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MATHEMATICAL FOUNDATION OF FLOW OF GLACIERS AND LARGE ICE MASSES

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In this lecture series on the mathematical foundation of flow of glaciers and large ice masses an attempt is made to present a rigorous review of the basic problems that are encountered when one is trying to formulate a rational model for the description of the physical behavior of these objects. Polythermal ice masses constitute cold and temperate zones, in which a one-component or a mixture concept are appropriate as the continuum mechanical model. Problems then arise what equations must be formulated in the two zones and at the cold-temperate transition surface. The first two sections are devoted to the deduction of the basic model from the general postulates of a continuum theory of binary mixtures. The results of these developments are new and prove those of earlier authors to be wrong. In the remaining four sections the cold ice model is then applied to the determination of stress and velocity distributions and of the surface profile. The application of the suggested approximate perturbation schemes to three-dimensional ice sheets is new; it offers interesting results which can easily be subject to verification in the field.

The lecture notes can, obviously not be complete, but mathematical problems are clearly stated and up to date as far as 1981/82. Important unsolved problems are indicated.

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Introduction

It is one purpose of glaciology to understand and describe how glaciers and ice sheets *flow* and in what sense their behavior can be related to that of the *geophysical environment*, the atmosphere and the substratum which the glacier is situated on. Glacier and ice sheet motion is primarily due to the action of gravity, however thermal effects are equally important as they cannot be ignored when stresses and velocities are determined. Typically, the interaction between the geophysical environment and the glacier is from the environment to the ice mass and not vice versa, in other words the thermal conditions of the environment affect the motion of the ice mass, but the feedback from the ice mass to the environment is generally ignored.

To date a realistic rational formulation of ice flow problems is still hampered by uncertainities in boundary and body flow conditions. For often there is only limited knowledge about the basal geometry; moreover, even if the geometry is known, it is not clear what mechanical or thermal boundary conditions should apply at the base. Second, ice in glaciers consists of clustered randomly oriented hexagonal ice crystals which have grown from snow by transformation processes through compaction under various stress states and thermal conditions. If one therefore assumes *creep flow* to be describable by an *isotropic fluid model*, this amounts to important simplifications, which often are found to be only qualitatively correct. Finally, uncertainities arise because it is very difficult to measure the quantities responsible for the driving of the system; these are accumulation due to snow fall, ablation due to melting and the geothermal heat.

Conceptually, glaciers and ice sheets are similar physical objects. They are usually grounded, but frequently they extend into the ocean or lake, and then they are referred to as a *shelf*. Distinction between glaciers and ice sheets is largely one of size, see Table I for scales, but typically glaciers are situated on a bed with non-negligible mean inclination; they flow in one direction, the downhill direction, and possess no ice divide. By contrast, ice sheets may often be assumed to lie on a horizontal bed; they have summits from which the ice flows in all directions. Such summits therefore act as ice divides.

Furthermore, glaciers are often confined to valleys; their flow is one-

Table I
Typical scales

	glacier	ice sheet
lenght	10–50 km	1000 km
depth	100–400 m	100–4000 m
celerity	100 m/a	a few cm/a

dimensional, and typical *non-surging* velocities are in the order of 100 m/a. Only rearly and when the glacier is going through a *surging state* are the celerities an order of magnitude larger. By contrast, ice sheets spread in two dimensions; their extent is in the order of 1000 km, and ice celerities are smaller, from a few centimeters per year to several meters per year, depending on geometrical and thermal conditions. The description of such broad a spectrum of physical conditions depends on a variety of parameters, and it is theoretically desirable that a single mathematical model is capable of explaining all pertinent velocity scales.

In order to arrive at a mathematical description of ice flow in large ice masses that allows a quantitative exploration, one must focus attention on a sufficiently simplified model. Local phenomena both in space and time, which manifest themselves in stress concentrations, crevasse formation etc., must be ignored. Time scales of relevance are years and not seconds, and this implies that seismic waves will not be covered. Exclusion of such phenomena delimits the aplication of the mathematical model, but it will be shown that even with such restriction the mathematical solutions of physically relevant problems will be very difficult.

Books and reviews on the topic treated in these lectures are by Colbeck (1980), Hutter (1982, 1983), Lliboutry (1964, 1965, 1971), Paterson (1981) and Shumskiy (1969). Paterson's book is a good introduction into the physics of glaciers, Hutter's work is more mathematical.

In what follows we shall use symbolic and Cartesian tensor notation interchangingly. Often, restriction will be made to *plane* flow. The Cartesian axes are then denoted by x and y, respectively; x will be measured parallel to a mean basal direction approximately coinciding with the forward velocity direction. Figure 1 is a sketch of an ice sheet or glacier with a grounded and floating portion.

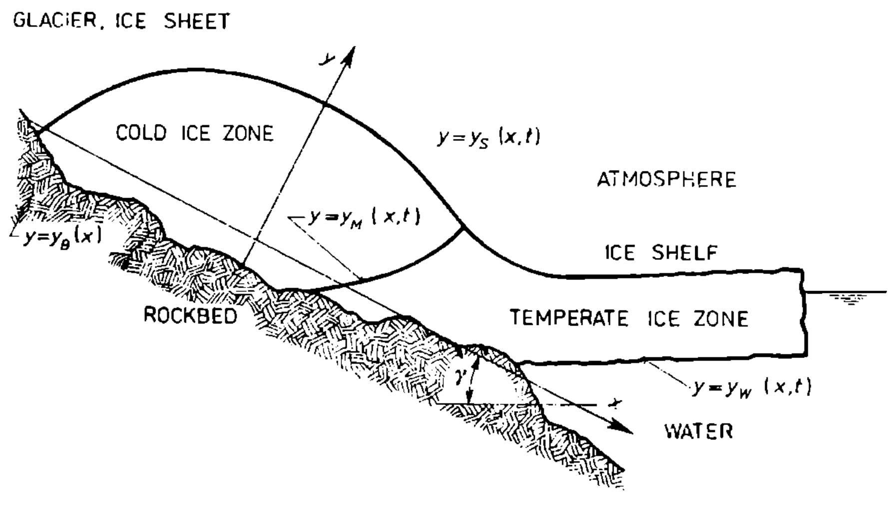

Fig. 1. Geometry of an ice sheet with grounded and floating portion (schematic)

2. Field equations and boundary conditions

The mathematical formulation of mechanical and thermodynamical statements for large ice masses are complicated by the fact that glaciers and ice sheets are in general *polythermal*: i.e., they constitute two zones, "cold" and "temperate", in which the ice is respectively below and at the melting point. In the cold zone heat generated by internal friction will affect the temperature distribution, and the latter in turn will influence the motion. In the temperate zone, on the other hand, frictional heat will melt some ice. Hence, whereas for cold glaciers a fluid model of a heat conducting viscous body may be an appropriate thermomechanical model, such cannot be for temperate ice whose description must bear some notion of a binary mixture of ice with percolating or trapped water. In a polythermal ice mass there are therefore four different boundaries (see Fig. 1), namely the base $y = y_B(x)$, the free surface $y = y_S(x, t)$, the ice-water interface at the floating portions, $y = y_W(x, t)$, and finally, the transition surface between cold and temperate ice, $y = y_M(x, t)$.

