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SOME THEOREMS OF SCORZA DRAGONI TYPE FOR MULTIFUNCTIONS WITH APPLICATION TO THE PROBLEM OF EXISTENCE OF SOLUTIONS FOR DIFFERENTIAL MULTIVALUED EQUATIONS

STANISŁAW ŁOJASIEWICZ, jr.

Instytut Matematyczny PAN, Kraków

Introduction

The paper consists of two parts. In the first part we present a unified approach to the theory of closed-valued multifunctions. We consider only the case of separable metrisable spaces. Our main aim is to study interrelations between measurability and various kinds of continuity of multifunctions (Lusin type theorems and its consequences such as the Scorza Dragoni type theorems). In the statements of definitions and theorems we remain in the framework of topology and measure theory (with only one exception: d-continuity). We fix our terminology carefully and with exact references (also in the proofs because even elementary terminology changes essentially (compare: [En], [Fe], [Ru], [Wa]). As our field of interest is rather elementary and simple (thanks to a suitable choice of definitions and statements) we do not state original references. The significant statements (either for the theory or for our applications) are called propositions. We present many remarks in order to justify our definitions and statements and to emphasize useful properties and consequences. The omitted proofs can be easily completed. In the second part of the paper we deal with multifunctions in more special spaces; however, the majority of our considerations from the first part do not simplify essentially even in the case of finite-dimensional Euclidean spaces.

The existence of solutions of the following differential relation is the subject of the second (main) part of the paper:

$$(1) \qquad \dot{x} \in F(t, x), \quad x(t_0) = x_0.$$

Here F stands for a multifunction from Ω (Ω is an open connected nonvoid subset of the Euclidean space $\boldsymbol{R}\times\boldsymbol{R}^n$) into $\boldsymbol{R}^n$ with nonempty closed values. The initial state (t_0, x_0) belongs to Ω. By a solution of the problem (1) we mean a continuous function $x(\cdot)$ defined in an open interval $J\subset\boldsymbol{R}$ and satisfying the following three conditions:

(i) $\big(t, x(t)\big)\in\Omega$ for $t\in J$ and $x(t_0) = x_0$,

(ii) the restriction of $x(\cdot)$ to any compact subinterval of J is absolutely continuous,

(iii) $\dot{x}(t) = dx(t)/dt \in F(t, x(t))$ for a.e. $t\in J$.

Under the assumption that $F(t, x)$ are convex and compact the existence problem has been solved: by Ważewski [Wż] for F continuous, by Pliś [Pl2] for F upper continuous in x and measurable in t. Assuming that $F(t, x)$ are merely compact the answer to the problem was given: by Filippov [Fi2] for F continuous, by Kaczyński and Olech [KO] and afterwards by Antosiewicz and Cellina [AC] for F continuous in x and measurable in t, and recently by Bressan [Br] and myself [Ło] for F lower continuous (let us observe that for F merely upper continuous the existence theorem does not hold [Fi2]). Here we present a natural extension of the results mentioned above. The statement of our existence theorem is close to the main theorem in [Ol]. The proof is based on [Ło], [Pl2] and on some results of the first part of the article.

1. Measurability and continuity of closed-valued multifunctions in separable metrisable spaces

Let X be an arbitrary set. By $P(X)$ we denote the family of all subsets of X. By a *multifunction F from X into Y* we mean a function from X into $P(Y)$ (we admit empty set as a value of F at some points). We denote $\operatorname{graph} F = \{(x, y)\in X\times Y\colon y\in F(x)\}$ and for $A\in P(Y)$ we put $F^-(A) = \{x\in X\colon F(x)\cap A\neq\emptyset\}$. By *outer measure* over X we mean the measure in the sense of [Fe, p. 53] and by *measure* on a σ-algebra of subsets of X [Ru, p. 9] we mean the positive measure in the sense of [Ru, p. 17]. Any outer measure μ, restricted to the σ-algebra of μ-measurable sets [Fe, pp. 54–55] is a complete [Wa, p. 864] measure on this σ-algebra. Conversely, let μ be a measure on a σ-algebra $\mathscr{T}$. Put for $A\in P(X)$, $\mu^*(A) = \inf\{\mu(B)\colon B\in\mathscr{T}$ and $A\subset B\}$. Then μ^* is another measure such that its restriction to $\mathscr{T}$ coincides with μ. Let X be a topological space. By $B(X)$ we denote the σ-algebra of Borel sets in X [En, p. 45]. We say that a measure μ on $B(X)$ is *locally finite* if every point in X has an open neighborhood U such that $\mu(U)$ is finite. From now on till the end of this part we suppose that all spaces considered are separable, metrisable and by μ we denote any measure on $B(X)$. Then the product σ-algebra $B(X)\times$

$\times B(Y)$ [Ru, p. 145] coincides with $B(X \times Y)$. By $B(X)^\mu$ we denote the σ-algebra of μ^*-measurable sets. Obviously $B(X) \subset B(X)^\mu$. Let p be the natural projection from $X \times Y$ onto Y.

Definition 1. We say that (X, μ, Y) is *projective* if $p\big(B(X \times Y)\big) \subset B(X)^\mu$.

We define a σ-algebra $B_\mu(X, Y)$ by putting $B_\mu(X, Y) = \big\{A \cup E: A \in B(X \times Y),\ E \in P(X \times Y) \text{ and } \mu^*\big(p(E)\big) = 0\big\}$.

Remark 1. If μ is locally finite, then $B(X)^\mu = \{A \cup E: A \in B(X), E \in P(X) \text{ and } \mu^*(E) = 0\}$. Hence $B(X)^\mu \times B(Y) \subset B_\mu(X, Y)$.

Proposition 1. *If Y is a Suslin space* [Wa, p. 863] *then (X, μ, Y) is projective.*

Proof. Let $\mathcal{N}$ be the set of all irrational numbers between 0 and 1. There exists a continuous mapping f such that $Y = f(\mathcal{N})$ [Wa, p. 864]. Let h be a homeomorphism from $\mathcal{N}$ onto $\mathcal{N} \times \mathcal{N}$ [Fe, pp. 63 and 65], I the identity on X, p_1 (p_2) the natural projection from $X \times \mathcal{N}$ onto $\mathcal{N}$ (from $(X \times \mathcal{N}) \times \mathcal{N}$ onto $X \times \mathcal{N}$), and $A \in B(X \times Y)$. Then $A_1 = (I \times f)^{-1}(A)$ belongs to $B(X \times \mathcal{N})$ as an inverse image of a Borel set by a continuous mapping. Since $(I \times f)(A_1) = A$, by the identity $p_1 = p \circ (I \times f)$ we get $p(A) = p_1(A_1)$. By [Fe, p. 66] A_1 is a Suslin subset of $X \times \mathcal{N}$ [Fe, p. 65]. Hence there exists a closed subset A_2 of $(X \times \mathcal{N}) \times \mathcal{N}$ such that $p_2(A_2) = A_1$. Put $B = (I \times h)^{-1}(A_2)$. Then B is a closed subset of $X \times \mathcal{N}$ and $(I \times h)(B) = A_2$. By the identity $p_1 = p_1 \circ p_2 \circ (I \times h)$ we get $p_1(B) = p_1 \circ p_2(A_2) = p(A)$ and consequently $p(A)$ is a Suslin subset of X. Hence $p(A)$ is μ^*-measurable by [Fe, p. 68] and so $p(A) \in B(X)^\mu$.

By $[-\infty, +\infty]$ we denote the extended real number system and for a and b in $[-\infty, +\infty]$ with $a \leqslant b$ we denote by $[a, b]$ ($]a, b[$) the closed (resp. open) interval with extremities a and b. We put $\boldsymbol{R} =]-\infty, +\infty[$ and $\boldsymbol{N}$ — the set of all positive integers. A function f from X into $[-\infty, +\infty]$ is called *lower* (*upper*) *semicontinuous* at $x \in X$ if for every sequence $\{x_n\}_{n \in \boldsymbol{N}}$ converging to x we have

$$\liminf_{n\to\infty} f(x_n) \geqslant f(x) \qquad \big(\limsup_{n\to\infty} f(x_n) \leqslant f(x)\big).$$

Then f is both l.s.c. and u.s.c. at x iff it is continuous at x. Put $\mathrm{Gr}^-(f) = \{(x, a) \in X \times \boldsymbol{R}: a < f(x)\}$ and $\mathrm{Gr}^+(f) = \{(x, a) \in X \times \boldsymbol{R}: a > f(x)\}$. Then f is l.s.c. (u.s.c.) at each point of X iff the set $\mathrm{Gr}^-(f)$ ($\mathrm{Gr}^+(f)$) is open in $X \times \boldsymbol{R}$. Let $\mathcal{T}$ be a σ-algebra of subsets of X. We say that f is $\mathcal{T}$-*measurable* if $f^{-1}(A) \in \mathcal{T}$ whenever $A \subset [-\infty, +\infty]$ is open. If $B(X) \subset \mathcal{T}$ then every l.s.c. or u.s.c. function is $\mathcal{T}$-measurable.

