ELEMENTARY AND ANALYTIC THEORY OF NUMBERS
BANACH CENTER PUBLICATIONS, VOLUME 17
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1985

QUASICRYSTALLOGRAPHY: SOME INTERESTING NEW PATTERNS

P. A. B. PLEASANTS

Department of Pure Mathematics, University College
Cardiff, Great Britain

1. Crystallographic patterns

A *crystallographic* pattern in n-dimensional space is one that is invariant under a set of translations that form a lattice of dimension n. Examples are the three 2-dimensional tilings in Figure 1. In each of them the plane is covered with non-overlapping congruent tiles of a single shape and a basis $\{x, y\}$ of the lattice of translations is indicated.

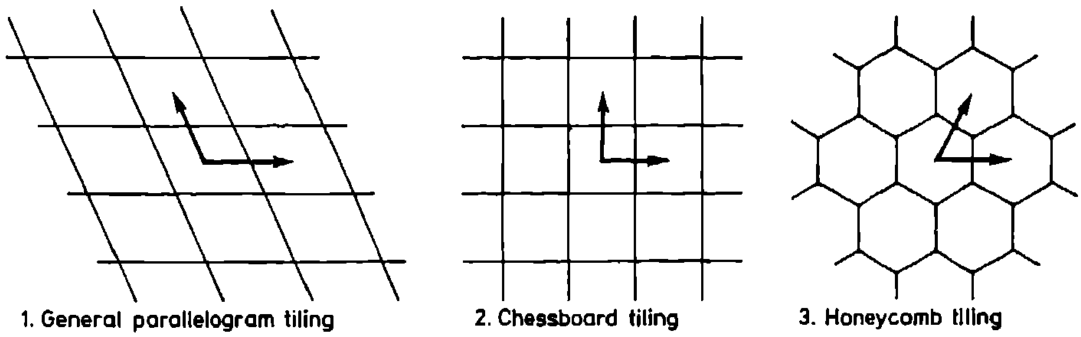

Fig. 1

In addition to its translation symmetries, each of these tilings has central symmetries consisting of the groups 2, $4m$ and $6m$, respectively. (In crystallographers' notation rm is the group generated by a rotation through an angle $2\pi/r$ and a mirror reflexion. In group theorists' notation these groups would be C_2, D_4 and D_6 – or perhaps D_8 and D_{12}.) The *crystallographic restriction* says that the latter two groups are maximal, in the sense that the group of central symmetries of every plane crystallographic pattern is a subgroup of either $4m$ or $6m$. There are, in fact, 17 different types of plane crystallographic patterns (or "wallpaper patterns", as they are called). There are 10 distinct subgroups of $4m$ and $6m$ altogether, but more than 10 types of pattern both because some symmetry groups can interact with the translation group in two essentially different ways and because reflexion, instead of acting as a mirror reflexion with its axis through the centre of rotation, can sometimes appear only in combination with a translation, in the form of a "glide reflexion". A more general form of the crystallographic restriction, applicable to space of any dimension, is the following. (We give the proof, derived from [8], as it is simple and elegant.)

CRYSTALLOGRAPHIC RESTRICTION. *If a crystallographic pattern in n-dimensional space is invariant under a symmetry of order m then*

$$\sum_{\substack{p^\alpha \| m \\ p^\alpha \neq 2}} \varphi(p^\alpha) \leqslant n,$$

where the sum is over maximal prime power divisors of m and φ is Euler's totient function.

For each dimension n this gives only finitely many possibilities for m.

Proof. Call the symmetry σ and take a fixed point of σ to be the origin, so that σ is linear. (Every symmetry of finite order has at least one fixed point.) The image of the origin under the lattice of translations is a lattice, L, of points in $\boldsymbol{R}^n$, and we take a basis of L to be our basis of $\boldsymbol{R}^n$. Then σ is a symmetry of L and so has an integer matrix with respect to this basis. The characteristic polynomial of σ therefore has the form $\prod f_i(x)$, where $f_i(x)$ is a power of the m_ith cyclotomic polynomial for some m_i dividing m. Let V_i be the kernel of $f_i(\sigma)$. By standard linear algebra, each V_i is σ-invariant, $\boldsymbol{R}^n$ is the direct sum of the V_i's, the characteristic polynomial of σ on V_i is $f_i(x)$, and the order of σ on V_i is m_i. Clearly the order m of σ is the least common multiple of the m_i's, so if $p^\alpha | m$ then $p^\alpha | m_i$ for some i. We now have

$$\begin{aligned} n = \sum_i \dim V_i \geqslant \sum_i \varphi(m_i) &= \sum_i \prod_{p^\alpha \| m_i} \varphi(p^\alpha) \\ &\geqslant \sum_i \sum_{\substack{p^\alpha \| m_i \\ p^\alpha \neq 2}} \varphi(p^\alpha), \text{ since } \varphi(p^\alpha) \geqslant 2 \text{ if } p^\alpha \neq 2, \\ &\geqslant \sum_{\substack{p^\alpha \| m \\ p^\alpha \neq 2}} \varphi(p^\alpha). \end{aligned}$$

R. L. E. Schwarzenberger drew my attention to the fact that the statement of the crystallographic restriction given in [8] (namely, that $\varphi(m) \leqslant n$) is incorrect without some extra hypothesis. For $n \leqslant 5$ it coincides with the correct version, but taking the direct sum of a 2-dimensional pattern with a symmetry of order 6 and a 4-dimensional pattern with a symmetry of order 5 gives a crystallographic pattern in $\mathbb{R}^6$ with a symmetry of order 30.

A good account of crystallography in space of any dimension can be found in [8]. Everything mentioned in this section is treated there in detail.