It is our goal to formulate, firstly, the field equations in the cold and temperate portions of the ice mass and secondly, to establish suitable boundary conditions for the four different bounding surfaces. Clearly, existence of cold and temperate subregions in the entire ice mass complicates the formulation. In the cold zone energy balance serves as an evolution equation for temperature and forms a crucial physical statement. In the temperate zone, on the other hand, energy balance is not as crucial except that production of internal energy governs the mass production of the constituents ice and water. Here it is the balance of mass of water, which replaces the energy equation. Further, the separating surface between cold and temperate ice in *non-material*, in general, and thus capable of propagating at its own speed. Depending on the thermal conditions, such surfaces may be created or annihilated. Strictly speaking, the remaining boundary surfaces are also non-material. For instance, at the free surface ice is added or subtracted by accumulation and surface ablation, respectively; a similar statement also holds for the ice-water interface where melting and freezing may occur depending on the thermal conditions in the ice and the ocean. To date, no clear formulation of this difficult boundary value problem exists, and we give an attempt of its derivation.

a) Body flow

α) Cold ice region. The common continuum mechanical model adopted for cold ice is a *non-Newtonian, viscous, heat conducting, incompressible fluid*. The field equation are therefore

$$
\begin{aligned}
&\operatorname{div} \boldsymbol{u} = 0, \\
&\varrho \dot{\boldsymbol{u}} = -\operatorname{grad} p + \operatorname{div} \boldsymbol{t}' + \varrho \boldsymbol{g}, \qquad (2.1)\\
&\varrho \dot{\varepsilon} = \operatorname{tr}(\boldsymbol{t}' \boldsymbol{D}) - \operatorname{div} \boldsymbol{q},
\end{aligned}
$$

and express local balance of mass, momentum and internal energy. In the above, $\boldsymbol{u}$ is the velocity vector, ϱ the density of ice, p pressure, $\boldsymbol{t}'$ symmetric Cauchy stress deviator, $\boldsymbol{g}$ vector of external forces, ε internal energy, $\boldsymbol{q}$ heat flux vector and $\boldsymbol{D} = \operatorname{sym}\operatorname{grad}\boldsymbol{u}$ the stretching tensor. In the subsequent analysis total stress will always be denoted by $\boldsymbol{t}$ without an accent, thus $\boldsymbol{t} = \boldsymbol{t}' - p\boldsymbol{1}$.

The balance laws (2.1) are complemented by constitutive relations for the rate of internal heat, heat flux and stress. Thermodynamic considerations then permit the deduction of the most general *admissible* constitutive relationships for the class of non-Newtonian fluids considered here, but for flow of large ice masses these are further reduced and simplified. The laws commonly adopted are

$$\begin{aligned} &\varrho\dot{\varepsilon} = \varrho c_p \dot{T} \qquad [\text{or } \varrho\varepsilon = \varrho\varepsilon(T_0) + \varrho c_p(T - T_0)], \\ (2.2) \qquad &\boldsymbol{q} = -\varkappa \operatorname{grad} T, \\ &\boldsymbol{D} = A(T) f(t'_{\mathrm{II}})\boldsymbol{t}', \qquad t'_{\mathrm{II}} = \tfrac{1}{2}\operatorname{tr}\boldsymbol{t}'^2, \end{aligned}$$

in which T is Kelvin temperature, T_0 a reference temperature, c_p specific heat at constant pressure and $\varkappa$ the heat conductivity of ice. Further, f is a creep response function, assumed to depend on the second stress-deviator invariant t'_{II} and A is a rate factor, which depends on temperature only. In view of the small temperature range occuring in this geophysical approximation a constant specific heat and a linear Fourier-type heat conduction law are sufficiently accurate. The constitutive relationship relating stress deviator, and stretching is solved for $\boldsymbol{D}$ as a function of $\boldsymbol{t}'$. It is a special case of the more general *Reiner–Rivlin type* constitutive law

$$(2.3) \qquad \boldsymbol{D} = -\tfrac{2}{3}\bar{g}(t'_{\mathrm{II}}, t'_{\mathrm{III}}, T)t'_{\mathrm{II}}\boldsymbol{1} + \bar{f}(-)\boldsymbol{t}' + \bar{g}(-)\boldsymbol{t}'^2$$

in which $\boldsymbol{1}$ is the identity tensor, and t'_{II} and t'_{III} denote the second and third invariants of the stress deviator $\boldsymbol{t}'$. It is known that in non-circular channels the constitutive relationship of the form (2.3) gives rise to secondary flow unless $\bar{g}(\cdot) \equiv 0$ and $\bar{f}(\cdot)$ does not depend on t'_{III}. Observations of valley glaciers do not provide any clue as to the existence of such secondary flow, (side and middle moraines do not spread horizontally). Provided that the constitutive relationship (2.3) reasonably models ice flow under creeping motion, it is therefore justified to set $\bar{g}(\cdot) \equiv 0$ and to ignore a depencence of $\bar{f}(\cdot)$ on t'_{III}. This brings us back to the law $(2.2)_3$. For time scales which are in the order of years, Equation $(2.2)_3$ is sufficiently accurate, see Hutter (1982, 1983), Morland and Spring (1982). The crucial material parameters are therefore $f(\cdot)$ and A. There is a considerable literature on the creep response function $f(\cdot)$ (see Glen (1955); Hawkes and Mellor (1972); Hobbs (1974); Mellor (1980); Michel (1979); Steinemann (1958)). Classically, $f(x^2) = |x|^{(n-1)}$ with $n = 1, 7 \rightarrow 4$, depending on stress range; this power law is called *Glen's flow law* but is better known among metallurgists as *Norton's* *law*, (1929). The slope singularity of the power law, $d[f(x^2)x]/dx \to \infty$ as $x \to 0$, causes unwanted singularities in many significant boundary value problems. This is one reason, the other being experimental evidence, that the power law is abandoned and a creep response function with finite slope at the origin is used. Polynomials (Meier (1960); Lliboutry (1969); Colbeck and Evans (1973)) and hyperbolic sines (Barnes et al. (1971); Assur (1980); Hutter (1980a, b)) have been suggested. Hutter (1980a) and Thompson (1979) use the simplest possible polynomial extension.

$$f(x) = x^{(n-1)/2} + \mathfrak{f} \tag{2.4}$$

accounting for quasi-Newtonian behavior at low stresses, $\mathfrak{f}$ having the dimension of the inverse of a viscosity.

The other important parameter in the stress-stretching law is the *rate factor* A. Usual functional relationships are of the Arrhenius-type, see Hobbs (1974),

$$A = \bar{A} \exp\left(-\frac{Q}{kT}\right), \tag{2.5}$$

where Q is activation energy, k Boltzmann's constant and $\bar{A}$ a constant. It is through this factor that a significant thermomechanical coupling occurs.

When the above constitutive relations are substituted into the balance laws (2.2) the *field equations* for the unknown fields $\boldsymbol{u}$ and T in the cold portion of the ice mass are obtained. For instance, the energy equation reads

$$\varrho c_p \dot{T} = \varkappa \nabla^2 T + 2A(T) f(t'_{II}) t'_{II}. \tag{2.6}$$

The second term on the right hand side is the *dissipation*, often called *strain heating*. When (2.6) is viewed as a balance equation for internal energy this term must be interpreted as the *internal energy production*.

β) *Temperate ice region*. Temperate ice is defined to be *at melting*, $T = T_M$. When such ice is subject to deformation, heat generated by viscous deformation cannot simply give rise to temperature changes, because temperature will be governed by the Claususs-Clapeyron equation. Most frictional heat produced by the ice will be used up by melting, implying that in the temperate zone there is not just temperate ice but a *binary mixture* of ice and molten water. A proper description must take account of this two phase character of the material, but, of course, since the ice-water ratio is very large, a mixture concept, in which the water content is treated as a *tracer* may adequately describe the processes in the temperate region. Further, at first glance an energy balance statement is not important either because the melting temperature of the mixture as a whole may be related to the total pressure.