Lusin's theorem is the main result relating measurability and continuity. We present its proof because our formulation differs from the classical one in two aspects. We do not assume any kind of compactness and do not restrict ourselves to the sets of finite measure. Local finiteness of the considered measure is our unique assumption relating topology and measure structure.

PROPOSITION 2 (Lusin's theorem). *Let* μ *be a locally finite measure on* $B(X)$ *and* f *a function from* X *into* $[-\infty, +\infty]$. *Then* f *is* $B(X)^{\mu}$-*measurable iff for every* $\varepsilon > 0$ *there exists a closed subset* K *of* X *such that* $\mu(X\setminus K) < \varepsilon$ *and* $f|_K$ *(the restriction of* f *to* K*) is continuous. The proposition remains true if we require only that* $f|_K$ *is l.s.c. or u.s.c.*

Proof. For every $x \in X$ take U_x to be an open neighborhood of x with $\mu(U_x) < +\infty$. Then $\{U_x\}_{x\in X}$ is an open cover of the paracompact space X [En, p. 273]. Let $\mathscr{A}$ be a locally finite open refinement [En, pp. 33 and 165] of the cover $\{U_x\}_{x\in X}$. As X is a Lindelöf space [En, pp. 247 and 320] we can choose a countable subcover $\{U_i\}_{i\in N}$ of $\mathscr{A}$. Then $\mu(U_i) < +\infty$ for every $i \in N$. Fix $\varepsilon > 0$. By [Fe, p. 76] there exists $K_i \subset U_i$, a closed subset of X, such that $\mu(U_i\setminus K_i) \leqslant \varepsilon/2^i$ and $f|_{K_i}$ is continuous.

Put $K = \bigcup_{i\in N} K_i$. Then K is closed and $f|_K$ is continuous because $\{K_i\}_{i\in N}$ is a locally finite family of closed sets. Conversely, let $\{K_n\}_{n\in N}$ be a family of closed sets such that $\mu(X\setminus K_n) \leqslant 1/2^n$ and $f|_{K_n}$ is l.s.c. or u.s.c. Let U be an open subset of $[-\infty, +\infty]$. Then

$$f^{-1}(U) = \bigcup_{n\in N} f|_{K_n}^{-1}(U) \cup f|_{X\setminus \bigcup K_n}^{-1}(U).$$

We claim that $f^{-1}(U) \in B(X)^{\mu}$. Since $\mu(X\setminus \bigcup_{n\in N} K_n) = 0$ we get $\mu^*\big(f|_{X\setminus\bigcup K_n}^{-1}(U)\big) = 0$. Hence it is enough to prove that $f|_{K_n}^{-1}(U) \in B(X)$ for every $n \in N$. For this purpose let us observe that without loss of generality we can replace the family of all open subsets of $[-\infty, +\infty]$ by any family generating the same σ-algebra. For every $a \in \boldsymbol{R}$, $f|_{K_n}^{-1}(]a, +\infty]) \in B(X)$ provided that $f|_{K_n}$ is l.s.c. because $f|_{K_n}^{-1}(]a, +\infty) = p\big(K_n \times]a, +\infty[\cap \mathrm{Gr}^-(f|_{K_n})\big)$ is an open set in K_n (with induced topology). Similarly $f|_{K_n}^{-1}([-\infty, a[)$ is open in K_n provided that $f|_{K_n}$ is u.s.c.

Remark 2. Lusin's theorem does not hold if we require only that μ is σ-finite (i.e. there exist $E_n \in B(X)$ with $\mu(E_n) < +\infty$ for $n \in N$ such that $X = \bigcup_{n\in N} E_n$) instead of the local finiteness of μ. For instance, take $X = [-\infty, +\infty]$, $\mu(A) = m(A\cap \boldsymbol{R})$ for $A \in B([-\infty, +\infty])$ where m is the one-dimensional Lebesgue measure and put $f(x) = 0$ for $x \in \boldsymbol{R}$ and $f(-\infty) = f(+\infty) = 1$.

Now, we are going to extend this Lusin theorem to the case of closed-valued multifunctions. From now on, till Remark 16, F denotes a closed-valued multifunction from X into Y and μ is locally finite.

DEFINITION 2. We say that F is *lower continuous* if $F^-(A)$ is open whenever A is open.

Remark 3. In opposition to [En, p. 89] we prefer to say "lower continuous" and not "lower semicontinuous" because in the case of single-valued F (i.e. $F(x) = \{f(x)\}$ for all $x \in X$ where f is a function from X into Y) and $Y = [-\infty, +\infty]$ F is lower continuous iff it is continuous (and lower semicontinuity of f does not imply lower continuity of F).

Let $\mathcal{T}$ be a σ-algebra of subsets of X.

DEFINITION 3. We say that F is *$\mathcal{T}$-measurable* if $F^-(A) \in \mathcal{T}$ whenever A is open.

Remark 4. This definition coincides with the clasical one for single-valued F. Some authors use the term "weakly measurable" instead of "measurable" [Wa, p. 862].

Let Y_0 be a dense subset of Y and d an admissible metric on Y (i.e. the topology induced by d coincides with the topology of Y). For $x \in X$, $y \in Y$ and $r > 0$ denote $B(y; r) = \{z \in Y\colon d(y, z) < r\}$ and $d_y(x) = \inf\{d(y, z)\colon z \in F(x)\}$. By the identity $F^-\big(B(y; r)\big) = d_y^{-1}(]-\infty, r[)$ and since Y is separable we get easily

Remark 5. F is $\mathcal{T}$-measurable iff d_y is $\mathcal{T}$-measurable for every $y \in Y_0$.

Since graph $F = \bigcap_{a \in Y_0} \{(x, y) \in X \times Y\colon d(y, a) \geqslant d_a(x)\}$ we obtain (assuming Y_0 countable)

Remark 6. graph $F \in \mathcal{T} \times B(Y)$ provided that F is $\mathcal{T}$-measurable.

Remark 7. F is $B(X)^\mu$-measurable iff graph $F \in B_\mu(X, Y)$ provided that (X, μ, Y) is projective.

By an operation on a family of multifunctions (from X into Y) we mean the multifunction (from X into Y) such that its value at $x \in X$ is the result of this operation on values at x of each of these multifunctions.

Remark 8. The closure of the union of a countable family of $\mathcal{T}$-measurable multifunctions is $\mathcal{T}$-measurable and if, moreover, (X, μ, Y) is projective and $\mathcal{T} = B(X)^\mu$, the same holds true for the intersection.

By the identity $F^-\big(B(y, r)\big) = p\big(X \times]-\infty, r[\cap \mathrm{Gr}^+(d_y)\big)$ (where p is the natural projection from $X \times \boldsymbol{R}$ onto X) we get

Remark 9. F is lower continuous iff d_y is upper semicontinuous for every $y \in Y_0$.

The last remark suggests the following

DEFINITION 4. We say that F is *upper d-continuous* if d_y is lower semicontinuous for every $y \in Y_0$.

DEFINITION 5. We say that F is *d-continuous* if it is lower continuous and upper d-continuous.

Remark 10. These two notions do not depend on the choice of Y_0; however (in contrast with measurability and lower continuity), they are not topological notions since they depend on the choice of an admissible metric d (this fact justifies our terminology). For instance, take $X = \{0\} \cup \{1/2^i: i \in N\}$ and $Y = N$ with topologies induced from R and define F by $F(0) = \emptyset$, $F(1/2^i) = \{n \in N: n \geqslant i\}$. Then F is d_1-continuous for d_1 given by $d_1(m, n) = |m-n|$ and is not upper d_2-continuous for d_2 given by

$$d_2(m, n) = \left|\frac{1}{m} - \frac{1}{n}\right|.$$

Proposition 2 leads to the following corollaries.

COROLLARY 1. *F is $B(X)^\mu$-measurable iff for every $\varepsilon > 0$ there exists a closed subset K of X such that $\mu(X \setminus K) < \varepsilon$ and such that $F|_K$ (the restriction of F to K) is lower continuous. The corollary remains true if we require that $F|_K$ is d-continuous or upper d-continuous instead of lower continuous.*

DEFINITION 6. We say that F is *upper continuous* (*strongly upper continuous*) if $F^-(A)$ is closed whenever A is a compact (closed) subset of Y.