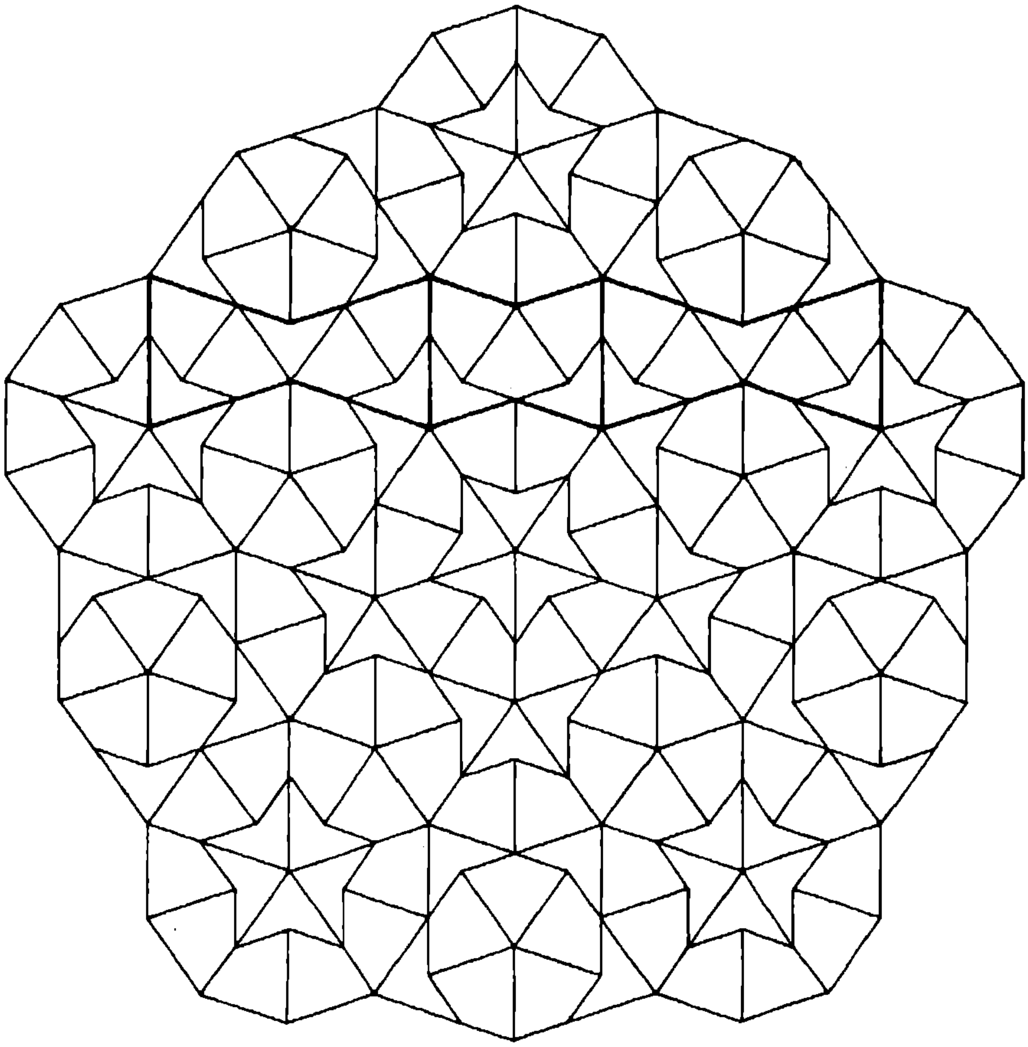

Fig. 2

2. Penrose's tiling

Figure 2 shows part of a non-crystallographic tiling of the plane constructed by R. Penrose (initially in a less simple form) in 1973. (The thicker lines have no significance at the moment: they are to aid explanation later.) It is composed of two shapes – the "kite" and the "dart" – which are derived from the regular pentagon, in that their angles are multiples of $\pi/5$ and the ratio of the long sides to the short is $\tau = (\sqrt{5}+1)/2$. The complete tiling has a good deal of the appearance of being crystallographic, even though it is not, because every configuration that occurs in it occurs everywhere throughout the plane with uniform frequency. In contrast with crystallographic tilings, the frequency is not independent of the configuration (larger configurations occur with lower frequency); but nevertheless the phenomenon gives the tiling the same kind of aesthetic appeal that crystallographic patterns have, which is, if anything, enhanced by not actually being crystallographic. The tiling also has pentagonal symmetry about the centre (and consequently, because of the phenomenon just described, local pentagonal symmetry everywhere). Such a symmetry is impossible for crystallographic patterns in the plane, because of the crystallographic restriction. Detailed accounts of Penrose's tiling and its properties can be found in [3], [6] and [7].

Clearly tilings of this sort are a Good Thing of which we cannot have too much. The purpose of this paper is to describe how to construct many other similar tilings. We start by making a list of what we see as the main properties of Penrose's tiling, that give it its interest and aesthetic appeal; then our aim is to construct other tilings with the same properties (but with different types of symmetry in Property 3). The properties (which follow) are not all independent of each other.

PROPERTY 1. *A finite number of shapes is used in the tiling.*

PROPERTY 2. *The tiling is quasicrystallographic.*

By this we mean that it satisfies the following two conditions:

QC1. *Every finite configuration of tiles that occurs in the tiling occurs everywhere throughout the tiling with constant frequency (the frequency depending on the configuration).*

QC2. *Only finitely many configurations of tiles of any given bounded size occur in the tiling.*

PROPERTY 3. *The tiling has pentagonal symmetry.*

PROPERTY 4. *The tiling is not periodic in any direction. In particular, it is not crystallographic.*

(In view of the crystallographic restriction, the fact that the tiling is not crystallographic is a consequence of its pentagonal symmetry.)

PROPERTY 5. *The tiling can be generated by an inflation operation.*

We explain what this means in the next section. It is the way Penrose constructed the tiling.

PROPERTY 6. *There are uncountably many essentially different ways of tiling the plane with kites and darts. These tilings are closely related, in that the same finite configurations of tiles occur in each of them.*

The necessity of making QC2 (which is clearly aesthetically desirable) a separate, explicit condition was pointed out by W. F. Lunnon, who noticed that certain tilings with heptagonal symmetry, constructed, independently, by J. F. Rigby and himself, failed to satisfy this condition in spite of having Properties 1, 4, 5 and 6 and satisfying QC1. In each case the tiles were polygons that did not always meet vertex-to-vertex and there were infinitely many positions relative to each other that a pair of tiles meeting along part of a side could take. So, somewhat unexpectedly, QC1 does not imply QC2. When the tiles are polygons and are always fitted together vertex-to-vertex (as is the case with the tilings we shall construct) then QC2 is necessarily satisfied.

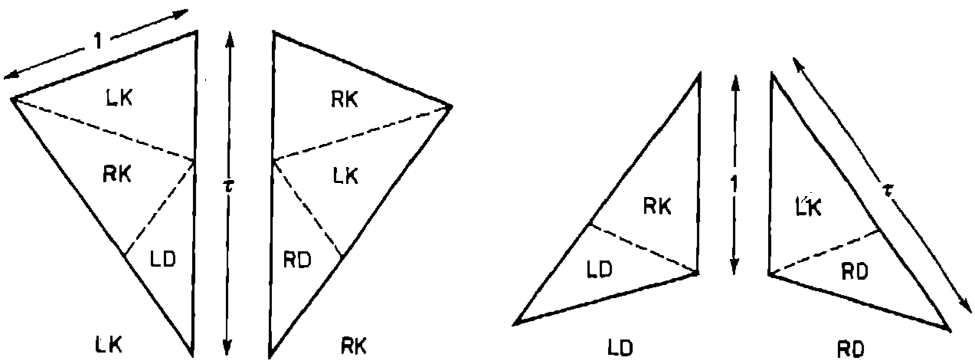

Fig. 3

3. Inflation

The operation of inflation for kites and darts is perhaps easier to explain if we cut both shapes into two congruent triangles along their axes of symmetry. We then have four shapes: a left half-kite (LK), a right half-kite (RK), a left half-dart (LD) and a right half-dart (RD). These shapes are, in fact, congruent in pairs, but for the present purpose it is important to regard them as distinct entities. Figure 3 shows how each half-kite can be subdivided into two half-kites and a half-dart and each half-dart can be subdivided into a half-kite and a half-dart, the new half-kites and half-darts being reduced in scale by a linear factor $1/\tau$ compared with the original ones. A tiling of the plane can now be generated as follows. First choose one of the four shapes and position it anywhere on the plane. Then choose a shape that has the previous one as a part in its subdivision, enlarge it by a factor τ and position it so that the part coincides with the previously placed shape. This has the effect of placing one or two more tiles with the one that has already been placed. Now choose a shape that has the shape last chosen as a part in its subdivision, enlarge it by a factor τ^2, carry out two stages of the subdivision process and position it appropriately. This adds more tiles to those that have already been placed. The process can be continued indefinitely: we choose an infinite sequence of shapes and the result is usually a tiling of the whole plane (though for certain special sequences of shapes only an infinite region of the plane is tiled). The left and right halves of kites and darts turn out always to come together correctly, so that the resulting tiling consists of whole kites and darts. At each stage of the construction there are either two or three possible choices for the next shape in the sequence, and it is this infinite sequence of choices that gives rise to the continuum of different tilings. Even when the shape and position of the starting tile are given, the construction still leads to uncountably many tilings (which can be shown to be different), and each of them is a translate of only countably many of the others. So there is a continuum of different Penrose tilings even after allowing for rigid motions. The sequence of shapes starting LD, LK, RD, RK, ... that is periodic with period 4 results in a tiling of only a sector of the plane with angle $\pi/5$, having the initial half-dart at the apex. The same is true of the mirror image sequence starting RD, RK, LD, LK, ... The plane can be covered by five sectors of each kind placed alternately round a point, and the resulting tiling is that shown in Figure 2.

There is a connexion between inflation and quasicrystallographicity, in that tilings generated by an inflation operation usually meet the first criterion of quasicrystallographicity. To make this statement more precise we need some definitions. A set of shapes has an *inflation* with *multiplier* λ if when each shape is scaled up by a linear factor λ (>1) it can be subdivided into shapes of the set (using the same shape more than once, if necessary). An inflation can be used to generate tilings by the construction just described. An inflation is *irreducible* if there is no proper subset of the shapes such that shapes from the subset subdivide into shapes from the subset only.

PROPOSITION. *A tiling generated by an irreducible inflation satisfies* QC1.

The proof uses Frobenius's theorem for non-negative matrices. The counting matrix of an inflation (the matrix whose (i, j) th element is the number of copies of the jth shape in the subdivision of the ith shape) is irreducible if the inflation is irreducible. A matrix all of whose powers are irreducible is called *primitive*. Every irreducible matrix has some power that is primitive. Frobenius's theorem says that a primitive non-negative matrix M has a unique eigenvalue λ of maximum modulus which is simple, real and positive and has a positive eigenvector. Also, for any non-negative real vector x, the sequence $\{\lambda^{-n} M^n x\}$ converges to an eigenvector of M with eigenvalue λ. Details of the proof of this proposition (and of all other results described here) can be found in [4].