An adequate mixture concept is therefore one with two mass balance equations for the mixture as whole and for constituent water, but only one balance equation of momentum for the total mixture. If, furthermore, it is assumed that *all energy production in the mixture is instantly used up by melting*, energy production and mass production can easily be interrelated with the aid of the *latent heat of fusion.*

It follows from general mixture theory (see article by I. Müller in this volume) that the blance equations of momentum and energy for the mixture as a whole have the form $(2.1)_{2,3}$, whereby all quantities are now those of the mixture. In particular, ϱ is total density, and the dot signifies *total derivative with respect to the barycentric velocity*,

$$\dot{f} = \frac{\partial f}{\partial t} + \operatorname{grad} f \cdot \boldsymbol{u}. \tag{2.7}$$

Balance of mass needs special consideration. To this end let ϱ_α and $\boldsymbol{u}_\alpha$ ($\alpha = 1, 2$) be the constituent densities and velocities, respectively, and define, as usual, total density and barycentric velocity as $\varrho = \sum_\alpha \varrho_\alpha$ and $\varrho\boldsymbol{u} = \sum_\alpha \varrho_\alpha \boldsymbol{u}^\alpha$. Then, a straightforward calculation shows that the balance laws of mass

$$\frac{\partial \varrho_\alpha}{\partial t} + \operatorname{div}(\varrho_\alpha \boldsymbol{u}^\alpha) = \mathfrak{C}_\alpha, \qquad \sum_\alpha \mathfrak{C}_\alpha = 0 \qquad (\alpha = 1, 2) \tag{2.8}$$

for the constituents, in which $\mathfrak{C}_\alpha$ are the *constituent mass productions*, can be written in the following form:

$$\begin{aligned} \varrho\dot{\omega}_\alpha &= -\operatorname{div} \boldsymbol{j}_\alpha + \pi_\alpha, \\ \boldsymbol{j}_\alpha &= \omega_\alpha(\boldsymbol{u}^\alpha - \boldsymbol{u}), \\ \pi_\alpha &= \mathfrak{C}_\alpha - \omega_\alpha \operatorname{div} \boldsymbol{u} - \omega_\alpha(\dot{\varrho} + \operatorname{grad} \varrho \cdot (\boldsymbol{u}^\alpha - \boldsymbol{u})), \end{aligned} \tag{2.9}$$

where $\omega_\alpha \equiv \varrho_\alpha/\varrho$ is the *mass fraction* of constituent α and $\boldsymbol{u}^\alpha - \boldsymbol{u}$ is the *diffusion velocity*. More convenient than working with two equations of the type $(2.9)_1$ is to use balance of mass for the mixture and that for constituent water. This then yields

$$\begin{aligned} &\dot{\varrho} + \varrho \operatorname{div} \boldsymbol{u} = 0, \\ &\qquad \varrho\dot{\omega} = -\operatorname{div} \boldsymbol{j} + \pi \quad \text{(for constituent water)}. \end{aligned} \tag{2.10}$$

Here and henceforth the subscript in the mass balance equation for constituent water will be deleted as the constituent water will always be ment. The quantity ω is then frequently referred to as *moisture content.*

A mixture is called incompressible when $\dot{\varrho} = 0$. Such an assumption simplifies the expression for π_α, but does still not imply $\pi_\alpha = \mathfrak{C}_\alpha$, see $(2.9)_3$. In order that such an indentity can hold the density must also be *uniform.* With these preliminary considerations we are now in a position to define the field equations that govern creep of ice in the temperate zone. The model that will be adopted is a *binary incompressible mixture with uniform density* for which balance of mass and momentum reduce to the equations

$$(2.11)\qquad \begin{aligned} \operatorname{div} \boldsymbol{u} &= 0, \\ \varrho \dot{\boldsymbol{u}} &= -\operatorname{grad} p + \operatorname{div} \boldsymbol{t}' + \varrho \boldsymbol{g}, \\ \varrho \dot{\omega} &= -\operatorname{div} \boldsymbol{j}_\omega + \mathfrak{C}_\omega . \end{aligned}$$

These balance laws must be complemented by constitutive relationships for the diffusive flux vector $\boldsymbol{j}_\omega$, mass production $\mathfrak{C}_\omega$ and stress deviator $\boldsymbol{t}'$. As far as diffusive mass flux is concerned it is most natural to postulate a *Fick-type constitutive relationship*, $\boldsymbol{j}_\omega = -\nu \operatorname{grad} \omega$, with a constant *diffusivity* ν. Considering the earlier assumption that all internal energy production of the mixture is used up by melting we have $\mathfrak{C}_\omega = \operatorname{tr}(\boldsymbol{t}' \boldsymbol{D})/L$, where L is *latent heat of fusion per unit volume of the mixture*. It remains to establish a constitutive relationship for stress deviator, which is assumed in the form $(2.2)_3$, where A now depends on moisture content rather than on temperature. Finally pressure and melting temperature are related to each other by the *linearized Clausius–Clapeyron-equation*, $dT/dp = -c_t$.

In summary, the constitutive relations completing the balance laws (2.11) are(¹)

$$(2.12)\qquad \mathfrak{C}_\omega = \operatorname{tr}(\boldsymbol{t}' \boldsymbol{D})/L, \qquad \boldsymbol{j}_\omega = -\nu \operatorname{grad} \omega, \qquad \boldsymbol{D} = A(\omega) f(t'_{\mathrm{II}}) \boldsymbol{t}'.$$

Comparing equations (2.1) and (2.2) which are valid for cold ice with (2.11) and (2.12), it is seen that, structurally, they are the same equations with the same type of couplings. The continuity equation and the momentum equations are coupled with an equation for a "state variable", temperature T and moisture content ω, respectively. Physically this equation is the energy equation or the balance of mass for the water, but mathematically both equations are of *parabolic* or *diffusion type*.

Practically there is nothing known about numerical values of the diffusivity ν; except for qualitative considerations saying that ν is small. Also, only incomplete knowledge exists about the function $A(\omega)$ see, however, Lliboutry (1976). Most practicioners therefore assume a constant value for A, achieving thereby a decoupling of the purely mechanical field equations from the diffusion equation.

As stated before the energy equation need not be considered in the temperate ice region, since temperature T_M follows from the Clausius–Clapeyron equation. Later we shall use an expression for the heat flux vector

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(¹) We may, of course also include a temperature or pressure dependency in the rate factor A; thus more generally $A = A(\omega, T_M)$.

in the temperate region. On the basis of a Fourier-type constitutive relation one obtains

$$(2.13) \qquad \boldsymbol{q} = -\varkappa \operatorname{grad} T = -\varkappa \frac{dT}{dp} \operatorname{grad} p = \varkappa c_t \operatorname{grad} p,$$

$\varkappa$ being the heat conductivity of the mixture.

b) Boundary conditions. The boundary surfaces (see Figure 1) $y = y_S$, $y = y_B$, $y = y_W$ and $y = y_M$ are not material, in general; they must therefore be regarded as surfaces of discontinuity. At such surfaces it is assumed that physical quantities may suffer a finite jump. In other words, whereas all quantities are assumed to be sufficiently differentiable in the regions on either side of the surface, this assumption is weakened when the surface is crossed. Given the balance law

$$\frac{d}{dt}\int_v \phi\, dv = \oint_{\partial v} \boldsymbol{\varphi}_\phi \cdot \boldsymbol{n}\, da + \int_v \pi_\phi\, dv + \int_v s_\phi\, dv$$

holding for the material domain v with boundary ∂v (unit outward normal vector $\boldsymbol{n}$, see Figure 2) it is shown in texts on continuum mechanics that the