Remark 11. If F is single-valued (given by f as in Remark 3) then F is strongly upper continuous iff f is continuous.

Remark 12. Our strong upper continuity coincides with the upper semicontinuity in the sense of [En, p. 88] and F is upper continuous iff graph F is a closed subset of $X \times Y$.

DEFINITION 7. We say that F is *continuous* if it is both lower and upper continuous.

Now, we are going to localize our definitions. Let $x_0 \in X$. We say that:

DEFINITION 8. F is *lower continuous at x_0* if for every $y_0 \in F(x_0)$ and for every sequence $\{x_n\}_{n \in N}$ converging to x_0 there exists a sequence $\{y_n\}_{n \in N}$ converging to y_0 such that $y_n \in F(x_n)$ for $n \in N$.

DEFINITION 9. F is *upper continuous at x_0* if for every sequence $\{x_n\}_{n \in N}$ converging to x_0 and for every sequence $\{y_n\}_{n \in N}$ converging to some $y_0 \in Y$ such that $y_n \in F(x_n)$ for $n \in N$ we have $y_0 \in F(x_0)$.

DEFINITION 10. F is *continuous at* x_0 if it is both lower and upper continuous at x_0.

Remark 13. F is (lower, upper) continuous iff it is (lower, upper) continuous at every $x_0 \in X$.

Remark 14. The following implications hold: F is strongly upper continuous $\Rightarrow$ F is upper d-continuous $\Rightarrow$ F is upper continuous. For Y compact these three notions coincide. However, Lusin's theorem (Corollary 1) is false whenever we use the strong upper continuity. Indeed, take $X = \boldsymbol{R}$, $Y = \boldsymbol{R}^2$ and μ the one-dimensional Lebesgue measure restricted to $B(\boldsymbol{R})$. For $x \in \boldsymbol{R}$ put $F(x) = \{(y, z) \in \boldsymbol{R}^2\colon y = t\sin x,\ z = t\cos x,\ t \in \boldsymbol{R}\}$. Then F is lower continuous (hence $B(X)$-measurable) but not strongly upper continuous. Moreover, if K is a closed subset of $\boldsymbol{R}$, then $F|_K$ is strongly upper continuous iff K is a discrete space [En, p. 31]. Hence K is countable and $\mu(K) = 0$.

Remark 15. If Y is locally compact (in this case Y is also a Suslin space) then there exists an admissible metric d such that upper continuity is equivalent to upper d-continuity. Conversely, if d is an admissible metric on Y such that upper continuity is equivalent to upper d-continuity and X is not a discrete space then Y is locally compact and any bounded closed subset of (Y, d) [En, p. 313] is compact.

Since this remark in not quite as easy to prove as the previous ones we include the proof.

Proof. If Y is compact the statement is trivial. If Y is locally compact but is not compact, take the Alexandroff compactification $\tilde{Y} = Y \cup \{\infty\}$ of Y [En, p. 222]. Let ϱ be an admissible metric on $\tilde{Y}$ and let d be a metric on Y defined by

$$d(x, y) = \varrho(x, y) + \left| \frac{1}{\varrho(x, \infty)} - \frac{1}{\varrho(y, \infty)} \right|.$$

Then d is an admissible metric and $A \in P(Y)$ is compact iff it is closed and bounded in (Y, d). Consequently, the upper d-continuity results from the upper continuity. Indeed, let F be an upper continuous multifunction from X into Y and suppose that for some $a \in Y$, d_a is not lower semicontinuous at some $x_0 \in X$. Then there exists a sequence $\{x_n\}_{n \in N}$ converging to x_0 and a positive constant M such that $\liminf_{n\to\infty} d_a(x_n) < M < d_a(x_0)$. Choose a subsequence $\{x_{n_k}\}_{k \in N}$ of $\{x_n\}_{n \in N}$ such that there exist $y_k \in F(x_{n_k})$ with $d(a, y_k) < M$ for $k \in N$. Since $\{y \in Y\colon d(a, y) \leqslant M\}$ is compact we get a $y_0 \in Y$ such that $y_0 = \lim_{m\to\infty} y_{k_m}$ where $\{y_{k_m}\}_{m \in N}$ is a subsequence of $\{y_k\}_{k \in N}$. Hence $y_0 \in F(x_0)$, because graph F is closed, and consequently $d_a(x_0) \leqslant M$ which contradicts our choice of M. Now, let us pass to the converse. Obviously, it is enough to prove that every bounded closed subset of (Y, d) is compact. Otherwise, let A be a bounded closed subset of Y which is not compact. Then there exists a sequence $\{y_n\}_{n\in N}$, $y_n \in A$ such that none of its subsequences is convergent.

As X is not a discrete space we can find a sequence $\{x_n\}_{n\in N}$ converging to a point x_0 such that $x_n \neq x_m$ for all $n \neq m$. We define a multifunction F from X into Y by $F(x_n) = \{y_n\}$ for $n \in N$ and $F(x) = \emptyset$ otherwise. Then graph$F$ is closed and for every $y \in Y$, d_y is not lower semicontinuous at x_0 because $\{y_n\}_{n\in N}$ is bounded.

COROLLARY 2 (cf. [Pl1] and [Ja, p. 476]). *F is $B(X)^\mu$-measurable iff for every $\varepsilon > 0$ there exists a closed subset K of X such that $\mu(X\setminus K) < \varepsilon$ and $F|_K$ is continuous. The corollary remains true if we require only that $F|_K$ is upper continuous provided that (X, μ, Y) is projective.*

Remark 16. The projectivity of (X, μ, Y) is essential in the last part of the corollary because an upper continuous multifunction (even single-valued) need not be $B(X)^\mu$-measurable. For instance, take $X = [0, 1]$, μ the one-dimensional Lebesgue measure restricted to $B(X)$. Let $A \in P(X)\setminus B(X)^\mu$ and $0 \in X\setminus A$. Put $Y = A \cup \{x \in X: -x \in X\setminus A\}$ and define F by $F(x) = \{x\}$ for $x \in A$ and $F(x) = \{-x\}$ for $x \in X\setminus A$. Then graph$F$ is closed in $X \times Y$, A is open in Y (obviously in X and Y we take the topologies induced from R) and $F^-(A) = A \notin B(X)^\mu$.

Now, we extend Definition 3 to the case of partially defined multifunctions. Let $X_0 \in P(X)$, let F be a closed-valued multifunction from X_0 into Y and $\mathcal{T}$ a σ-algebra in X.

DEFINITION 11. We say that F is *$\mathcal{T}$-measurable* if $F^-(A) \in \mathcal{T}$ whenever $A \subset Y$ is open.

Remark 17. F is $\mathcal{T}$-measurable iff $F^-(Y) \in \mathcal{T}$ and F is $\mathcal{T}_0$-measurable ($\mathcal{T}_0$ is the σ-algebra on $F^-(Y)$ induced by $\mathcal{T}$). Moreover, F is $\mathcal{T}$-measurable iff $\hat{F}: X \to Y$ (defined by $\hat{F}(x) = F(x)$ for $x \in X_0$ and $\hat{F}(x) = \emptyset$ otherwise) is $\mathcal{T}$-measurable. Remarks 5, 6, 7 and 8 remain true for partially defined multifunctions.

From now on, till the end of this section, F is a closed-valued multifunction from $X \times Y$ into Z and μ is locally finite.

COROLLARY 3 (Scorza Dragoni type theorem — the upper continuous case, cf. [Rz]). *Define the multifunction G from X into $Y \times Z$ by putting $G(x) =$ graph$F(x, \cdot)$ if graph$F(x, \cdot)$ is closed and $G(x) = \emptyset$ otherwise. Suppose that $F(x, \cdot)$ is upper continuous for a.e. $x \in X$ and that G is $B(X)$-measurable. Then for every $\varepsilon > 0$ there exists a closed subset K of X with $\mu(X\setminus K) < \varepsilon$ such that $F|_{K\times Y}$ is upper continuous.*

Proof. As at the beginning of the proof of Proposition 2, we use the local finiteness of μ and [Fe, p. 62] to get a closed $X_0 \in P(X)$ with $\mu(X \setminus X_0) < \varepsilon/2$ such that $F(x, \cdot)$ is upper continuous for every $x \in X_0$. Corollary 2 applied to G gives K_0 closed in X with $\mu(X \setminus K_0) < \varepsilon/2$ such that $G|_{K_0}$ is upper continuous. Put $K = K_0 \cap X_0$ and observe that graph $F|_{K \times Y} = \operatorname{graph} G|_K$ is closed.