Figure 4 shows two inflations each based on a single shape: an "L-shape" and a "sphinx". (In the case of the sphinx, we have to allow mirror

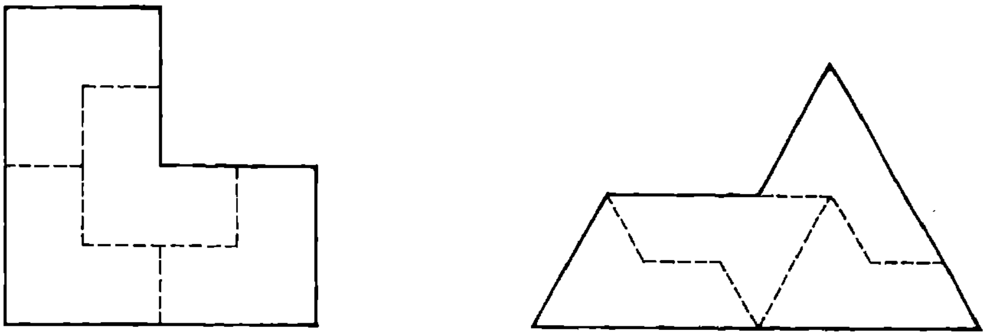

Fig. 4

images of the original shape.) These were known before Penrose's tiling (and the sphinx is mentioned in [3]). They both give rise to tilings that are non-periodic but are quasicrystallographic (even though the tiles do not always fit together vertex-to-vertex). Their symmetry is less interesting than that of the Penrose tiling, however, in that neither has a symmetry that is not possible for crystallographic patterns. (This is inevitable, since the shapes and their inflations are derived from tilings of the plane by squares and equilateral triangles, respectively.) There is a quasicrystallographic tiling of the plane by L-shapes with square symmetry ($4m$) and there are quasicrystallographic tilings by sphinxes with mirror symmetry (m) and with the symmetry of rotation through π (2) (but no tiling with both these symmetries). These (and certain proper subgroups of $4m$, in the case of L-shapes) are the only symmetries that occur in inflation-generated tilings by these shapes.

4. Species of tilings

Any two Penrose tilings by kites and darts have the property that any finite configuration of tiles that occurs in either of them also occurs in the other. We shall describe tilings related in this way as being of the same *species*. The kites and darts tilings then form a complete species.

For crystallographic tilings the concept of a species is not very interesting, as it is not hard to see that the species of a crystallographic tiling simply consists of all tilings obtained from it by rigid motions. This does give the species a topological structure, however. For a 1-dimensional crystallographic tiling the only rigid motions are translations, and since the tiling is periodic the set of translates is topologically a circle. So the species is topologically a circle in this case. The translates of a 2-dimensional crystallographic tiling are in one-one correspondence with the points in a period parallelogram of its lattice of translation symmetries. So the set of translates (that is, the species modulo rotations) is topologically a torus. More generally, the species of an n-dimensional crystallographic tiling modulo rotations is topologically an n-dimensional torus. Since this torus is compact and the set of rotations of n-dimensional space about a fixed point is also compact, the species itself is compact.

5. 1-dimensional quasicrystallographic tilings

Our aim is to take the Penrose tiling apart to find out what makes it tick. In this we have been successful, in that we have found a two-stage construction method for a tiling that (although it is not one of the tilings constructed by Penrose) has all the Properties 1 to 6 and is a close relative of the kites and darts tiling. We have also been lucky, in that the same construction method can be used to construct similar tilings with any kind of symmetry. (It was like taking a Soviet watch apart and finding you have enough pieces to make two watches when you put it together again.)

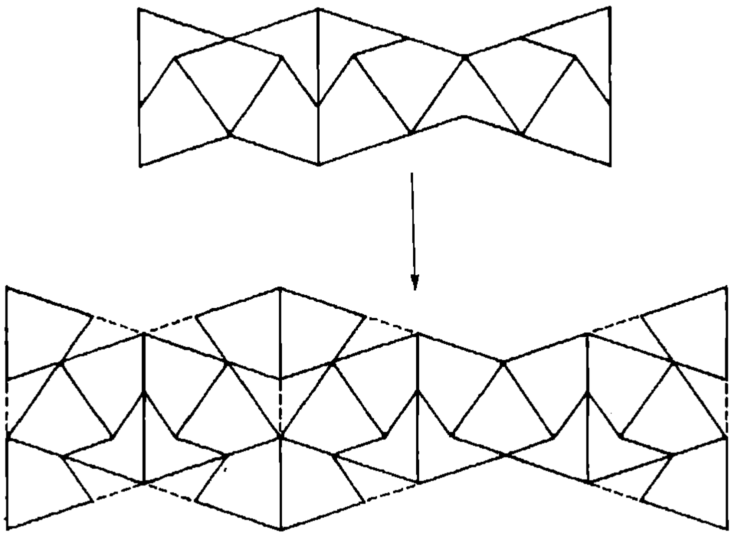

Fig. 5

We begin by simplifying the problem by looking at 1-dimensional quasicrystallographic tilings. The two stages of our construction method will be first to construct suitable 1-dimensional tilings, then to derive higher-dimensional tilings as "products" (of a certain sort) of 1-dimensional tilings. The Penrose tilings themselves contain 1-dimensional tilings, which seem to play an important part in making them work. The thick lines in Figure 2 outline a short "worm" consisting of a "short bow tie" sandwiched between two "long bow ties". A long bow tie is τ times the length of a short one, and Figure 5 shows that the inflation of a short bow tie contains two halves of long bow ties and the inflation of a long bow tie contains a short bow tie and two halves of long bow ties, in such a way that the inflation of a worm contains a worm τ times the length. Consequently, every kites and darts tiling contains arbitrarily long worms, and some tilings contain worms that are infinite in both directions. Such an infinite worm gives a tiling of its centre line by intervals L and S (corresponding to the long and short bow ties) of two lengths in the ratio τ to 1. This tiling is generated by the inflation $S \to L$, $L \to LS$. (We have introduced a shift by an amount $\frac{1}{2}L$, so that the intervals are subdivided into complete intervals in accordance with our definition of an inflation.) It has no two consecutive S's and no three consecutive L's. Part of such a tiling is as follows (where the brackets indicate inflation from another tiling of the species):

$$\ldots(LS)\,L(LS)\,L(LS)(LS)\,L(LS)(LS)\,L(LS)\,L(LS)(L\ldots$$

Two-symbol sequences of this sort have been studied extensively. In particular, we refer to [9] and [1], which (especially [9]) provide many references to earlier literature. This tiling has many other interesting properties, of which we mention only the "2-distance property". Let $\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$ be the boundary points of the tiling in their natural order measured from a fixed origin (so that $\{x_n\}$ is an increasing sequence). The distance $x_{n+1}-x_n$ between consecutive points takes only the two values L and S as n varies. The 2-distance property is that, more generally, for every r the distances $x_{n+r}-x_n$ between points r apart take only two values, L_r and S_r as n varies, where L_r and S_r depend only on r.

If we change the scale so that L has length τ and S has length 1, and take one of the points x_n as the origin, then each x_n is $a+b\tau$, for some integers a and b, so is an integer of the field $Q(\tau)$. Since the tiling is very interesting, the x_n's must be a very interesting subset of the integers of $Q(\tau)$. What subset is it? The integers of $Q(\tau)$ can be represented by the integer points in the plane, (a, b) representing $a+b\tau$. When this is done the x_n's correspond to the integer points in the infinite strip (outlined with continuous lines in Figure 6) that has slope τ and meets the x-axis in the interval $[-\frac{1}{2}, \frac{1}{2}]$. (The significance of the broken lines in Figure 6 will be explained in the next section.) The relation between the strip and the tiling is very close: if the strip is cut at right angles through the integer points in it, as indicated in Figure 6, then the resulting rectangles are of two lengths (in the ratio τ to 1) and are arranged in the same way as the long and short intervals in the tiling; the only difference is that the scale is reduced by a factor $\sqrt{3-\tau}$. (The reason for this is that the strip is perpendicular to the lines $x+\tau y = \text{constant}$, so the distance of (x, y) along the strip is proportional to the size of $x+\tau y$.)