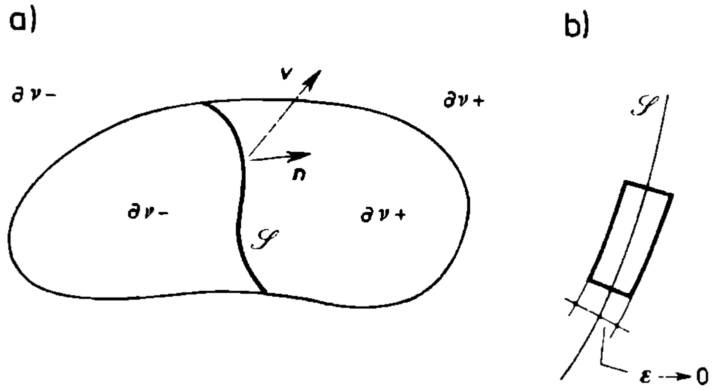

Fig. 2. a) Body volume separated into two parts v^+ and v^- by a surface $\mathscr{S}$ at which fields may suffer a finite jump. b) Small "pillbox"-like part of a body enclosing singularity surface, explaining jump conditions

following jump condition must hold at the surface of discontinuity $\mathscr{S}$ moving with *normal speed* $\boldsymbol{v} \cdot \boldsymbol{n}$:

$$(2.14) \qquad [\![\boldsymbol{\varphi}_\phi \cdot \boldsymbol{n}]\!] - [\![\phi(\boldsymbol{u} - \boldsymbol{v}) \cdot \boldsymbol{n}]\!] = 0.$$

Here the bracket stands for the difference $[\![\phi]\!] = \phi^+ - \phi^-$, and $\phi^\pm$ denote the values of ϕ as the surface is approached from the positive (v^+) and the negative (v^-) side of the surface, respectively. On *material surfaces*(1) $\boldsymbol{u} = \boldsymbol{v}$

(1) $\boldsymbol{v} \cdot \boldsymbol{n}$ is the normal speed, the singular surface is moving with.

and the second term of (2.14) vanishes. Balance of mass, momentum, energy and entropy obey relations of the form (2.14) and can be written as[1]

$$
\begin{aligned}
&[\![\varrho(\boldsymbol{u}-\boldsymbol{v})\cdot\boldsymbol{n}]\!] = 0,\\
&[\![\boldsymbol{t}\cdot\boldsymbol{n}]\!] - [\![\varrho\boldsymbol{u}(\boldsymbol{u}-\boldsymbol{v})\cdot\boldsymbol{n}]\!] = 0,\\
(2.15)\qquad &[\![(\boldsymbol{u}\cdot\boldsymbol{t}-\boldsymbol{q})\cdot\boldsymbol{n}]\!] - [\![\varrho(\varepsilon+\tfrac{1}{2}u^2)(\boldsymbol{u}-\boldsymbol{v})\cdot\boldsymbol{n}]\!] = 0,\\
&\left[\!\!\left[\frac{\boldsymbol{q}\cdot\boldsymbol{n}}{T}\right]\!\!\right] + [\![\varrho\eta(\boldsymbol{u}-\boldsymbol{v})\cdot\boldsymbol{n}]\!] = 0,
\end{aligned}
$$

in which η is entropy. The first of these equations determines the mass flux from v^- into v^+. Notice that when $\boldsymbol{u}=\boldsymbol{v}$, then $\mathscr{S}$ is material. On the other hand $a_\perp := (\boldsymbol{u}^- - \boldsymbol{v}^-)\cdot\boldsymbol{n}$ is the *volume flux through the surface*, so that $(2.15)_1$ may also be written as

$$\varrho^-(\boldsymbol{u}-\boldsymbol{v})^-\cdot\boldsymbol{n} = \varrho^- a_\perp = \varrho^+(\boldsymbol{u}-\boldsymbol{v})^+\cdot\boldsymbol{n}.$$

The diffusion equation for the moisture content $(2.11)_3$ is also a local balance equation which can be put into the form (2.13). Its corresponding jump condition has the form

$$(2.16)\qquad [\![\boldsymbol{j}\cdot\boldsymbol{n}]\!] + [\![\varrho\omega(\boldsymbol{u}-\boldsymbol{v})\cdot\boldsymbol{n}]\!] = \tilde{\omega}$$

where $\tilde{\omega}$ is the *surface production* of ω.

Before we proceed we *define* a *surface of phase change* as a singular surface at which the *temperature is continuous*. Thus, $[\![T]\!]=0$, characterizing *a diathermic wall.* For such surfaces jump of energy and entropy can be put into a different form, see Hutter (1983), namely[2]

$$
(2.17a)\qquad
\begin{aligned}
[\![\boldsymbol{q}\cdot\boldsymbol{n}]\!] &= [\![\quad]\!]\varrho^- a_\perp,\\
[\![\boldsymbol{q}\cdot\boldsymbol{n}]\!] &= -T[\![\eta]\!]\varrho^- a_\perp
\end{aligned}
$$

where the exact form of the term $[\![\quad]\!]$ on the right hand side of the first equation is of no relevance here because it can be eliminated. By definition the latent heat of fusion is given by $L = T[\![\eta]\!]$, so that from either one of equations (2.17a) we obtain

$$(2.17b)\qquad [\![\boldsymbol{q}\cdot\boldsymbol{n}]\!] = L\varrho^- a_\perp \qquad \text{(on a surface of phase change)}$$

This is a very appealing form of the energy jump.

---

[1] One additional and essential assumption in deriving (2.14) and in deducing (2.15) is that there is no surface production term. It is thus assumed that the singularity surfaces are such that neither mass, nor momentum, nor energy, nor entropy is producted. If there would be a surface production the right hand side of equation (2.14) would carry this surface production term.

[2] Usual definitions of surfaces of phase change also have continuous tangential velocities. This is, however, not essential for (2.17) to be valid.

The above statements are of *dynamic nature*. However, for every moving surface there is also a *kinematic* statement. If $S(x, t) \equiv 0$ is the equation of such an orientable surface, and this surface is *material*, it is known that $dS/dt \equiv 0$ represents this kinematic statement and forms an evolution equation for S. For a *non-material* surface dS/dt cannot vanish, as there must be a mass flux through the surface. However, the derivative of S *following the surface* must vanish, implying that

$$\frac{\partial S}{\partial t}+\text{grad } S\cdot \boldsymbol{v} = \frac{\partial S}{\partial t}+\text{grad } S\cdot \boldsymbol{u}^- +\text{grad } S(\boldsymbol{v}-\boldsymbol{u})^- \equiv 0$$

or

$$\text{(2.18)} \qquad \left(\frac{dS}{dt}\right)^- = -\text{grad } S\cdot(\boldsymbol{v}-\boldsymbol{u})^- = \|\text{grad } S\|\,((\boldsymbol{u}-\boldsymbol{v})^-\cdot \boldsymbol{n}) = \|\text{grad } S\|\, a_\perp,$$

where $\boldsymbol{n} = \text{grad } S/\|\text{grad } S\|$ points into the positive side v^+ (thus defining the sign of $S(\cdot)$). Here the superscript $(-)$ is important as the formula would change, if it would be referred to quantities on the right side of the surface of discontinuity.

