applying the ε-inequality. This yields

(4.10) $$(0.5\gamma_0 - \varepsilon_0 - \varepsilon_4) \|Z^k\|^2_{H^1(\Omega)} + \tau \sum_{n=0}^{k-1} (1 - \varepsilon_1 - \varepsilon_2) \|Z_t^n\|^2_{L^2(\Omega)} + \\ + \frac{\theta^2\tau^2}{1+\theta\tau} (1 - \varepsilon_5) \|D_1 D_2 Z_t^n\|^2_{L^2(\Omega_2)} + \tau(\theta - 0.5\gamma_1) \|Z_t^n\|^2_{H^1(\Omega_2)}$$
$$\leqslant M \left\{ \sum_{n=0}^{k-1} \tau \left\{ \|Z^n\|^2_{H^1(\ )} + \frac{\tau^2}{4\varepsilon_3} \|W_{tt}^n\|^2_{H^1(\Omega_2)} + \tau^2 \|D_1 D_2 W_t^n\|^2_{L^2(\Omega_2)} + \right.\right. \\ \left. + \|u^n - W^n\|^2_{H^1(\Omega)} + \|u_t^n - W_t^n\|^2_{L^2(\Omega)} + \tau^2 \right\} + \\ \left. + \|Z^0\|^2_{H^1(\Omega)} + \tau^2 \|W_t^{k-1}\|^2_{H^1(\Omega_2)} + \tau^2 \|W_t^0\|^2_{H^1(\Omega_2)} \right\}.$$

Choosing respectively ε_i, $i = 0, \ldots, 5$, and using the discrete Gronwall inequality we get the estimate of the type (4.10) without the term $\tau \sum_{n=0}^{k-1} \|Z^n\|^2_{H^1(\Omega)}$ and with in general different M. To obtain (4.6) we proceed as follows. We put $W^n = \tilde{u}^n$, where $\tilde{u}^n \in V_h(\Omega_2)$ interpolates the extension of u to $\bar{\Omega}_2$. For example, from [7] we know that

$$\|u^n - \tilde{u}^n\|_{H^1(\Omega_2)} \leqslant Mh \|u^n\|_{H^2(\Omega)}.$$

Applying the triangle inequality, we finally get (4.6).

5. Remarks

1. For the methods (2.1) and (2.2) we can also find an error estimate in the norms of the spaces $L_\tau^\infty(L^2)$ and $L_\tau^2(L^2)$. In the space $L_\tau^2(L^2)$ the error is $O(\tau + h^2)$ whenever $f \in L^2(0, T; L^2(\Omega))$ and $u_0 \in H^1$ (see [5], [8]).

2. The same results are also true for the problem (2.1) and (2.2) without the term $\theta^2\tau^2(1+\theta\tau)^{-1}(D_1D_2U_t^n, D_1D_2v)_{L^2(\Omega_2)}$.

In this case a system of linear equations can be solved by FFT together with the capacitance matrix technique. The cost of this algorithm is of order $O(\tau^{-1}h^{-2}\ln h^{-1})$ whenever τ is proportional to h.

3. The presented ADG method can also be used to the numerical solution of some nonlinear parabolic and hiperbolic problems; see [3], [4] and [9].

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