This strip model sheds much light on the worm tilings and accounts for many properties that previously seemed unrelated. It can be seen that the configuration of tiles within a given distance of a boundary point x_n depends

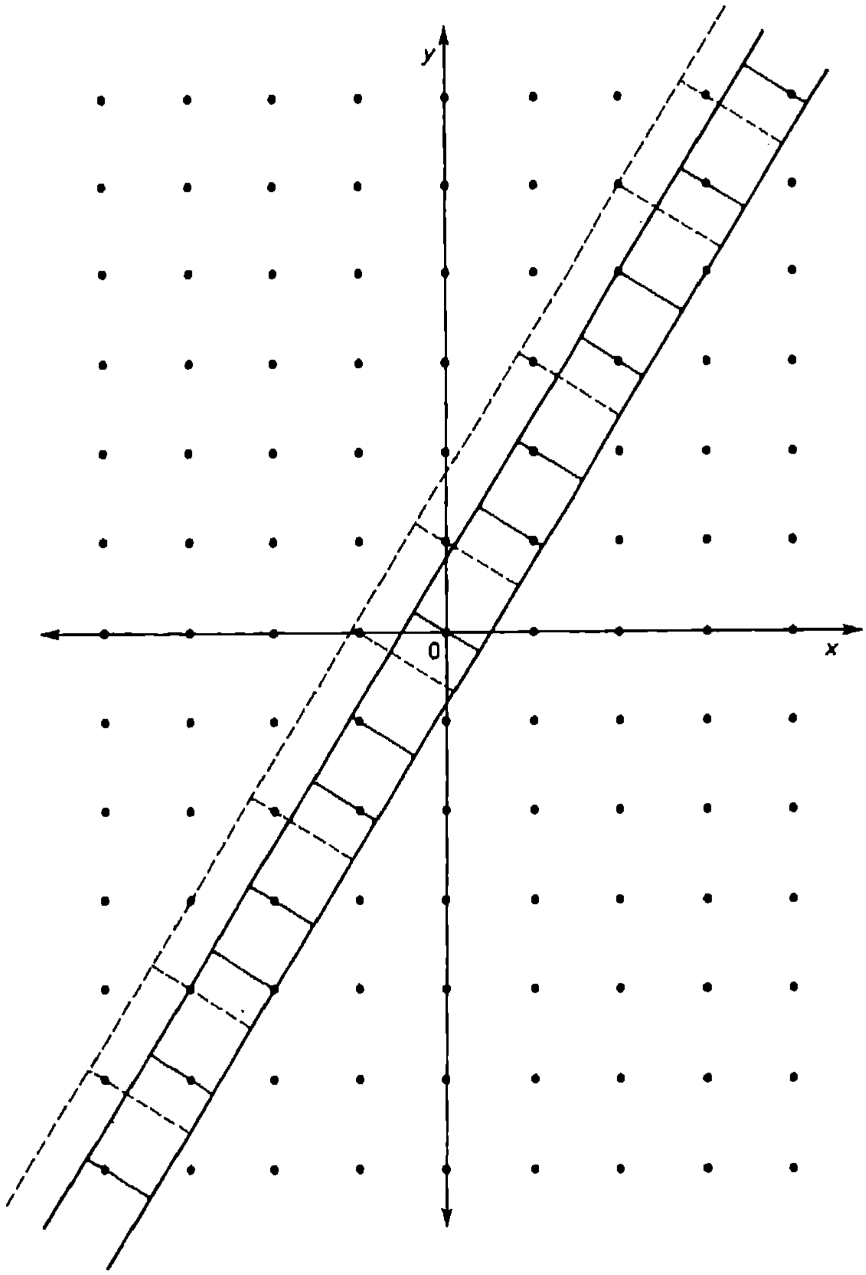

Fig. 6

on the position of the corresponding integer point relative to the edges of the strip and that only finitely many configurations of given length are possible. Consequently QC2 is satisfied for the worm tilings, and to prove QC1 it is enough to show that the integer points in the strip are uniformly distributed across its width. This is equivalent to the fractional part of $n\tau$ being uniformly distributed mod 1, which is so since τ is irrational. It can be seen from this that not only is this particular worm tiling quasicrystallographic, but so are the tilings obtained in the same way from parallel strips of the same width, and these tilings are all of the same species. If the strip is translated by an integer vector we clearly get exactly the same tiling, and consequently the tilings derived from parallel strips of the same width are in one-one correspondence with the points on a torus. (It can be shown that translations of the strip that are distinct modulo the lattice of integer vectors give different tilings.) One-dimensional tilings can be given a topology in which tilings that nearly coincide on a large neighbourhood of the origin are close, and in this topology a species is the closure of the set of translates of any one of its members. For the tilings derived from the strips this topology is the same as the natural topology of the torus, so since the torus is closed and contains all translates of the original tiling it contains all tilings of the species. Consequently the tilings derived from parallel strips of the same width are all the tilings of the species. (An appropriate convention is needed for the strips that have an integer point on each edge, if they are not to be exceptions to some of these statements, but we shall not go into such detail here.)

The 2-distance property is also easily accounted for by the strip model. Since the strip meets horizontal lines in intervals of length 1, there is exactly one integer point in the strip on each line $y = n$ (n an integer), and this corresponds to the point x_n of the tiling. It follows that (for fixed r and variable n) there are only two possibilities for the vector joining the integer points corresponding to x_n and x_{n+r}, because the y-component of this vector is r and there are only two possibilities for the x-component (namely, $[r/\tau]$ and $[r/\tau]+1$). The construction can be varied a bit without affecting the proof of the 2-distance property. Strips of any slope can be allowed (no special property of the slope τ has been used) and it is not necessary to cut the strip at right angles through the integer points in it – any angle of cut (provided it is the same for every integer point) will do. What is important for the 2-distance property is that the strip meets the x-axis in an interval of length 1. For the statement of the 2-distance property the points must be labelled with the y-coordinate of the corresponding integer point in the strip. For some slopes of strip and angles of cut this labelling may not exactly correspond to the linear ordering of the points. W. F. Lunnon has shown that this construction gives essentially all sequences with the 2-distance property.

LUNNON'S THEOREM. *The construction just described gives all sequences with the 2-distance property (up to scale) apart from a certain well defined class of exceptions. Each exceptional sequence consists of a periodic sequence with a section of finite length removed and the gap closed up.*

The strip construction suggests the general idea of making patterns that are quasicrystallographic but not crystallographic out of lower dimensional "slices" of crystallographic patterns (the integer lattice in this case). We shall not use this idea in its full generality, but only in the case when the slice and the derived pattern are 1-dimensional.

DEFINITION. Take any prism (infinite in both directions) in $\boldsymbol{R}^n$ whose cross-section is an $(n-1)$-dimensional polytope and whose axis is in "general direction" (i.e. the coordinates of the axis are linearly independent over $\boldsymbol{Q}$). Also take any hyperplane (i.e. $(n-1)$-dimensional subspace) in $\boldsymbol{R}^n$ not parallel to the prism. Cut through the integer points inside the prism with hyperplanes parallel to the chosen one. The resulting 1-dimensional tiling is called a *prism pattern.*

As with the worm tiling, the configuration of a prism pattern within a given distance of one of its points depends on the whereabouts of the corresponding integer point in the cross-section of the prism, and QC2 is satisfied for prism patterns. Also all the tilings in the species of a prism pattern are obtained by translating the prism (keeping the directions of its axis and of the cutting hyperplanes fixed), and hence the species of a prism pattern is topologically an n-dimensional torus. Because of the stipulation that the axis of the prism is in general direction, prism patterns are not crystallographic (which, since they are 1-dimensional, is the same as saying they are non-periodic). (Again, a convention is needed to fit the prisms with an integer point on the boundary into this framework. It is to facilitate this that we restrict ourselves to prisms with polytopic cross-section, which can have only finitely many integer points on the boundary – at most one on each face.)

LEMMA. *If a prism in general direction has area of cross-section A then the number of integer points in a section of length X is $AX+o(X)$.*

Outline of proof. The number of integer points in a polytope of volume V and surface area S is $V+O(S)$. A section of prism of length X has volume AX, but its surface area also has order of magnitude X. The idea of the proof is to find an integral unimodular transformation that takes the section of prism into a shorter section of a wider prism. Such a transformation preserves integer points and does not change the volume of the prism, but it will reduce its surface area and hence reduce the error in estimating the number of integer points inside it. By the usual box-principle argument we can find arbitrarily long integer vectors as close as we like to the axis of the prism. If the axis is in general direction then it is not parallel to the hyperplane containing any $n-1$ such vectors, and so we can find n such vectors that are linearly independent. It is also possible to arrange that these vectors form a basis of the integer lattice. The transformation that takes these vectors to the unit coordinate vectors then does what we want.