Consider, for instance plane flow with $S = y_S(x, t)-y \equiv 0$. In this case (2.18) has the form

$$\text{(2.19)} \qquad \frac{\partial y_S}{\partial t}+\frac{\partial y_S}{\partial x}u_x^- - u_y^- = \sqrt{1+\left(\frac{\partial y_S}{\partial x}\right)^2}\, a_\perp = a,$$

where a is the volume flux through the surface per unit lenght in the x-direction and u_x^- and u_y^- are the velocity components in the x- and y-direction, respectively. Finally, note that if $\Sigma(x, t) \equiv 0$ is an identity for a physical quantity which must hold on the singularity surface, then an equation of the form (2.18) will also hold for Σ. This will be used later.

With the above preliminary derivations presentation of boundary conditions is straightforward. Two types of conditions must be considered, the kinematic and the dynamic conditions, and these must be established for both, cold and temperate ice.

α) *At the free surface.* Because of accumulation (nourishment due to snowing) and ablation (wastage due to surface melting the free surface is non-material, implying that the jump conditions (2.15) (2.17) and the kinematic surface equation (2.18) or (2.19) apply.

Letting the ice region be v^- and the atmospheric side be v^+ the above mentioned equations yield

$$\text{(2.20)} \qquad \frac{\partial S_S}{\partial t}+\text{grad } S_S\cdot \boldsymbol{u}^- = \|\text{grad } S_S\|\, a_\perp \qquad \text{on} \qquad S_S(x, t) \equiv 0$$

as kinematic equation in which $a_\perp$ is, in general a prescribed function of position and time, and for a *cold* boundary

(2.21a)
$$[\![\boldsymbol{t}\cdot\boldsymbol{n}]\!] = [\![\boldsymbol{u}]\!]\varrho^- a_\perp,$$
$$[\![\boldsymbol{q}\cdot\boldsymbol{n}]\!] = -[\![\varepsilon(T_0)+c_p(T-T_0)+\tfrac{1}{2}u^2]\!]\varrho^- a_\perp + [\![\boldsymbol{u}]\!]\cdot\{\boldsymbol{t}^+\cdot\boldsymbol{n}+\boldsymbol{u}^-\varrho^- a_\perp\}$$

as jump conditions of momentum and energy. For the derivation of these equations no simplifying assumptions have yet been invoked. As evident, neither traction nor normal heat flux are continuous. Both are affected by the accumulation function and the jump in velocity. We have already seen in $(2.15)_1$ that $a_\perp \neq 0$ implies $[\![\boldsymbol{u}\cdot\boldsymbol{n}]\!] \neq 0$ and therefore $[\![\boldsymbol{u}]\!] \neq \boldsymbol{0}$. The physical interpretation of $(2.21)_1$ is that the jump of traction equals the product of velocity jump and mass flux. On the other hand, $(2.21)_2$ states that heat flux normal to the boundary is balanced by atmospheric heat flux, a diffusive flux of internal and kinetic energy and a term which can be interpreted as a power due to diffusive momentum flux. In practice the terms on the right hand side of (2.21) are ignored so that

(2.21b)
$$[\![\boldsymbol{t}\cdot\boldsymbol{n}]\!] \simeq 0, \qquad [\![\boldsymbol{q}\cdot\boldsymbol{n}]\!] \simeq 0$$

is obtained.

In glacier and ice sheet problems the first of these is the stress free condition. The second condition is often the result of a boundary value problem for which on the surface $T = T_S$ and $a_\perp > 0$ are prescribed functions of position and time. On a *temperate* free surface $T = T_S = T_M$, and $a_\perp$ can be related to the heat flux. For the *temperate* free surface is a surface of phase change at which (2.20) and $(2.21)_1$ are still valid; $(2.21)_2$, however, must be replaced by (2.17), or

(2.22)
$$[\![\boldsymbol{q}\cdot\boldsymbol{n}]\!] = -\varrho L a_\perp,$$

relating energy jump and ablation rate with latent heat. Furthermore, the jump condition of moisture (2.16) must hold, which reads

(2.23)
$$[\![\boldsymbol{j}\cdot\boldsymbol{n}]\!] = \varrho\omega a_\perp + \tilde{\omega}$$

where the flux $\boldsymbol{j}^+\cdot\boldsymbol{n}$ is assumed known from measurements of total melting rate and surface discharge. Expected values for $\boldsymbol{j}^+\cdot\boldsymbol{n}$ are negative.

There are no physical arguments that would support the assumption of a non-zero moisture production at the free surface. Hence we set $\tilde{\omega} = 0$ and thus obtain the mixed type boundary condition

(2.24)
$$\boldsymbol{j}\cdot\boldsymbol{n} + \varrho\omega a_\perp = \boldsymbol{j}^+\cdot\boldsymbol{n},$$

valid at the temperate free surface. Quantities not carrying an index $(-)$ are those on the ice side. This convention will be observed throughout.

In summary, at a cold free surface (2.20) and (2.21) must hold, whereas for a temperate free surface $(2.21)_2$ is replaced by (2.24). Here (2.22) also serves to the climatologist as an equation in obtaining ablation rates from an energy budget.

β) *Along the ice-water interface.* Conceptually this surface is not much different from the free surface as it separates again two fluids. Since melting and freezing may occur the ice-water interface is also nonmaterial, but such that $[\![T]\!] = 0$.

Let $S_W(x, t) \equiv 0$ be the equation of the surface, then

$$\frac{\partial S_W}{\partial t} + \operatorname{grad} S_W \, \boldsymbol{u} = \|\operatorname{grad} S_W\| \, a_W^{\perp} \qquad \text{on} \qquad S_W(x, t) \equiv 0 \tag{2.25}$$

is the kinematic surface equation, where $a_W^{\perp}$ is the melting-freezing rate which is positive for freezing.

With the same simplifying assumptions as in the last section the jump conditions of momentum and energy for cold and temperate ice are

$$[\![\boldsymbol{t}\cdot\boldsymbol{n}]\!] = \boldsymbol{0} \qquad \text{and} \qquad [\![\boldsymbol{q}\cdot\boldsymbol{n}]\!] = -\varrho L a_W^{\perp}, \tag{2.26}$$

where it is assumed that traction and energy flux on the water side are known. Typically,

$$\begin{aligned} (\boldsymbol{t}\cdot\boldsymbol{n})_{\text{water}} &= -p_w \boldsymbol{n}, \\ (\boldsymbol{q}\cdot\boldsymbol{n})_{\text{water}} &= h(T_{\text{ice}} - T_{\text{water}}), \end{aligned} \tag{2.27}$$

in which h is a *heat transfer coefficient* whose value depends on the flow conditions within the boundary layer beneath the ice. With Nusselt-, Prandtl- and Reynolds-numbers defined as

$$\mathrm{N} = \frac{hx}{\varkappa}, \qquad \mathrm{P} = \left(\frac{\nu \varrho c_p}{\varkappa}\right)_{\text{water}}, \qquad \mathrm{R} = \left(\frac{Ux}{\nu}\right)_{\text{water}} \ (^1)$$

such relations are of the form

$$\mathrm{N} = a \mathrm{P}^{\alpha} \mathrm{R}^{\beta}, \tag{2.28}$$

(a, α, β) being constants whose values depend on the flow nature (laminar versus turbulent) of the boundary layer current, see Schlichting (1978). Note that when $T_{\text{ice}} = T_M$, where T_M is melting temperature, melting must occur, $a_W^{\perp} < 0$; alternatively, when $T_{\text{ice}} = T_{\text{water}} < T_M$, one has freezing, $a_W^{\perp} > 0$. In the interval $T_{\text{water}} < T_{\text{ice}} < T_M$ the ice is adjusting its temperature to the energy

(1) U is the ocean current below the boundary layer, x fetch, ν viscosity, c_p specific heat, ϱ density and $\varkappa$ heat conductivity, all for water at its given salinity.

budget; there is neither melting nor freezing, $a_W^\perp = 0$. In summary, the thermal boundary condition at the ice-water interface has the form

$$L\varrho a_W^\perp = \begin{cases} -\boldsymbol{q}\cdot\boldsymbol{n}|_{T_{\text{ice}}=T_{\text{water}}}, & a_W^\perp > 0, \\ 0 = h(T_{\text{ice}} - T_{\text{water}}) - \boldsymbol{q}\cdot\boldsymbol{n}|_{T_{\text{ice}}}, & a_W^\perp = 0, \\ h(T_M - T_{\text{water}}) - \boldsymbol{q}\cdot\boldsymbol{n}|_{T_{\text{ice}} - T_M}, & a_W^\perp < 0. \end{cases} \tag{2.29}$$

Evidently, either temperature is prescribed and then (2.29) relates the heat flux on the ice side normal to the interface to the freezing or melting rate, or else $a_W^\perp = 0$ is known, and then the temperature on the ice side and its normal derivative are related to the (given) water temperature.