Remark 18. If $(X, \mu, Y \times Z)$ is projective then the converse holds. Namely, if the conclusion of Corollary 3 is fulfilled then $F(x, \cdot)$ is upper continuous for a.e. $x \in X$ and G is $B(X)^\mu$-measurable (or equivalently $\operatorname{graph} F \in B_\mu(X, Y \times Z)$).

COROLLARY 4 (Scorza Dragoni type theorem — d-continuous case). *Let d be an admissible metric on Z. Then the following two conditions are equivalent*:

(i) $F(X, \cdot)$ *is d-continuous for a.e. $x \in X$ and $F(\cdot, y)$ is $B(X)^\mu$-measurable for all $y \in Y$.*

(ii) *For every $\varepsilon > 0$ there exists a closed subset K of X with $\mu(X \setminus K) < \varepsilon$ such that $F|_{K \times Y}$ is d-continuous.*

Proof. Fix $\varepsilon > 0$. Let F_0 be a multifunction from $X \times Y$ into Z defined by $F_0(x, y) = F(x, y)$ if $F(x, \cdot)$ is d-continuous and $F_0(x, y) = \emptyset$ otherwise. Let $\{z_i\}_{i \in N}$ be a sequence dense in Z. Put $f_i(x, y) = \inf\{d(z_i, a): a \in F_0(x, y)\}$.

Then $f_i(x, \cdot)$ is continuous for all $x \in X$ and $f_i(\cdot, y)$ is $B(X)^\mu$-measurable for all $y \in Y$. Define a single-valued multifunction F_i from $X \times Y$ into $[-\infty, +\infty]$ by $F_i(x, y) = \{f_i(x, y)\}$. We will apply Corollary 3 to F_i. Let G_i correspond to F_i (as G corresponds to F in Corollary 3); then $G_i(x) = \operatorname{graph} F_i(x, \cdot)$ for all $x \in X$. Moreover, since $F_i(x, \cdot)$ is lower continuous we have

$$G_i(U \times V) = \bigcup_{y \in A \cap U} F_i(\cdot, y)^-(V)$$

for any A dense in Y and $U \times V$ open in $Y \times [-\infty, +\infty]$. Therefore the $B(X)^\mu$-measurability of G_i follows from the $B(X)^\mu$-measurability of $F_i(\cdot, y)$. Hence (by Corollary 3) we get a closed $K_i \subset X$ with $\mu(X \setminus K_i) < \varepsilon 2^{-i-1}$ such that $F_i|_{K_i \times Y}$ is upper continuous. Since $[-\infty, +\infty]$ is compact, $F_i|_{K_i \times Y}$ is strongly upper continuous. Hence $f_i|_{K_i \times Y}$ is continuous (for every $i \in N$) and consequently $F_0|_{K_0 \times Y}$ is d-continuous where $K_0 = \bigcap_{i \in N} K_i$. By the same argument as in the proof of Corollary 3 we can find a closed $X_0 \subset X$ with $\mu(X \setminus X_0) < \varepsilon/2$ such that $F(x, \cdot)$ is d-continuous for every $x \in X_0$. Put $K = X_0 \cap K_0$. Then $F|_{K \times Y}$ is d-continuous. The implication (ii) $\Rightarrow$ (i) is obvious.

Remark 19 (cf. [HV]). If in Corollary 4 we assume additionally that Z is locally compact then (by Remark 15) we can replace in the statement d-continuity by continuity.

Remark 20. If F in Corollary 4 is single-valued (given by $F(x,y) = \{f(x,y)\}$) then we can replace the d-continuity of F by the continuity of f, and the measurability of F by that of f. In this way we get a generalization of the pioneering result [SD].

LEMMA 1. *Let f be a $B_\mu(X, Y)$-measurable function from $X\times Y$ into $[-\infty, +\infty]$ and suppose that $(X, \mu, Y\times \boldsymbol{R})$ is projective. Then $E = \{x\in X\colon f(x,\cdot)$ is upper semicontinuous$\}$ belongs to $B(X)^\mu$.*

Proof. By the formulas $G(x) = Y\times\boldsymbol{R}\setminus \mathrm{Gr}^+(f(x,\cdot))$ and $\bar{G}(x) = \mathrm{cl}\,G(x)$ we define two multifunctions G and $\bar{G}$ from X into $Y\times\boldsymbol{R}$. Let H be the closed-valued multifunction from $X\times Y$ into $\boldsymbol{R}$ given by $H(x,y) = \{r\in\boldsymbol{R}\colon r\leqslant f(x,y)\}$. Then H is $B_\mu(X, Y)$-measurable and $G(x) = \mathrm{graph}\, H(x,\cdot)$. The equalities $\bar{G}^-(U\times V) = G^-(U\times V) = p\big(H^-(V)\cap X\times U\big)$ hold for any $U\times V$ open in $Y\times\boldsymbol{R}$ where p is the natural projection from $X\times Y$ onto Y. Hence G is $B(X)^\mu$-measurable because the projectivity of $(X, \mu, Y\times\boldsymbol{R})$ implies that of (X, μ, Y). Therefore $\mathrm{graph}\,\bar{G}\in B_\mu(X, Y\times\boldsymbol{R})$. Since

$$X\times Y\times\boldsymbol{R}\setminus\mathrm{graph}\,G = \bigcup_{a\in A} f^{-1}([-\infty, a[)\times]a, +\infty[$$

for any A dense in $\boldsymbol{R}$, we have $\mathrm{graph}\,G\in B_\mu(X, Y)\times B(\boldsymbol{R})\subset B_\mu(X, Y\times\boldsymbol{R})$. Since $(X, \mu, Y\times\boldsymbol{R})$ is projective we have

$$X\setminus E = \{x\in X\colon G(x)\neq\bar{G}(x)\} = q(\mathrm{graph}\,\bar{G}\setminus\mathrm{graph}\,G)\in B(X)^\mu$$

(here q is the natural projection from $X\times Y\times\boldsymbol{R}$ onto X).

PROPOSITION 3. *Suppose that F is $B_\mu(X, Y)$-measurable and that $(X, \mu, Y\times\boldsymbol{R})$ is projective. Define $A(F) = \{(x,y)\in X\times Y\colon$ there exists an open neighborhood U of y such that $F(x,\cdot)|_U$ is lower continuous$\}$. Then $F|_{A(F)}$ is $B(X, Y)$-measurable.*

Proof. As $F|_{A(F)}^-(U) = A(F)\cap F^-(U)$ we need only to prove that $A(F)\in B_\mu(X, Y)$. Let $\{B_i\}_{i\in\boldsymbol{N}}$ be a countable base of the topology on Y. Define $E_i = \{x\in X\colon F(x,\cdot)|_{B_i}$ is lower continuous$\}$. Then $A(F) = \bigcup_{i\in\boldsymbol{N}} E_i\times B_i$. Put $F_i = F|_{X\times B_i}$. Then F_i is $B_\mu(X, B_i)$-measurable. Let $\{z_j\}_{j\in\boldsymbol{N}}$ be a dense sequence in Z and d an admissible metric in Z. Define $f_{i,j}\colon X\times B_i\to[-\infty, +\infty]$ by $f_{i,j}(x,y) = \inf\{d(z_j, a)\colon a\in F_i(x,y)\}$ and put $E_{i,j} = \{x\in X\colon f_{i,j}(x,\cdot)$ is upper semicontinuous$\}$. As $(X, \mu, Y\times\boldsymbol{R})$ is projective, $(X, \mu, B_i\times\boldsymbol{R})$ is also projective. Therefore applying Lemma 1 to $f_{i,j}$ we obtain $E_{i,j}\in B(X)^\mu$. As $E_i = \bigcap_{j\in\boldsymbol{N}} E_{i,j}$ we get $E_i\in B(X)$ and hence $A(F)\in B_\mu(X, Y)$.