The hypothesis that the prism is in general direction is essential for this lemma: for example, it is possible for a narrow enough prism in a rational direction to avoid integer points altogether.

COROLLARY. *Prism patterns satisfy* QC1, *and hence are quasicrystallographic.*

Proof. Apply the lemma to the subprism of the original prism that contains the integer points that give rise to points of the pattern having a given desired configuration as a neighbourhood.

6. Inflation of prism patterns

The broken lines in Figure 6 show a wider strip that meets the x-axis in an interval of length τ with its right-hand edge coinciding with the right-hand edge of the original strip. When the wider strip is cut at right angles through the extra integer points it contains the effect is to divide each long interval of the worm tiling in the ratio $\tau{:}1$. This is the inflation we have seen already.

Why does this work? The reason is that the matrix

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

is unimodular (so preserves integer points) and has one eigenvector parallel to the strip with eigenvalue τ and one eigenvector at right angles to the strip with eigenvalue $-1/\tau$. So M transforms the wide strip into a translate of the original strip and preserves the direction of the cutting lines at right angles to the strip. Hence the tiling derived from the wide strip, when magnified by a factor τ, is in the same species as the tiling derived from the original strip. Also, since the wide strip contains the original strip, the wide strip tiling is a refinement of the original tiling. When the integer points (a, b) are regarded as numbers $a+b\tau$ in $Q(\tau)$ the effect of M on integer points is the same as multiplying by τ.

DEFINITION. A prism pattern P_2 is an *inflation with multiplier* λ of a prism pattern P_1 if it is in the same species as P_1 and refines λP_1.

This is more general than our previous concept of an inflation of a set of shapes, because it does not require a tile to subdivide in the same way at every occurrence. It has the drawback that it may not be possible to use the inflation to generate the tiling, since if the same tile subdivides in more than one way there may be no way of deciding which method of subdivision to use at each stage of the construction in order to arrive at the desired tiling (or even at a tiling of the same species).

Figure 7 shows two other inflations of the worm tiling. The upper half is the same as Figure 6 except that the wide strip is positioned so that its centre line coincides with the centre line of the original strip, instead of the right-hand edges coinciding. The result is an inflation with multiplier τ in which S becomes L and L becomes either LS or SL, but not the same every time. In the lower half of Figure 7 the original strip is enclosed in a strip that meets the x-axis in an interval of length $\tau^2/2$ and extra cuts are inserted not

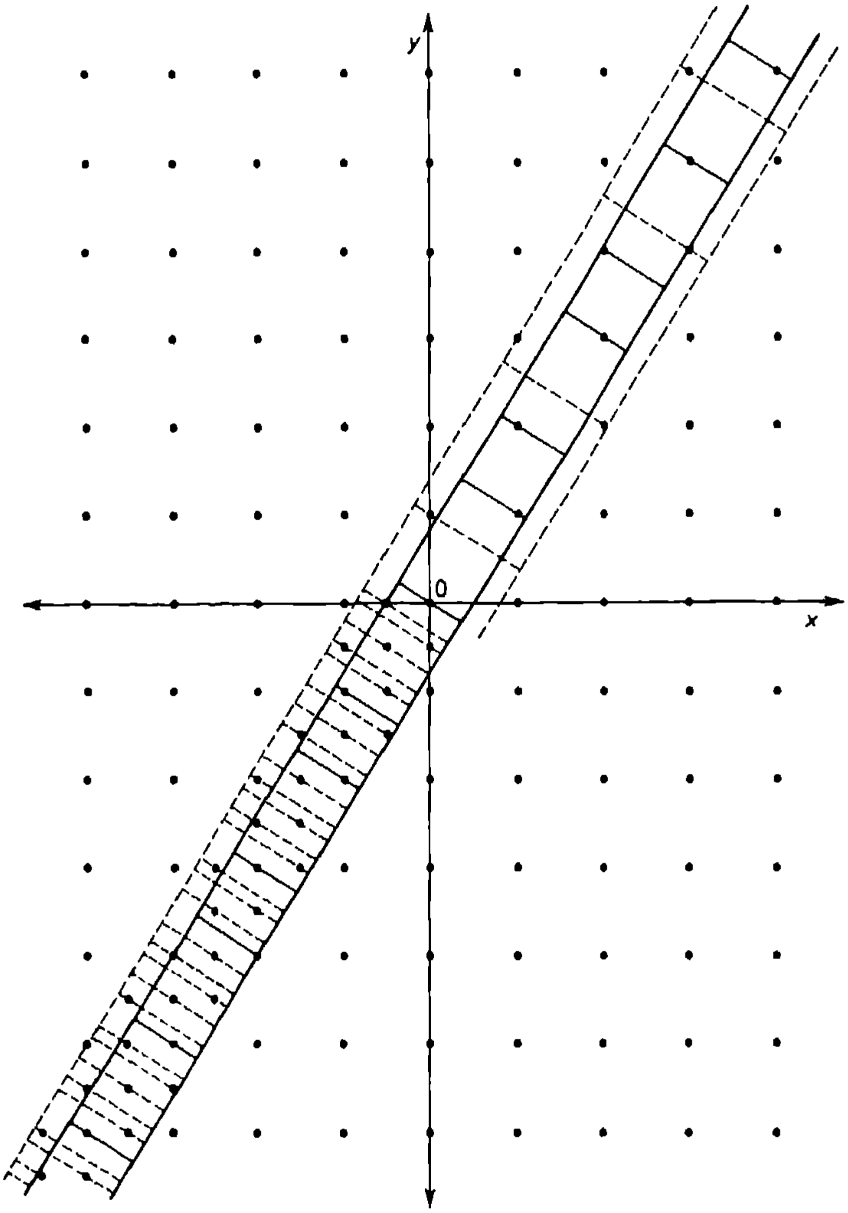

Fig. 7

only through the additional integer points in the wider strip but through all half-integer points in the wider strip as well. The result is an inflation with multiplier $2\tau^2$ in which neither tile subdivides in the same way at every occurrence. This example shows that an inflation can be derived from a matrix that is not unimodular – merely integral. The matrix is

$$\begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix},$$

which has an eigenvector parallel to the strip with eigenvalue $2\tau^2$, an eigenvector at right angles to the strip with eigenvalue $2/\tau^2$, and takes half-integer points to integer points. The effect of the matrix on the integers of $Q(\tau)$ is to multiply them by $2\tau^2$.

To get a prism pattern with an inflation with multiplier λ by this process we need an integer matrix M with λ as a simple eigenvalue and all other eigenvalues < 1 in modulus. It is then possible to find a prism, with its axis in the direction of the eigenvector for λ, that has the property that it is a subset of its image under M^{-1}. For the cutting hyperplane we use the hyperplane containing all the other eigenvectors of M. (This is a real hyperplane because complex eigenvectors occur in conjugate pairs.) For this construction λ must be an algebraic integer with all its conjugates < 1 in modulus (i.e. a Pisot–Vijayaraghavan number). Conversely, for any PV-number λ we can find at least one such matrix M: namely, the matrix of multiplication by λ, regarded as a linear operator on the ring of integers of $Q(\lambda)$. (Instead of the ring of integers, any full module in $Q(\lambda)$ that is closed with respect to multiplying by λ will do.) It can be shown that the inflations constructed in this way are the only inflations of prism patterns there are, and that inequivalent full modules give different inflations. Since every real algebraic number field can be generated by a PV-number, it follows that there is a prism pattern with an inflation with multiplier λ such that $Q(\lambda)$ is any desired real field. The dimension of the prism used is equal to the degree of λ as an algebraic number.

We look briefly at the question of which of these inflations can be used to generate the corresponding tiling. The answer is: almost none.

Definition. An inflation is *context-free* if each kind of tile is subdivided in only one way. It is *bounded-context* if the way a tile is subdivided can be decided by looking at a bounded neighbourhood of it in the tiling.