The assumption in the above was that the ice at the lower boundary is cold, or just reaches the melting point. This requires that $T_{\text{water}} \leqslant T_M$. When $T_{\text{water}} > T_M$, there is a layer of temperate ice close to the ice-water interface. In that case only $(2.29)_3$ applies, but $\boldsymbol{q}\cdot\boldsymbol{n}$ is negligibly small; thus the heat transfer law determines the melting rate in this case. The jump condition for moisture (2.16) becomes $[\![\boldsymbol{j}\cdot\boldsymbol{n}]\!] = -[\![\omega]\!] a_W^\perp$ provided that there is no surface moisture production. Since on the water side we must have $\omega \equiv 1$ throughout, whence $\boldsymbol{j} \equiv \boldsymbol{0}$, the boundary condition of moisture at the ice water interface must be

$$\boldsymbol{j}\cdot\boldsymbol{n} - \varrho(1-\omega)a_W^\perp = 0 \qquad \text{(on the ice side).} \tag{2.30}$$

This completes the boundary conditions for the ice water interface.

γ) *Along the cold-temperate transition surface.* A complete description of polythermal glaciers must also include transition conditions at inner surfaces where cold ice reaches the melting point. Let $S_M(\boldsymbol{x}, t) \equiv 0$ be the equation of this surface. As a surface of phase change, it is non-material and thus the same jump conditions as before apply with $[\![T]\!] = 0$. Let v^- be the cold ice zone and v^+ the corresponding temperate zone. Hence, transition conditions are

$$\begin{gathered} [\![T]\!] = 0, \qquad [\![\boldsymbol{u}-(\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}]\!] = \boldsymbol{0}, \\ [\![\boldsymbol{t}\cdot\boldsymbol{n}]\!] = [\![\boldsymbol{u}]\!]\varrho^- a_M^\perp \simeq 0, \\ [\![\boldsymbol{q}\cdot\boldsymbol{n}]\!] = -L\varrho^- a_M^\perp, \\ [\![\boldsymbol{j}\cdot\boldsymbol{n}]\!] + [\![\omega]\!]\varrho^- a_M^\perp = \tilde{\omega}. \end{gathered} \tag{2.31}$$

The first two equations are the conditions which must hold at a surface of phase change when there is no jump in tangential velocity, the remaining three conditions are the momentum, energy ans moisture jump conditions on such surfaces. With $\tilde{\omega} = 0$, which seems to be a natural assumption, $(2.31)_{3,4}$ can be combined as

$$[\![\boldsymbol{q}\cdot\boldsymbol{n}]\!] = \frac{L}{[\![\omega]\!]}[\![\boldsymbol{j}\cdot\boldsymbol{n}]\!],$$

relating energy jump, moisture jump and moisture flux. Since $\omega^- \equiv 0$ and $\boldsymbol{j}^- \equiv \mathbf{0}$ one has

$$(\boldsymbol{q}\cdot\boldsymbol{n})^+ - (\boldsymbol{q}\cdot\boldsymbol{n})^- = \frac{L}{\omega^+}(\boldsymbol{j}\cdot\boldsymbol{n})^+, \tag{2.32}$$

where $(\pm)$ stand for the temperate and cold ice regions, respectively. $(\boldsymbol{q}\cdot\boldsymbol{n})^+$ can be expressed by the pressure gradient, if the relation (2.13) is adopted. Alternatively this flux is very small and can safely be ignored.

The above dynamic conditions must be complemented by kinematic conditions. A first is

$$\left(\frac{dS_M}{df}\right)^- = \frac{\partial S_M}{\partial t} + \operatorname{grad} S_M \cdot \boldsymbol{u}^\perp = \|\operatorname{grad} S_M\|\, a_M^\perp \tag{2.33}$$

and forms the usual kinematic surface condition. Further conditions can be deduced from any function $f(x, t)$ which is continuous and satisfies the condition $f = \text{constant}$ on $S_M = 0$. Such a statement holds for temperature, $T - T_M = 0$, so that

$$\left(\frac{d(T-T_M)}{dt}\right)^- = \|\operatorname{grad}(T-T_M)\|\, a_M^\perp \tag{2.34}$$

with $dT_M/dp = -c_t$. Notice that an equation of the form (2.34) is necessary as with (2.31)–(2.33) $a_M^\perp$ is still undetermined; (2.34) is its defining equation.

We mention that Fowler and Larson (1978) have established a theory of polythermal ice with the same field equations except that they ignore $\boldsymbol{j}$ in the moisture balance equation and omit (2.34).

These authors also require at the cold-temperate transition surface that apart from $(2.31)_{1,2,3}$ also $\omega^{+-} = 0$ and $[\![\boldsymbol{q}\cdot\boldsymbol{n}]\!] = 0$ must hold. It is evident from $(2.31)_4$ that these transition conditions contradict continuum mechanical principles. Further, notice that with $\omega^+ = 0$, Equation (2.32) becomes singular and therefore meaningless. In this case the last two equations (2.31) uncouple. Thus, a *second set of possible boundary conditions* is

$$\begin{gathered}
[\![T]\!] = 0, \qquad [\![\boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}]\!] = \mathbf{0}, \qquad [\![\omega]\!] = 0,\\
[\![\boldsymbol{t}\cdot\boldsymbol{n}]\!] = \mathbf{0},\\
[\![\boldsymbol{q}\cdot\boldsymbol{n}]\!] = -L\varrho^-\, a_M^\perp,\\
[\![\boldsymbol{j}\cdot\boldsymbol{n}]\!] = \tilde{\omega},
\end{gathered} \tag{2.35}$$

with $T = T_M$ and $\omega^\pm = 0$. Moreover, the kinematic conditions (2.33), (2.34) must hold, and a further condition would be

$$\left(\frac{d\omega}{dt}\right)^- = \|\operatorname{grad}\omega\|\, a_M^\perp. \tag{2.36}$$

This system is, clearly, overdetermined even if (2.36) is interpreted as an evolutionary equation for the non-vanishing moisture production $\tilde{\omega}$. For because the temperature and moisture field equations are parabolic, it must follow from (2.35) that $[\![\boldsymbol{q}\cdot\boldsymbol{n}]\!]$ and $[\![\boldsymbol{j}\cdot\boldsymbol{n}]\!]$ cannot be simultaneously prescribed. Transition conditions with $\omega^{+}=0$ are therefore wrong.