PROPOSITION 4. *Suppose that (X, μ, Y) is projective. Then the following two conditions are equivalent*:

(i) $F|_{A(F)}$ *is* $B_\mu(X, Y)$*-measurable* (*where* $A(F)$ *is defined as in Proposition* 3)

(ii) *For every* $\varepsilon > 0$ *there exists* $K \subset X$ *closed with* $\mu(X \setminus K) < \varepsilon$ *and* $\Omega \subset X \times Y$ *open such that* $A(F) \cap K \times Y = \Omega \cap K \times Y$ *and* $F|_{\Omega \cap K \times Y}$ *is lower continuous.*

Proof. Implication (ii) $\Rightarrow$ (i) is obvious, so we pass to the converse. Fix $\varepsilon > 0$. Let G be the closed-valued multifunction from X into Y defined by $G(x) = \{y \in Y\colon (x, y) \in X \times Y \setminus A(F)\}$. Since $\operatorname{graph} G = X \times Y \setminus A(F) \in B_\mu(X, Y)$, it follows that G is $B(X)^\mu$-measurable. By Corollary 1 applied to G we can choose $X_0 \subset X$ closed with $\mu(X \setminus X_0) < \varepsilon/2$ such that $G|_{X_0}$ is upper continuous. Define a closed-valued multifunction F_0 from $X \times Y$ into Z by $F_0(x, y) = F(x, y)$ for $(x, y) \in A(F) \cap X_0 \times Y$ and $F_0(x, y) = \emptyset$ otherwise. Then F_0 is $B_\mu(X, Y)$-measurable and $F_0(x, \cdot)$ is lower continuous for all $x \in X$.

Let $\{z_i\}_{i \in N}$ be a dense sequence in Z and let d be an admissible metric in Z. Define $f_i\colon X \times Y \to [-\infty, +\infty]$ by $f_i(x, y) = \inf\{d(z_i, a)\colon a \in F_0(x, y)\}$. Let G_i be the closed-valued multifunction from X into $Y \times \boldsymbol{R}$ given by $G_i(x) = Y \times \boldsymbol{R} \setminus \mathrm{Gr}^+\big(f_i(x, \cdot)\big)$. It follows (by an argument as in the proof of Lema 1) that G_i is $B(X)^\mu$-measurable. Use Corollary 1 to get a closed $K_i \subset X$ with $\mu(X \setminus K_i) < \varepsilon 2^{-i-1}$ such that $G_i|_{K_i}$ is upper continuous. Put $K_0 = \bigcap_{i \in N} K_i$. The function $f_i|_{K_0 \times Y}$ is upper semicontinuous because

$$K_0 \times Y \times \boldsymbol{R} \setminus \mathrm{Gr}^+(f_i|_{K_0 \times Y}) = \operatorname{graph}(G_i|_{K_0})$$

is closed. Hence $F_0|_{K_0 \times Y}$ is lower continuous. Put $K = K_0 \cap X_0$ and $\Omega = X \times Y \setminus \operatorname{graph} G|_K$. Since

$$X \times Y \setminus \operatorname{graph} G|_K = (X \setminus K) \times Y \cup A(F)$$

we have $\Omega \cap K \times Y = A(F) \cap K \times Y$. For U open in Z the set $(F|_{\Omega \cap K \times Y})^-(U) = (F_0|_{K \times Y})^-(U)$ is open in $K \times Y$ and hence in $\Omega \cap K \times Y$.

COROLLARY 5 (Scorza Dragoni type theorem — lower continuous and continuous cases, cf. [Fr] and [Rz]). *Suppose that (X, μ, Y) is projective. Then the following two conditions are equivalent*:

(i) F is $B_\mu(X, Y)$*-measurable and* $F(x, \cdot)$ *is* (*lower*) *continuous for a.e.* $x \in X$.

(ii) *For every* $\varepsilon > 0$ *there exists* $K \subset X$ *closed with* $\mu(X \setminus K) < \varepsilon$ *such that* $F|_{K \times Y}$ *is* (*lower*) *continuous.*

Proof. Implication (ii) $\Rightarrow$ (i) is obvious in both cases. To get the converse choose closed $X_0 \subset X$ (by the same argument as in the proof of Corollary 3) with $\mu(X \setminus X_0) < \varepsilon/2$ such that $F(x, \cdot)$ is (lower) continuous for all $x \in X_0$. In the lower continuous case apply Proposition 4 with $F_0 = F|_{X_0 \times Y}$ and with μ restricted to $B(X_0)$. In the continuous case use, moreover, Corollary 3 with F_0 and with μ restricted to $B(X_0)$. In order to verify the $B(X_0)^\mu$-measurability of G_0 observe that since $F_0(x, \cdot)$ is upper continuous it follows that $G_0(x) = \operatorname{graph} F_0(x, \cdot)$, and since $F_0(x, \cdot)$ is lower continuous we have $G_0^-(U \times V) = \bigcup_{y \in A \cap U} F_0(\cdot, y)^-(V)$ for any A dense in Y and $U \times V$ open in $Y \times Z$.

Now, we will state two useful properties of the operations on multifunctions which are not closely related with Lusin's theorems. We use the convention of Remark 8.

The intersection of a countable family of $\mathcal{T}$-measurable multifunctions is $\mathcal{T}$-measurable provided that Y is σ-compact. If, moreover, Y is a finite-dimensional Euclidean space, then the closed convex hull of a $\mathcal{T}$-measurable multifunction is $\mathcal{T}$-measurable.

The above properties result easily from [CV, pp. 63, 67 and 70].

2. Existence Theorem

By ab we denote the scalar product of two vectors a and b in $\mathbf{R}^n$ and by $|a| = \sqrt{aa}$ the corresponding norm. For $x \in \mathbf{R}^n$ and X and Y in $P(\mathbf{R}^n)$ we put $\operatorname{dist}(x, Y) = \inf\{|x-y|: y \in Y\}$ and $q(X, Y) = \sup\{\operatorname{dist}(x, Y): x \in X\}$. By $\operatorname{co} X$ we denote the convex hull of X. We put $h(Y, x) = \sup\{yx: y \in Y\}$. If Y is bounded and A is dense in $\mathbf{R}^n$ then $\operatorname{clco} Y = \{x \in \mathbf{R}^n: xa \leqslant h(Y, a)$ holds for $a \in A\}$. By $C(I)$ we denote the Banach space of continuous functions (from a compact interval $I = [a, b] \subset \mathbf{R}$ into $\mathbf{R}^n$) with the norm $\|u\| = \max\{|u(t)|: t \in I\}$. The following lemma or its variants have frequently been used in the proofs of existence of solutions for (1) [Fi1], [Pl2], [Da] or in the related closure theorems [Au]. We present a simple proof of it, which avoids the use of infinite-dimensional functional analysis.

LEMMA 2. *Let $\{x_k\}_{k \in \mathbf{N}}$ be a sequence in $C(I)$ converging to x_0. Suppose that x_k are absolutely continuous and that $|\dot{x}_k(t)| \leqslant m(t)$ holds for a.e. $t \in I$ (for $k \geqslant 1$) where m is integrable on I. Then x_0 is absolutely continuous and $\dot{x}_0(t) \in \operatorname{clco} F(t)$ for a.e. $t \in I$ where F is the multifunction from I into $\mathbf{R}^n$ defined by $F(t) = \{\dot{x}_k(t): k \geqslant 1\}$ if $\dot{x}_k(t)$ exists for $k \geqslant 1$ and $F(t) = \{0\}$ otherwise.*

Proof. We have

$$|x_k(s)-x_k(t)| \leqslant \int_s^t m(r)\,dr \quad \text{for } a \leqslant s \leqslant t \leqslant b \text{ and } k \geqslant 1,$$

and so for $k = 0$. Hence x_0 is absolutely continuous. Since $\dot{x}_k(t) \in F(t)$ for a.e. $t \in I$ and $k \in \boldsymbol{N}$ we have (for given $p \in \boldsymbol{R}^n$ and $k \geqslant 1$) $p\dot{x}_k(t) - h(F(t), p) \leqslant 0$ almost everywhere in I. Hence the absolutely continuous functions $t \to px_k(t) - \int_a^t h(F(r), p)\,dr$ are decreasing for $k \geqslant 1$ and so for $k = 0$. Consequently $p\dot{x}_0(t) \leqslant h(F(t), p)$ holds for a.e. $t \in I$. Using it with p in a countable dense subset of $\boldsymbol{R}^n$ we get $\dot{x}_0(t) \in \operatorname{cl}\operatorname{co} F(t)$ for a.e. $t \in I$.

By μ we denote the one-dimensional Lebesque measure restricted to $B(E)$ where $E \in B(\boldsymbol{R})$. We will say "measurable" instead of "$B(E)^\mu$-measurable" and we denote $\mathscr{T} = B_\mu(\boldsymbol{R}, \boldsymbol{R}^n)$. Let f be a function from $\boldsymbol{R} \times \boldsymbol{R}^n$ into $[0, +\infty]$. Then f is called ***locally integrably bounded*** if for every bounded subset K of $\boldsymbol{R} \times \boldsymbol{R}^n$ there exists a function p integrable on $\boldsymbol{R}$ such that $f(t, x) \leqslant p(t)$ holds for $(t, x) \in K$. We say that a multifunction from $\boldsymbol{R} \times \boldsymbol{R}^n$ into $\boldsymbol{R}^n$ is an *orientor field* if its values are closed and nonempty.