Superficially, a bounded-context inflation is more general than a context-free one, but in fact the two concepts are virtually identical. A bounded-context inflation can be regarded as context-free by regarding tiles that subdivide differently as distinct, even though they are the same shape and size. The tiles of the inflated tiling must be given a corresponding finer classification, and for a bounded-context inflation this can be done in such a way that the inflated tiling is of the same species as the original tiling even after taking account of these distinctions between geometrically identical tiles. A context-free inflation can be used to generate the tilings of the species it acts on, and consequently the same goes for bounded-context inflations too.

The harsh truth is that most inflations of prism patterns are neither context-free nor bounded-context, however. A prism pattern with an inflation has a continuum of different inflations, corresponding to the continuum of different ways of positioning the wider prism so that it encloses the narrower one. At most countably many of these can be bounded-context (finitely many for each size of context). Furthermore, it can be shown that inflations of prism patterns derived from prisms of dimension greater than 2 are never bounded-context, and neither are inflations derived from matrices that are not unimodular. Consequently, algebraic numbers of degree 3 or more are never multipliers of bounded-context inflations of prism patterns, and neither are quadratic non-units.

Neither of the two inflations illustrated in Figure 7 is bounded-context. The inflation given by the upper half of the figure has a neat description in terms of unbounded contexts, however. For each long tile L take the longest possible section of the tiling that is symmetric about the centre of this L. Because of the maximality of this section, it is flanked by tiles of different kinds: either $L\{\text{symmetric}\}\,S$ or $S\,\{\text{symmetric}\}\,L$. The L at the centre inflates correspondingly as LS or SL. Since there is an infinite worm tiling that is symmetric about a central L (the inflated tiling in Fig. 6) every tiling of the species has arbitrarily long sections that are symmetric about a central L, and consequently there is no fixed size of neighbourhood of L that decides the inflation.

Although an inflation of a prism pattern is not context-free in general, an obvious way of deriving a context-free inflation with the same multiplier is to choose just one of the finitely many possible ways of subdividing each tile. The resulting context-free inflation can then be used to generate a tiling. A tiling obtained in this way is not a prism pattern, in general, but it is not far from being a prism pattern: it can be obtained by cutting with hyperplanes parallel to the original ones through *some* (not all) of the integer points in a *wider* prism parallel to the original prism.

7. A square quasicrystallographic tiling

Prism patterns are a rich source of 1-dimensional quasicrystallographic tilings with inflation. The next stage in our construction is to use them to create higher-dimensional quasicrystallographic tilings, and the simplest way to do this is by direct products.

In Figure 8 the horizontal and vertical continuous straight lines are drawn so as to cut each axis into a symmetric worm tiling. The result is a quasicrystallographic tiling of the plane with square symmetry that uses three kinds of tile: a large square, a small square and a rectangle. The broken lines correspond to inflation of the two worm tilings, and hence give an inflation of the 2-dimensional tiling. We have chosen a context-free inflation of the worm tilings, and so the inflation of the 2-dimensional tiling is context-free and can be used to generate the tiling. This tiling has Properties 1 to 6, except that pentagonal symmetry is replaced by square symmetry in Property 3.

In the same way, taking the direct product of n copies of the worm

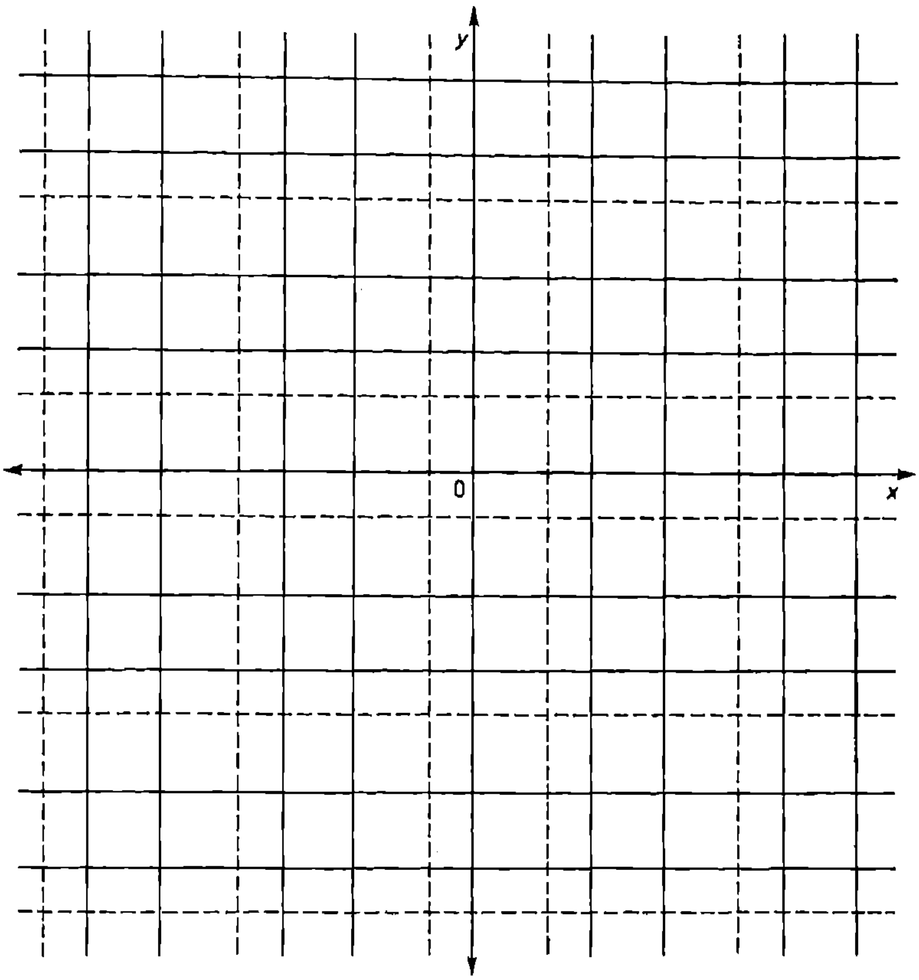

Fig. 8

tiling (or any other 1-dimensional quasicrystallographic tiling with a context-free inflation) gives a tiling of n-dimensional space with hypercubic symmetry having Properties 1, 2, 4, 5 and 6. Constructing tilings with other kinds of symmetry is not so trivial, although essentially the same idea works. If we take a system of parallel straight lines that intersect a transversal in the worm tiling and superpose five copies of it, one parallel to each side of a regular pentagon (see Figure 9), then the result is a tiling of the plane with pentagonal symmetry having Properties 1 to 6, each tile being a convex polygon with not more than five sides. The difficulty lies in proving that the tiling contains only finitely many different shapes and that it is quasicrystallographic. It is for this purpose that the lemmas in the next section are needed. This construction method can be used to give quasicrystallographic tilings with any symmetry. In the general case the 1-dimensional tiling used must be chosen to have an inflation multiplier that generates a field containing the cosines of all the angles between the symmetry directions. A very similar construction is used by de Bruijn [2] in analyzing the Penrose tiling. The difference is that instead of being equally spaced, like the grids that make up de Bruijn's "pentagrids", our systems of lines are spaced according to a prism pattern.

8. The linear form lemmas

FIRST LINEAR FORM LEMMA. *Let $P_1, \ldots, P_r, P_{r+1}$ be prism patterns derived from parallel prisms with parallel cutting hyperplanes, and let*

$$L = a_1 x_1 + \ldots + a_r x_r$$

be a linear form with rational coefficients. Fix a radius ϱ and let x_i run through the points of P_i ($i = 1, \ldots, r$). Then the points with distances L along P_{r+1} have only finitely many different kinds of ϱ-neighbourhood in P_{r+1}.