δ) *At the base.* Physically the boundary conditions at the base are the least understood of all. Depending on whether the glacier is cold or temperate, different boundary conditions apply. On the cold portion it is assumed that *the ice adheres to the rock bed.* Hence the no-slip condition applies and there is no jump of momentum and heat flux,

$$[\![\boldsymbol{u}]\!]=\boldsymbol{0},\qquad [\![\boldsymbol{t}\cdot\boldsymbol{n}]\!]=\boldsymbol{0},\qquad [\![\boldsymbol{q}\cdot\boldsymbol{n}]\!]=0. \tag{2.37}$$

On a *rigid bed* this implies

$$\boldsymbol{u}=\boldsymbol{0},\qquad \boldsymbol{q}\cdot\boldsymbol{n}+Q^{\mathrm{geoth}}=0, \tag{2.38}$$

where Q^{geoth} is the geothermal heat flow *into* the ice. A jump condition for $\boldsymbol{t}$ needs not be written down as basal traction can be determined from $(2.37)_2$.

Strictly, equations (2.37) and (2.38) are correct only on the *true basal surface*. This true surface is, however, never known, since one has usually no information about the roughness on all scales. Neither is it important that the true surface be known, since sufficiently far distant from a small boundary layer the flow will not feel the small scale undulations. Let $S_B(x,t)\equiv 0$ be the *mean basal surface* on which small scale roughness is smoothed out, see Figure 3. The question then arises what the correct boundary

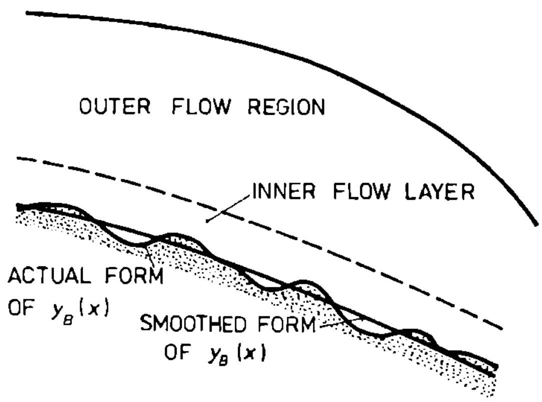

Fig. 3. Ice-rock bed, its actual and smoothed-out forms, explaining sliding friction

conditions should be on this mean surface in order that on the true surface the conditions (2.37) and (2.38) apply. An exact solution of this problem is not possible, as boundary conditions on the mean surface can approximate those on the true surface only in some asymptotic sense. The problem is one of matched asymptotic expansions. Indeed, boundary layer theory shows that to lowest order, equations (2.38) are asymptotically correct if the base is rigid and the ice adheres to it.

Consider now that portion of the base where the ice is *temperate*. Here the heat generated by deformation causes melting giving rise to the existence of a layer of water separating ice sole and ice bed. Often this layer is just a few microns thick and acts as a lubricant with negligible film thickness; its effect is to change the boundary condition from no-slip to *sliding*. On the true surface and for an ideal lubricant this sliding is frictionless. The boundary condition at the mean surface is, however, a viscous sliding law, as the outer flow far from the base feels the inner flow as a viscous drag whose form and magnitude will depend on the roughness of the bed. When sufficient water is available, ice sole and rock bed may separate causing the glacier to become partly afloat. Water filled cavities are formed reducing thereby the frictional resistance. Sliding *with* and without *cavity formation* characterize therefore two types of boundary conditions that can apply at the base, and the form of the boundary condition that applies must depend on the type of sliding that applies.

The derivation of the frictional law is a matter of *regelation physics*. Here we simply mention that it amounts to establishing a relationship between sliding velocity and basal traction. Clearly the derivation of the remaining boundary conditions must follow similar lines, but this has never been done to my knowledge. For the purposes of this paper we therefore *assume* the mean rock bed, $S_B \equiv 0$, to be a *rigid* singular surface along which the ice is sliding, and we explore the inferences which follow from the jump conditions. To this end it should be observed that because sliding is permitted the basal surface does not fulfill the usual conditions of a surface of phase change([1]), but the occurrence of sliding at the basal boundary was reason for us to define here a surface of phase change as a singularity surface with no temperature jump.

With $\boldsymbol{u}^+ = \boldsymbol{0}$ it is an easy exercise to derive from (2.15)–(2.17) the following jump conditions

$$
\begin{aligned}
(\boldsymbol{u}\cdot\boldsymbol{n})^- &= \frac{[\![\varrho]\!]}{\varrho^+}\, a_B^\perp, \\
[\![\boldsymbol{t}\cdot\boldsymbol{n}]\!] &= -\boldsymbol{u}^-\, \varrho^-\, a_B^\perp, \\
[\![\boldsymbol{q}\cdot\boldsymbol{n}]\!] &= -L\varrho^-\, a_B^\perp, \\
[\![\boldsymbol{j}\cdot\boldsymbol{n}]\!] + [\![\omega]\!]\,\varrho^-\, a_B^\perp &= 0;
\end{aligned}
\tag{2.39}
$$

([1]) Such surfaces are defined as singular surfaces for which $[\![T]\!] = 0$ and $[\![u_\parallel]\!] = 0$ where $u_\parallel$ is the tangential velocity.

in which surface moisture production has been set to zero, and $(+)$ and $(-)$ stand for rock and ice, respectively. The first of these relates melting rate to normal velocity and the second the traction at the rock to the stress at the ice sole.

It remains to derive the sliding law. To this end let

$$t^* = t-(n\cdot t\cdot n)\,1 = t'-(n\cdot t'\cdot n)\,1 \tag{2.40}$$

be the tensor whose associated vector $t^*\cdot n$ is tangential to the mean basal surface $S_B \equiv 0$. A viscous sliding law which is compatible with the kinematic condition of the bed must have a form

$$u = -F\left((t^*\cdot n)^2, \cdot\right) t^*\cdot n + \frac{[\![\varrho]\!]}{\varrho^+} a_B^\perp n, \tag{2.41}$$

where F is a scalar valued function, which depends upon shear traction vector and other quantities characterizing the sliding, as e.g. ω, p, T, etc. The term on the right-hand side involving $a_B^\perp$ is added, because the sliding law must satisfy $(2.39)_1$.

Very little is known about the function F. For *sliding without cavity formation* $F(x) = Cx^{(m-1)/2}$, with C and m depending on the roughness of the bed and on the creep response function, see Lliboutry (1979), Weertmann (1979). For *sliding with cavity formation* rigorous derivations of functional relationships for F do not exist, but the work of Lliboutry (1979) suggests that F may become *multi-valued*, or at least singular, see Figure 4, thus giving rise to a run-away instability.(¹)

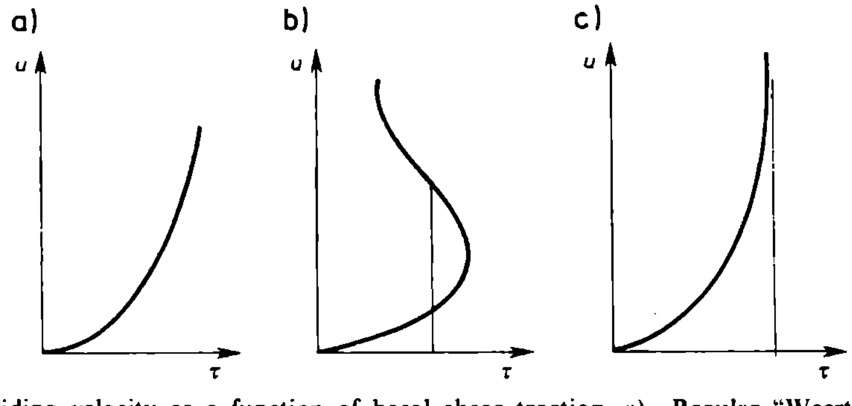

Fig. 4. Sliding velocity as a function of basal shear traction. a) Regular "Weertmann-type" sliding law. b) Sliding law with a range of basal traction for which u is multi-valued. c) For a critical value of shear traction sliding velocity grows fast

Note that the law (2.41) includes both the no-slip and the viscous sliding boundary conditions. When crossing the transition point "cold-temperate", see Figure 1, F must drop from a finite value to zero. This can be achieved

(¹) A rigorous treatment of sliding without cavity formation is given by Fowler (1981).

in various different ways yielding substantially different results (see Fowler and Larson (1978), Hutter and Olunloyo (1981a, (1981b)).