THEOREM 1. *Let F be an orientor field. Then the problem* (1) *has a solution provided that F satisfies the following regularity assumptions*:

(i) *The following condition holds for a.e. $t \in \boldsymbol{R}$. For all $x \in \boldsymbol{R}^n$ either $F(t, \cdot)$ is upper continuous at x and $F(t, x) = \operatorname{co} F(t, x)$ or $F(t, \cdot)$ restricted to some neighborhood of x is lower continuous.*

(ii) *F is $\mathscr{T}$-measurable and* $\operatorname{dist}(0, F(\cdot, \cdot))$ *is locally integrably bounded.*

The following proposition generalizes the essential part of the proof (presented in [Ło]) of a particular case of our theorem.

PROPOSITION 5. *Let $\{F_k\}_{k \in \boldsymbol{N}}$ be a sequence of lower continuous orientor fields and $\{D_k\}_{k \in \boldsymbol{N}}$ a sequence of closed subsets of $\boldsymbol{R}^{n+1}$ such that $F_k|_{D_k} = F_{k+1}|_{D_k}$ for $k \in \boldsymbol{N}$. Let X be a compact set in $C(I)$ and $\{\varepsilon_k\}_{k \in \boldsymbol{N}}$ a sequence of positive constants with $\sum_{k=1}^{\infty} \varepsilon_k < +\infty$.*

Then there exists a sequence $\{f_k\}_{k \in \boldsymbol{N}}$ of mappings from X into the space of piecewise constant functions from I into $\boldsymbol{R}$ such that

$$\operatorname{dist}\big(f_k(u)(t), F_k(t, u(t))\big) \leqslant \varepsilon_k$$

holds for $k \geqslant 1$, $t \in I$, $u \in X$, and if, moreover, $(t, u(t)) \in D_k$ then $|f_k(u)(t) - f_{k+1}(u)(t)| \leqslant \varepsilon_k$.

Additionally, for every $u \in X$ there exists a countable subset Z_u of I such that for each $t \in I \setminus Z_u$ and $k \geqslant 1$ the mapping $v \to f_k(v)(t)$ is continuous at u.

A multifunction F from $\boldsymbol{R}$ into $\boldsymbol{R}$ is called *simple* if $\operatorname{graph} F$ is a finite union of sets of the type $E \times \{x\}$ where E is a compact interval and $x \in \boldsymbol{R}$.

We omit an elementary proof of the following

LEMMA 3. *Let V and W be two simple multifunctions and v a piecewise constant function (defined on an interval I) such that $v(t) \in V(t)$ holds for $t \in I$. Suppose that $I \subset W^-(\boldsymbol{R}^n)$. Then there exists a piecewise constant function w such that $w(t) \in W(t)$ and $|v(t)-w(t)| \leqslant q\big(V(t), W(t)\big)$ holds for $t \in I$.*

LEMMA 4. *Let F and G be two lower continuous orientor fields and D a closed subset of $\boldsymbol{R}^{n+1}$ such that $F|_D = G|_D$. Choose $u \in C(I)$, $\beta > 0$ and a simple multifunction V with $I \subset V^-(\boldsymbol{R}^n)$. Suppose that $q\big(V(t), F\big(t, u(t)\big)\big) \leqslant \beta$ holds for $t \in I$. Then for every $a > 0$ there exists $\delta > 0$ and a simple multifunction W with $I \subset W^-(\boldsymbol{R}^n)$ such that $q\big(W(t), G\big(t, v(t)\big)\big) \leqslant a$ holds for all $t \in I$ and $v \in \operatorname{cl} B(u;\delta)$ and such that if $\big(t, v(t)\big) \in D$ for some $t \in I$ and some $v \in B(u;\delta)$ then $q\big(V(t), W(t)\big) \leqslant \beta$.*

Proof of Lemma 4. Put $X_0 = \bigcup_{t \in I} V(t)$ and $E(v) = \big\{t \in I\colon \big(t, v(t)\big) \in D\big\}$ for $v \in C(I)$. Let $x \in X_0$ and $S \in E(u) \cap V^-(\{x\})$. Choose $y_{x,s} \in F\big(s, u(s)\big)$ such that $|x - y_{x,s}| = \operatorname{dist}\big(x, F\big(s, u(s)\big)\big)$. Then $|x - y_{x,s}| \leqslant \beta$ and $y_{x,s} \in G\big(s, u(s)\big)$. Moreover, there exist $\varepsilon_{x,s} > 0$ such that $\operatorname{dist}\big(y_{x,s}, G\big(t, v(t)\big)\big) \leqslant a$ for $|t-s| \leqslant \varepsilon_{x,s}$ and $\|v-u\| \leqslant \varepsilon_{x,s}$ because the mapping $(t, v) \to \big(t, v(t)\big)$ is continuous and $\operatorname{dist}\big(y_{x,s}, G(\cdot,\cdot)\big)$ is u.s.c. By the compactness of $E(u) \cap V^-(\{x\})$ we can choose $i(x) \in \boldsymbol{N}$ and $s_i = s_{i,x} \in E(u) \cap V^-(\{x\})$ (with $i = 1, 2, \ldots, i(x)$) such that $E(u) \cap V^-(\{x\}) \subset J(x)$ where

$$J(x) = \bigcup_{i=1}^{i=i(x)} \{t\colon |t - s_{i,x}| < \varepsilon_{x,s_i}\}.$$

For $s \in I \setminus J(x)$ take $y_{x,s} \in G\big(s, u(s)\big)$. By similar arguments there exist $\varepsilon_{x,s} > 0$ such that $\operatorname{dist}\big(y_{x,s}, G\big(t, v\big)\big) \leqslant a$ for $|t-s| \leqslant \varepsilon_{x,s}$ and $\|v-u\| \leqslant \varepsilon_{x,s}$; moreover, we can choose $j(x) \in \boldsymbol{N}$ and $s_{i,x} \in I \setminus J(x)$ (with $i = i(x)+1, \ldots \ldots, j(x)$) such that

$$I \setminus J(x) \subset \bigcup_{i=i(x)+1}^{i=j(x)} \{t\colon |t - s_{i,x}| < \varepsilon_{x,s_i}\}.$$

Define W by putting

$$\operatorname{graph} W = \bigcup_{x \in X_0} \bigcup_{i=1}^{i=j(x)} [s_{i,x} - \varepsilon_{x,s_i}, s_{i,x} + \varepsilon_{x,s_i}] \times \{y_{x,s_i}\}.$$

Obviously $I \subset W^-(\boldsymbol{R}^n)$, W is simple and $q\big(W(t), G\big(t, v(t)\big)\big) \leqslant a$ for $t \in I$ and $\|v-u\| \leqslant \varepsilon$ where $\varepsilon = \min\{\varepsilon_{x,s_i}: i \leqslant j(x), x \in X_0\}$. The multifunction $v \to E(v) \cap V^-(\{x\})$ is strongly upper continuous because its graph is closed and I is compact. Hence there exists $\delta_x > 0$ such that $E(v) \cap V^-(\{x\}) \subset J(x)$ for $\|v-u\| \leqslant \delta_x$. Put $\delta = \min\{\varepsilon, \delta_x: x \in X_0\}$. Fix v and t satisfying $\big(t, v(t)\big) \in D$ and $\|v-u\| < \delta$. Since for $x \in V(t)$ we have $t \in J(x)$, it follows that dist $\big(x, W(t)\big) \leqslant |x - y_{x,s_i}| \leqslant \beta$ for some $i \leqslant i(x)$ and so $q\big(V(t), W(t)\big) \leqslant \beta$.

Proof of Proposition 5. Define an orientor field F_0 by putting $F_0(t, x) = \{0\}$. Choose $\varepsilon_0 > 0$ and put $D_0 = \emptyset$. We will construct by induction a sequence $\{A_k\}_{k \geqslant 0}$ of finite subsets of N^{k+1} and for every $\gamma \in A_k$ we will choose $r_\gamma > 0$, $u_\gamma \in X$ satisfying $X \subset \bigcup_{\gamma \in A_k} B(u_\gamma; r_\gamma)$ and a simple multifunction V_γ with $I \subset V_\gamma^-(\boldsymbol{R}^n)$ such that $q\big(V_\gamma(t), F_k\big(t, u(t)\big)\big) \leqslant \varepsilon_k$ holds for $t \in I$ and $u \in \mathrm{cl}\, B(u_\gamma; r_\gamma)$. Moreover, the above sequences will enjoy the following two additional properties:

I. $X \cap \mathrm{cl}\, B(u_\gamma; r_\gamma) \subset \bigcup_{(\gamma,i) \in A_{k+1}} B(u_{(\gamma,i)}; r_{(\gamma,i)})$ holds for $k \geqslant 0$ and $\gamma \in A_k$.