Proof. Translating each P_i has the effect of adding a constant to L, which clearly does not affect the conclusion of the lemma. So without loss of generality we can suppose that 0 is a point of each P_i. Since x_i is a point of P_i, there is an integer vector $\boldsymbol{x}_i$ such that $\boldsymbol{x}_i = x_i \boldsymbol{u} + y_i$, where $\boldsymbol{u}$ is the unit vector parallel to the axes of the prisms and y_i is a vector parallel to the cutting hyperplanes whose length is no greater than the maximum diameter of the cross-section of P_i's prism by a cutting hyperplane. First suppose that the a_i's are integers. Then $a_1 \boldsymbol{x}_1 + \ldots + a_r \boldsymbol{x}_r$ is an integer vector and has the form $L\boldsymbol{u} + y$, where y is parallel to the cutting hyperplanes and has bounded length. Since there is an integer point inside P_{r+1}'s prism on the cutting hyperplane through the point 0 on its axis, it follows that there is an integer point within a bounded distance of the axis on the parallel hyperplane through the point L on the axis. The position of this integer point determines the position of nearby integer points, so it follows that there are only finitely many parallel hyperplanes within a bounded distance of the point L that can possibly contain an integer point inside the prism for P_{r+1} and their relative positions are independent of the value of L. Hence there are only finitely many possibilities for the ϱ-neighbourhood in P_{r+1} of any of the points L. (The latter part of this argument is essentially the same as the proof that QC2 holds for prism patterns.) When the a_i's are not integers, the same argument again gives a point a bounded distance from the axis of P_{r+1}'s prism on the hyperplane through the point L on the axis, but it may not be an integer point. It is, however, a rational point with denominator equal to the least common denominator of $a_1, \ldots, a_r$, at worst; so there are finitely many possibilities for it modulo the integer lattice. The rest of the argument now carries through as before.

SECOND LINEAR FORM LEMMA. *If each of the prism patterns P_i in the previous lemma has an inflation with multiplier λ and the coefficients a_i lie in the field $\boldsymbol{Q}(\lambda)$ then the same conclusion holds.*

Proof. Let $\mathscr{P}_i$ be the prism that gives rise to the prism pattern P_i and let M_i be the matrix that gives the inflation of it. Then the prism $M_i \mathscr{P}_i$ is parallel to $\mathscr{P}_i$ and, with cutting hyperplanes in the same direction as those for $\mathscr{P}_i$, it gives rise to the prism pattern λP_i. Similarly $M_i^2 \mathscr{P}_i$ is parallel to $\mathscr{P}_i$ and gives rise to the prism pattern $\lambda^2 P_i$, and so on. So $\lambda^k P_i$, for all i and k, are prism patterns derived from parallel prisms with parallel cutting planes. Since each a_i is a polynomial in λ with rational coefficients, the lemma follows from the first linear form lemma applied to a larger number of prism patterns.

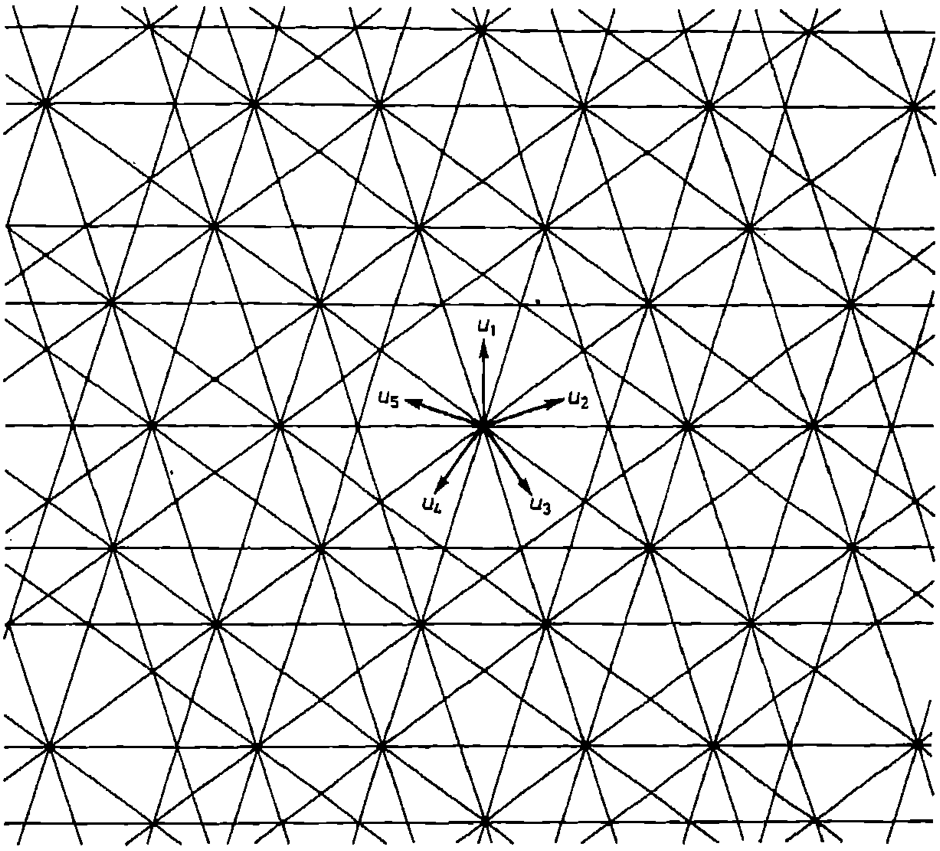

Fig. 9

9. Construction of quasicrystallographic tilings with arbitrary symmetry

Let $u_1, u_2, \ldots, u_s$ be equally spaced unit vectors in the plane emanating from the origin (so that the angle between consecutive vectors is $2\pi/s$). Take a prism pattern P, and for each u_j lay a copy of P along the direction u_j and draw the system of lines perpendicular to u_j through the points of P. (Figure 9 illustrates the case $s = 5$.) The lines divide the plane into polygons of bounded size.

Our first task is to ensure that these polygons are of only finitely many shapes and sizes. Take any such polygon and choose one of its vertices $\boldsymbol{x}$. We shall suppose for simplicity that $\boldsymbol{x}$ is on the intersection of a line perpendicular to $\boldsymbol{u}_1$ and a line perpendicular to $\boldsymbol{u}_2$ (although a vertex on any other pair of lines could be treated in just the same way). Then $\boldsymbol{x}\cdot\boldsymbol{u}_1$ and $\boldsymbol{x}\cdot\boldsymbol{u}_2$ are both in P. Using $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ as a basis, we have

$$\boldsymbol{u}_j = a_j\boldsymbol{u}_1 + a_j\boldsymbol{u}_2 \qquad (j = 1, \ldots, s),$$

where the a_j's and b_j's are in $\boldsymbol{Q}(\cos 2\pi/s)$. We now choose for P a prism pattern with an inflation with multiplier λ such that $\boldsymbol{Q}(\lambda) \supseteq \boldsymbol{Q}(\cos 2\pi/s)$. Then, by the second linear form lemma, there are only finitely many possibilities for the ϱ-neighbourhood in P of

$$\boldsymbol{x}\cdot\boldsymbol{u}_j = a_j(\boldsymbol{x}\cdot\boldsymbol{u}_1) + b_j(\boldsymbol{x}\cdot\boldsymbol{u}_2),$$

where ϱ is the maximum distance between consecutive points of P. Clearly the shape and size of the chosen polygon is determined by the ϱ-neighbourhoods of $\boldsymbol{x}\cdot\boldsymbol{u}_j$ $(j = 1, \ldots, s)$ in P, and hence only finitely many shapes and sizes occur.

Also, tilings constructed in this way are quasicrystallographic. The proof of QC1 is along the same lines as the above proof that only finitely many shapes of tile occur, but uses a different pair of linear form lemmas. Again, it is necessary for P to have an inflation with multiplier λ such that $\boldsymbol{Q}(\lambda) \supseteq \boldsymbol{Q}(\cos 2\pi/s)$. Since, by construction, the polygons fit together vertex-to-vertex, QC2 is automatic.

Finally, the tiling has the symmetry of the regular s-gon, by construction, and is not crystallographic because P is not periodic. Also it has an inflation that is inherited from the inflation of P. The inflation is not usually bounded-context, however, and, indeed, it can only be chosen to be bounded-context when $s = 3, 4, 5, 6, 8, 10$ or 12 (so that $\boldsymbol{Q}(\cos 2\pi/s)$ is quadratic).

Modifying this construction to make the inflation context-free (so that it can be used to generate the tiling) is the main difficulty in the proof of our main theorem. Applying the simple method of making an inflation context-free, described in the last paragraph of Section 6, directly to the 2-dimensional tiling results in a tiling that may not satisfy QC2. Applying it at an earlier stage to the prism pattern P (replacing it by a pattern with a context-free inflation) does not affect the proof that the number of shapes is finite but makes it difficult to ensure that the 2-dimensional tiling satisfies QC1. Hybrid constructions that attempt to avoid both these pitfalls can lose the symmetry. The difficulty can be overcome, however, and the inflation can be replaced by a context-free inflation without sacrificing the other desirable properties of the tiling.

The construction can be carried out in higher dimensions too, starting from a set of unit vectors that is closed and transitive under the action of the desired symmetry group and dividing up space by means of hyperplanes at right angles to these vectors. As before, it is necessary to use a prism pattern with an inflation with multiplier λ such that $Q(\lambda)$ contains the cosines of all the angles between the unit vectors. There are very few irreducible finite symmetry groups in dimensions ≥ 3, and for all of them the cosines of the angles between the unit vectors are algebraic. In dimensions ≥ 5 there are only the hypercubic and simplicial groups, and for these the unit vectors can be chosen so that the cosines are rational. In dimension 3 there is also the dodecahedral group and in dimension 4 there are two other groups, but in each case the unit vectors can be chosen so that the cosines lie in $Q(\tau)$ at worst. (In the plane the only finite symmetry groups are n and nm, and we have dealt with all of these.)

We thus have the following

MAIN THEOREM. *Given any finite symmetry group (in any number of dimensions) there are tilings with that symmetry having Properties* 1, 2, 4 *and* 5. *The inflation multiplier can be any PV-number* λ *such that* $Q(\lambda)$ *contains the cosines of all the angles between the symmetry directions.*

10. Examples

Pentagonal group. Our construction gives a tiling with more than two shapes that is closely related to Penrose's kites and darts tiling. It is less aesthetically pleasing than Penrose's tiling, partly because it uses more shapes but partly because it is a criss-cross of straight lines (a drawback of all tilings constructed in this way). See Figure 9.

Dodecahedral group. We obtain a quasicrystallographic tiling of 3-dimensional space with dodecahedral symmetry and with plane sections that have the above pentagonal tiling (related to Penrose's). It has an inflation with multiplier τ.

A similar tiling of 3-space that uses just two rhombohedral shapes has recently been discovered by R. Ammann (see [5]).

Heptagonal group, etc. Our construction gives quasicrystallographic tilings of the plane with the symmetry of any regular polygon. These tilings can be generated by inflation.

Cubic and simplicial groups. For the cubic and simplicial groups in any number of dimensions the cosines of the angles between the symmetry directions are all rational. So the second linear form lemma is not needed for these groups, and quasicrystallographic tilings can be constructed with these symmetries having any PV-number as inflation multiplier. Perhaps more interestingly, there are quasicrystallographic tilings with these symmetries that have no inflation at all.

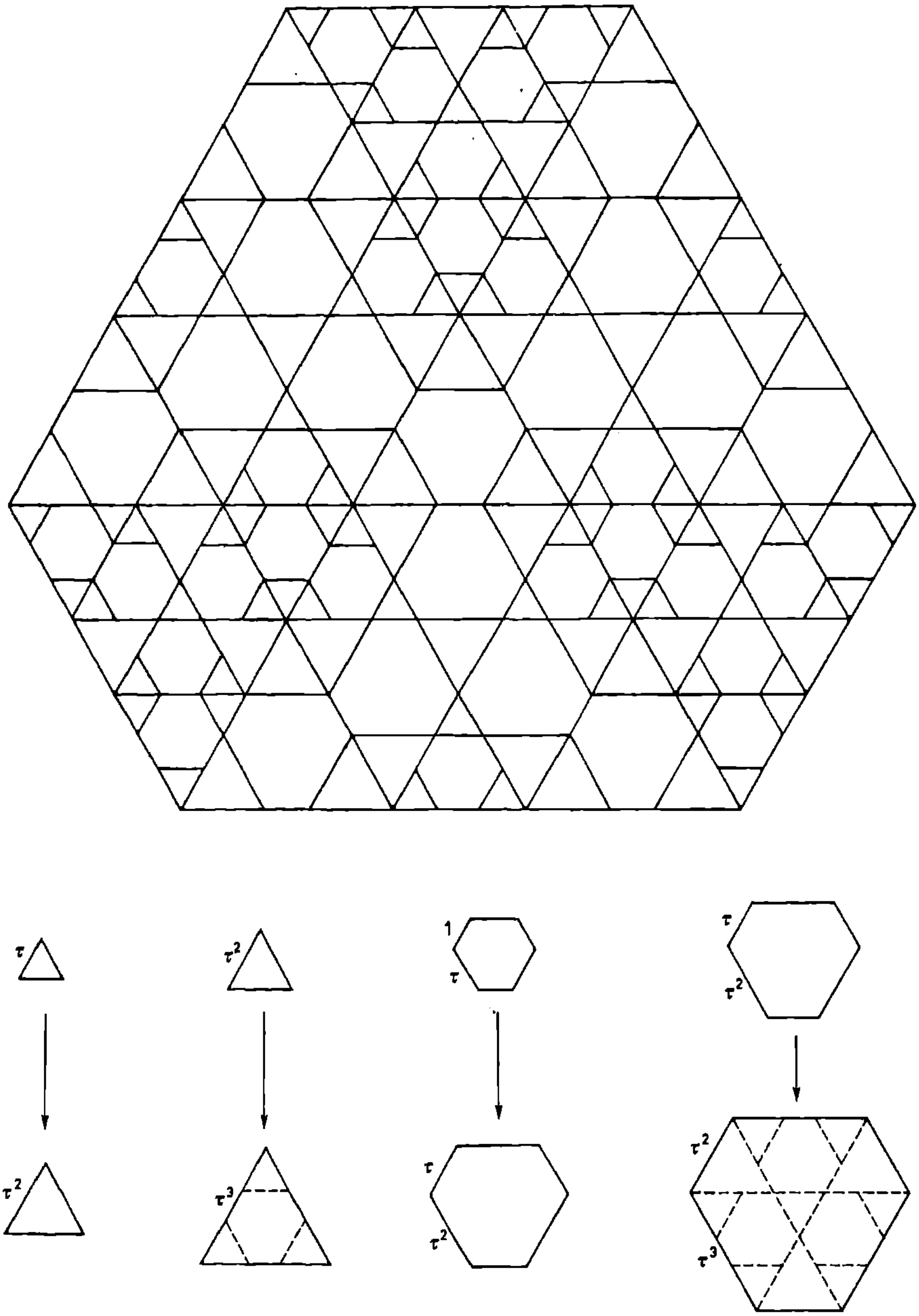

Fig. 10

It would be inexcusable to end this paper without showing at least one new tiling, although, as we have already said, the tilings we construct all consist simply of criss-cross lines. Accordingly, Figure 10 shows a tiling in our final category of examples: it has simplicial symmetry in the plane, i.e.

symmetry $3m$. It was obtained by using the worm tiling to give a quasicrystallographic tiling with simplicial symmetry and an inflation with multiplier τ, according to our construction. The inflation of the resulting shapes was then slightly modified and simplified to give an inflation of a set of only four shapes: two sizes of equilateral triangle and two sizes of similar hexagon. (The way the enlarged shapes are subdivided is shown at the bottom of Figure 10.) The simplified inflation was then used to generate the tiling shown. The criss-cross lines are still strongly in evidence.

Most of the work presented here is in the course of publication in [4], where fuller details can be found.

References

[1] N. G. de Bruijn, *Sequences of zeros and ones generated by special production rules*, Kon. Nederl. Akad. Wetensch. Proc. Ser. A 84 (1981), 27–37.

[2] –, *Algebraic theory of Penrose's non-periodic tilings of the plane*, ibid. 84 (1981), 39–66.

[3] M. Gardner, *Extraordinary nonperiodic tiling that enriches the theory of tiles*, Scientific American, January 1977, 110–121.

[4] W. F. Lunnon and P. A. B. Pleasants, *Quasicrystallographic tilings*, to appear.

[5] A. L. Mackay, *De Nive Quinquangula: On the pentagonal snowflake*, Kristallografiya 26 (1981), 910–919; Sov. Phys. Crystallogr. 26 (1981), 517–522.

[6] R. Penrose, *The role of aesthetics in pure and applied mathematical research*, Bull. Inst. Math. and Appl. 10 (1974), 266–271.

[7] –, *Pentaplexity: a class of non-periodic tilings of the plane*, Eureka 39 (1978). (Reprinted in The Mathematical Intelligencer 2 (1979), 32–37.)

[8] R. L. E. Schwarzenberger, *N-dimensional crystallography*, Pitman, London 1980.

[9] K. B. Stolarsky, *Beatty sequences, continued fractions and certain shift operators*, Canad. Math. Bull. 19 (1976), 473–482.

Presented to the Semester
Elementary and Analytic Theory of Numbers
September 1–November 13, 1982