II. If $\big(t, u(t)\big) \in D_k$ for some $k \geqslant 0$, $t \in I$ and $u \in B(u_{(\gamma,i)}; r_{(\gamma,i)})$ with some $\gamma \in A_k$ and $(\gamma, i) \in A_{k+1}$ then $q\big(V_\gamma(t), V_{(\gamma,i)}(t)\big) \leqslant \varepsilon_k$.

For this purpose put $A_0 = \{1\}$ and choose $u_1 \in X$ and $r_1 > 0$ such that $X \subset B(u_1; r_1)$. Define V_1 by putting $V_1(t) = \{0\}$ for $t \in I$ and $V_1(t) = \emptyset$ otherwise. Next, fix $k \geqslant 0$ and suppose that we have defined A_k and $r_\gamma, u_\gamma, V_\gamma$ with $I \subset V_\gamma^-(\boldsymbol{R}^n)$ for $\gamma \in A_k$ satisfying $X \subset \bigcup_{\gamma \in A_k} B(u_\gamma; r_\gamma)$ and such that $q\big(V_\gamma(t), F_k\big(t, u(t)\big)\big) \leqslant \varepsilon_k$ holds for $t \in I$ and $U \in \mathrm{cl}\, B(u_\gamma; r_\gamma)$. Applying Lemma 4 (with $F = F_k$, $G = F_{k+1}$, $D = D_k$, $V = V_\gamma$, $u \in \mathrm{cl}\, B(u_\gamma, r_\gamma)$, $\beta = \varepsilon_k$, $a = \varepsilon_{k+1}$) we get $\delta = \delta_u^\gamma$ and $W = W_u^\gamma$ satisfying the assertion of the lemma. By compactness of $X \cap \mathrm{cl}\, B(u_\gamma; r_\gamma)$ we can choose $m(\gamma) \in N$ and $u_i = u_{i,\gamma} \in X \cap \mathrm{cl}\, B(u_\gamma; r_\gamma)$ (with $i = 1, 2, \ldots, m(\gamma)$) such that

$$X \cap \mathrm{cl}\, B(u_\gamma; r_\gamma) \subset \bigcup_{i=1}^{i=m(\gamma)} B(u_i; \delta_{u_i}^\gamma).$$

Define $A_{k+1} = \{(\gamma, i) \in N^{k+2}: \gamma \in A_k \text{ and } i \leqslant m(\gamma)\}$ and for $(\gamma, i) \in A_{k+1}$ put $r_{(\gamma,i)} = \delta_{u_i}^\gamma$, $u_{(\gamma,i)} = u_i$ and $V_{(\gamma,i)} = W_{u_i}^\gamma$. As

$$X \cap \mathrm{cl}\, B(u_\gamma; r_\gamma) \subset \bigcup_{(\gamma,i) \in A_{k+1}} B(u_{(\gamma,i)}; r_{(\gamma,i)})$$

holds for $\gamma \in A_k$ from the induction assumption, we get $X \subset \bigcup_{\delta \in A_{k+1}} B(u_\delta; r_\delta)$. The other required properties follow from the assertion of Lemma 4.

Next, for $k \geqslant 0$ let p_γ (with $\gamma \in A_k$) be a continuous partition of unity on X subordinate to the open covering given by $X \subset \bigcup_{\gamma \in A_k} B(u_\gamma; r_\gamma)$

and satisfying additionally $p_\gamma = \sum_{(\gamma,i)\in A_{k+1}} p_{(\gamma,i)}$. We are going to construct it. For every $\gamma \in A_k$, $k \geqslant 0$ let q_γ^i be a continuous partition of unity on $X \cap \mathrm{cl}\, B(u_\gamma; r_\gamma)$ subordinate to the open covering given by

$$X \cap \mathrm{cl}\, B(u_\gamma; r_\gamma) \subset \bigcup_{(\gamma,i)\in A_{k+1}} B(u_{(\gamma,i)}, r_{(\gamma,i)}).$$

Put $q_\delta = q_\gamma^i$ for $\delta = (\gamma, i) \in A_{k+1}$ and $q_\delta(u) = 1$ for $u \in X$ and $\delta \in A_0$. Let $\hat{q}_\delta$ be a continuous extension of q_δ on X. We define p_γ by putting

$$p_\gamma(u) = \hat{q}_{\gamma_1}(u)\hat{q}_{(\gamma_1,\gamma_2)}(u) \ldots \hat{q}_{(\gamma_1,\ldots,\gamma_{k+1})}(u) \text{ for } u \in X$$

where $\gamma = (\gamma_1, \ldots, \gamma_{k+1}) \in A_k$. Obviously $p_{(\gamma,i)}(u) \neq 0$ implies $p_\gamma(u) \neq 0$. We claim that $p_\gamma(u) \neq 0$ implies $u \in B(u_\gamma; r_\gamma)$. For $\gamma \in A_0$ this follows since $X \subset B(u_1; r_1)$. Fix $k \geqslant 0$ and suppose that our claim holds for all $\gamma \in A_k$. Let $\delta \in A_{k+1}$ and $u \in X$ satisfy $p_\delta(u) \neq 0$. Take $\gamma \in A_k$ and $i \in N$ such that $\delta = (\gamma, i)$. Then $p_\gamma(u) \neq 0$ and $\hat{q}_\delta(u) \neq 0$. By the induction assumption we get $u \in B(u_\gamma; r_\gamma)$. Hence $0 \neq \hat{q}_\delta(u) = q_\delta(u) = q_\gamma^i(u)$ and therefore $u \in B(u_\delta; r_\delta)$. From our claim we get $p_\gamma(u) = q_{\gamma_1}(u) \ldots q_{(\gamma_1,\ldots,\gamma_{k+1})}(u)$ provided that $p_\gamma(u) \neq 0$. Hence $p_\gamma(u) \geqslant 0$ and

$$p_\gamma(u) = \sum_{(\gamma,i)\in A_{k+1}} p_{(\gamma,i)}(u)$$

holds for all $u \in X$ because for $p_\gamma(u) = 0$ the equality is obvious. It follows that

$$\sum_{\gamma\in A_{k+1}} p_\gamma(u) = \sum_{\gamma\in A_k} p_\gamma(u) = \ldots = \sum_{\gamma\in A_0} p_\gamma(u) = 1.$$

Define for every $k \geqslant 0$ and $u \in X$ a partition $I_\gamma(u)$ (with $\gamma \in A_k$) of the interval $[a, b[$ by putting

$$I_\gamma(u) = \Big[a + (b-a) \sum_{\substack{\delta\in A_k\\ \delta<\gamma}} p_\delta(u),\ a + (b-a) \sum_{\substack{\delta\in A_k\\ \delta\leqslant\gamma}} p_\delta(u)\Big[.$$

where the lexicographic order in A_k is used. Then

$$I_\gamma(u) = \bigcup_{(\gamma,i)\in A_{k+1}} I_{(\gamma,i)}, \quad I_\gamma(u) \neq \emptyset \text{ implies } u \in B(u_\gamma; r_\gamma)$$

and if $t \in \mathrm{int}\, I_\gamma(u)$ then $t \in \mathrm{int}\, I_\gamma(v)$ for v close to u. Using Lemma 3 we can choose a family v_γ (with $\gamma \in A_k$, $k \geqslant 0$) of piecewise constant functions such that $v_\gamma(t) \in V_\gamma(t)$ and $|v_\gamma(t) - v_{(\gamma,i)}(t)| \leqslant q\big(V_\gamma(t), V_{(\gamma,i)}(t)\big)$ holds for $t \in I$ and $(\gamma, i) \in A_{k+1}$.

For $k \in N$ and $u \in X$ define on I a piecewise constant function $f_k(u)$ by putting $f_k(u)(t) = v_\gamma(t)$ if $t \in I_\gamma(u)$ for some $\gamma \in A_k$ and $f_k(u)(b)$ by the following requirement: $f_k(u)(b) = f_{k-1}(u)(b)$ if $\big(b, u(b)\big) \in D_{k-1}$ and $f_k(u)(b) \in F_k\big(b, u(b)\big)$ otherwise. Then $\{f_k\}_{k\in N}$ satisfies the first part of the conclusion of the proposition. For fixed $u \in X$, $k \in N$ and $t \in I$ the mapping $v \to f_k(v)(t)$ is constant in some neighborhood of u provided that $t \in \bigcup_{\gamma \in A_k} \operatorname{int} I_\gamma(u)$. Therefore we put

$$Z_u = \bigcup_{k \in N} \big(I \setminus \bigcup_{\gamma \in A_k} \operatorname{int} I_\gamma(u)\big).$$

Proof of Theorem 1. Without loss of generality we can assume $t_0 = 0$ and $x_0 = 0$. Let m be an integrable bound for the function $1 + \operatorname{dist}(0, F(\cdot, \cdot))$ restricted to $[-1, 1] \times B(0; 1)$. Choose $-1 < a < 0 < b < 1$ so small that $\int_a^b m(t)\,dt < 1$. Denote $I = [a, b]$ and $Y = B(0; 1)$. Apply Proposition 3 to F and define a multifunction G from $I \times Y$ into R by putting $G(t, x) = F(t, x) \cap \operatorname{cl} B(0; m(t))$ for $(t, x) \in I \times Y \setminus A(F)$ and $G(t, x) = \operatorname{cl}[F(t, x) \cap \cap B(0; m(t))]$ for $(t, x) \in A(F) \cap I \times Y$. Then G is $\mathscr{T}$-measurable and $G(t, x) \neq \emptyset$ for $(t, x) \in I \times Y$. Fix $c > 0$. Applying Proposition 4 to G we get a compact subset $K = K(c)$ of I and a subset $\Omega = \Omega(c)$ of $I \times Y$ (open in $I \times Y$) such that $\mu(I \setminus K) < c$, $K \times Y \cap A(G) = K \times Y \cap \Omega$ and $G|_{K \times Y \cap \Omega}$ is lower continuous. Since the set $(R \times Y) \cap \Omega$ is open in R^{n+1} we can choose a sequence of compact subsets of R^{n+1} such that $L_k \subset \operatorname{int} L_{k+1}$ and $\bigcup L_k = (R \times Y) \cap \Omega$. Put $D_k = K \times R \cap L_k$ for $k \in N$. Then $\bigcup D_k = K \times Y \cap \Omega$. Define an orientor field F_k by putting $F_k(t, x) = G(t, x)$ for $(t, x) \in D_k$ and $F_k(t, x) = R^n$ otherwise. Since D_k is closed F_k is lower continuous. Define

$$X = \{v \in C(I)\colon v \text{ is absolutely continuous, } v(0) = 0 \text{ and } |\dot v(t)| \leqslant m(t) \text{ for a.e. } t \in I\}.$$

Then by Lemma 2 and the Arzelà–Ascoli theorem X is compact. Apply Proposition 5 and define g_c for all $u \in X$ and $t \in I$ satisfying $(t, u(t)) \in \in K \times Y \cap A(G)$ by $g_c(u)(t) = \lim_{k \to \infty} f_k(u)(t)$. Choose $c = c_i \to 0$ and define E_i by $E_1 = K(c_1)$ and $E_{i+1} = K(c_{i+1}) \setminus \bigcup_{j \leqslant i} E_j$ for $i \in N$. For $(t, u) \in I \times X$ define $H(u)(t)$ by putting

$$H(u)(t) = \begin{cases} \{g_{c_i}(u)(t)\} & \text{if } (t, u(t)) \in E_i \times Y \cap A(G), \\ G(t, u(t)) & \text{otherwise}. \end{cases}$$

One can easily check the measurability of the multifunction $t \to H(u)(t)$. We claim that for given $u \in X$ and a.e. $t \in I$ the multifunction $H(\cdot)(t)$ is upper continuous at u. Fix $u \in X$. Since, for given $c > 0$, $g_c(\cdot)(t)$ is continuous at u for $t \in I \setminus Z_u$ (Z_u from Proposition 5) satisfying $(t, u(t)) \in \in K(c) \times Y \cap A(G)$ (because, since $L_k \subset \operatorname{int} L_{k+1}$, for k_0 sufficiently large we have $(t, v(t)) \in D_k$ for $k \geqslant k_0$ and v close to u, and because the limit passage determining $g_c(\cdot)(t)$ is uniform), $\mu(I \setminus \bigcup E_i) = 0$ and t-sections of $A(G)$ are open; therefore $H(\cdot)(t)$ is continuous at u for a.e. $t \in I$ satisfying $(t, u(t)) \in A(G)$. Observe that $I \times Y \cap A(F) \subset A(G)$ and that for a.e.

$t \in I$ if $(t, x) \notin A(F)$ then F and hence G is upper continuous at x and $G(t, x)$ is convex. Since, moreover, $g_c(u)(t) \in G(t, u(t))$ whenever defined, it follows that $H(\cdot)(t)$ is upper continuous at u for a.e. $t \in I$ satisfying $(t, u(t)) \notin A(G)$. So our claim is verified and, moreover, the values of $H(u)(\cdot)$ are closed, convex and nonempty for a.e. $t \in I$. Define a multifunction S from X into X by putting $S(u) = \{v \in X: \dot{v}(t) \in H(u)(t)$ for a.e. $t \in I\}$ for $u \in X$. The sets X and $S(u)$ are convex (trivially) and compact by Lemma 2 and the Arzelà–Ascoli theorem. Let $f(t)$ be the point in $H(u)(t)$ uniquely determined for a.e. $t \in I$ by $\operatorname{dist}(0, H(u)(t)) = |f(t)|$.

Then f is measurable and the function $t \to \int_0^t f(r)\,dr$ belongs to $S(u)$. So $S(u) \neq \emptyset$. Let us prove that graph S is closed. Take $u_k \to u_0$ and $v_k \to v_0$ with $v_k \in S(u_k)$. By Lemma 2 applied to $\{v_k\}_{k \geqslant i}$ we get

$$\dot{v}_0(t) \in \bigcap_{i \geqslant 1} \operatorname{cl} \operatorname{co} \{\dot{v}_k(t): k \geqslant i\} \subset \bigcap_{i \geqslant 1} \operatorname{cl} \operatorname{co} \bigcup_{k \geqslant i} H(u_k)(t) \subset H(u_0)(t)$$

for a.e. $t \in I$ (the last inclusion follows from the upper continuity of $H(\cdot)(t)$ since $H(u_0)(t)$ is a nonempty, convex, compact set for a.e. $t \in I$). Hence $v_0 \in S(u)$. By Kakutani–Ky-Fan fixed point theorem [Be, p. 270] there exists $x \in X$ such that $x \in S(x)$, i.e. x is absolutely continuous, $x(0) = 0$ and $\dot{x}(t) \in H(x)(t) \subset F(t, x(t))$ for a.e. $t \in I$. So x is a solution of the problem (1).

COROLLARY 6 (cf. [Ol]). *Let F be an orientor field. Suppose that* (i) *holds true and in* (ii) *replace the $\mathcal{T}$-measurability of F by the measurability of $F(\cdot, x)$ for $x \in \mathbf{R}^n$. Assume, moreover, that $F(t, \cdot)$ is upper continuous for a.e. $t \in \mathbf{R}^n$. Then the problem* (1) *has a solution.*

Proof. Let p be a locally integrably bounded function from $\mathbf{R} \times \mathbf{R}^n$ into $[1, +\infty[$ continuous in x, measurable in t and such that for a.e. $t \in \mathbf{R}$, $\operatorname{dist}(0, F(t, x)) < p(t, x)$ holds for all $x \in \mathbf{R}^n$. Let Y be a dense countable subset of $\mathbf{R}^n$. We define two multifunctions G and H by putting:

$$G(t, x) = \bigcap_{k \in N} \operatorname{cl} \bigcup_{\substack{y \in Y \\ |y - x| < 1/k}} F(t, y) \quad \text{for } (t, x) \in \mathbf{R}^{n+1},$$

$$H(z) = \begin{cases} \operatorname{cl}\big(G(z) \cap B(0; p(z))\big) & \text{if } z \in A(G), \\ \operatorname{co}\big(G(z) \cap \operatorname{cl} B(0; p(z))\big) & \text{if } z \in \mathbf{R}^{n+1} \setminus A(G), \end{cases}$$

where $A(\cdot)$ is defined in Proposition 3.

Then H satisfies (i) and (ii) from the theorem and for a.e. $t \in \mathbf{R}$ we have $\emptyset \neq H(t, x) \subset F(t, x)$ for all $x \in \mathbf{R}^n$.

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