In summary, the boundary conditions at the base of a temperate ice region are (2.39) and (2.41). In practical applications they are further simplified. For the normal velocity is much smaller than the tangential velocity and the impulsive term $\varrho^- u^- a_B^\perp$ may also be ignored in the boundary condition of stress. Further, geothermal heat is much larger than the conductive heat flux in the ice. (2.39) and (2.41) are therefore generally used in the simplified form

$$\begin{aligned} &[\![\boldsymbol{t}\cdot\boldsymbol{n}]\!] = 0, \qquad \boldsymbol{u} = -F\left((\boldsymbol{t}^*\cdot\boldsymbol{n})^2, \cdot\right)\boldsymbol{t}^*\cdot\boldsymbol{n}, \\ &Q_{\text{geoth}} = L\varrho a_B^\perp, \qquad [\![\boldsymbol{j}\cdot\boldsymbol{n}]\!] + [\![\omega]\!]\,\varrho a_B^\perp = 0, \end{aligned} \tag{2.42}$$

where $(\boldsymbol{j}\cdot\boldsymbol{n})_{\text{rock}}$ is the *moisture flux into the rock* and ω_{rock} is the moisture content on the rock side, both assumed as known.

3. Dimensionless variables

The non-dimensionalization of the basic problem involves the finding of typical dimensional quantities to serve as scaling factors for the various dimensional variables in the problem. We shall now restric considerations to plane flow; and shall further only treat wholly cold ice regions in which the base may just reach the melting point. Boundary value problems for polythermal ice masses have not been attacked so far. The problems illustrated here however serve as a guide-line for further studies. Non-dimensional variables will be introduced as follows

$$\begin{aligned} (x, y) &\Rightarrow D(x, y) \qquad t \Rightarrow \frac{D}{U}t, \\ (t'_{xx}, t'_{xy}, t'_{yy}, p) &\Rightarrow \varrho g D(\sigma'_x, \tau', \sigma'_y, p), \\ (u, v, a) &\Rightarrow U(u, v, a), \\ T &= T_f + T_0\,\vartheta \end{aligned} \tag{3.1}$$

in which

$$\begin{aligned} D &= \text{typical depth, 100–500 m}, \\ U &= \text{typical forward speed, 100 m/a}, \\ T_f &= \text{freezing temperature at atmospheric pressure}, \\ T_0 &= \text{typical temperature range, 20°C}. \end{aligned} \tag{3.2}$$

With the choice (3.2) non-dimensional variables have order unity, except those for shear stress and accumulation rate, which are smaller. Substituting (3.1) into (2.1) and (2.2) yields

$$
(3.3)\qquad
\begin{aligned}
\operatorname{div} \boldsymbol{u} &= 0,\\
\mathrm{F}\dot{\boldsymbol{u}} &= -\operatorname{grad} p + \operatorname{div} \boldsymbol{\sigma}' + \boldsymbol{g},\\
\boldsymbol{D} &= \mathrm{G} \exp(\mathrm{A}\theta)\, \mathfrak{f}(\sigma'_{\mathrm{II}})\, \boldsymbol{\sigma}',\\
\dot{\vartheta} &= \mathrm{D}\nabla^2 \vartheta + 2\mathrm{E}\mathfrak{f}(\sigma'_{\mathrm{II}})\sigma'_{\mathrm{II}},
\end{aligned}
$$

where operators are with respect to dimensionless (x, y, t). In (3.3)

$$
\theta = \frac{1+\mathrm{Z}}{1+\mathrm{Z}\vartheta}\vartheta, \qquad \sigma'_{\mathrm{II}} = \tfrac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}')^2, \qquad \mathfrak{f}(x) = \frac{f(\varrho^2 g^2 D^2 x)}{f(\varrho^2 g^2 D^2)}
$$

and Roman letters are dimensionless characteristic quantities defined as follows.

$$
\begin{aligned}
\mathrm{Z} &= \frac{T_0}{T_f} = O(10^{-1}) \quad \text{(temperature ratio)},\\
\mathrm{F} &= \frac{U^2}{gD} \ll 10^{-8} \quad \text{(Froude number)},\\
\mathrm{G} &= \frac{D}{U} A \exp\left(\frac{-Q}{kT}\right) \varrho g D \mathfrak{f}(\varrho^2 g^2 D^2) \quad \text{(Glen number)},\\
\mathrm{A} &= \frac{Q}{kTf}\frac{\mathrm{Z}}{1+\mathrm{Z}} = O(1) \quad \text{(Arrhenius number)},\\
\mathrm{D} &= \frac{\varkappa}{\varrho c D U} < 10^{-2} \quad \text{(thermal diffusion number)},\\
\mathrm{E} &= \frac{gD}{cT_0} < 10^{-1} \quad \text{(energy dissipation number)}.
\end{aligned}
$$

The names given to these are only suggestions and are not commonly used. We have also assigned orders of magnitudes, which are obtained when typical values for the characteristic parameters and for the physical constants are substituted. These suggest that acceleration terms may safely be ignored in $(3.3)_2$ and show in the energy equation that dissipation may be an important term, perhaps more important than the conduction term. This latter term can not be ignored, however, because it would change the energy equation from parabolic to hyperbolic causing inconsistencies with boundary conditions. To the Glen number no value has been assigned above. This number depends inversely on the forward velocity and must therefore be large for flat ice sheets, but smaller for steep glaciers. Typically, $\mathrm{G} = O(10^2)$ or large for ice sheets, but $\mathrm{G} \lesssim O(10^1)$ for glaciers. This difference in order of magnitude of G for the two types of large ice masses will result in different asymptotic solution procedures, one allowing for ice divides, the other not.

In what follows attention will be limited to grounded, cold ice sheets. For these the boundary conditions *at the base* $S_B \equiv 0$, read

$$\begin{aligned} &\boldsymbol{u} = -\mathscr{F}\big((\boldsymbol{\sigma}^* \cdot \boldsymbol{n})^2, \cdot\big)\boldsymbol{\sigma}^* \cdot \boldsymbol{n}, \qquad \boldsymbol{u} \cdot \boldsymbol{n} = 0, \\ &\frac{\partial \vartheta}{\partial n} = -\mathrm{Q}_{\text{geoth}}, \qquad \text{when heat flux is prescribed} \end{aligned} \tag{3.4}$$

and *at the free surface* $S_S \equiv 0$

$$\begin{aligned} &\frac{\partial S_S}{\partial t} + \operatorname{grad} S_S\, \boldsymbol{u} = \|\operatorname{grad} S_S\|\, a_S^{\perp}, \\ &\boldsymbol{\sigma} \cdot \boldsymbol{n} = -p^{\text{atm}}\boldsymbol{n}, \\ &\vartheta = \vartheta_S, \qquad \text{when temperature is prescribed.} \end{aligned} \tag{3.5}$$

In these expressions $\mathscr{F}(\cdot)$ is dimensionless sliding law, $\mathscr{F} = F/U$, and $\mathrm{Q}_{\text{geoth}}$ a given dimensionless geothermal heat flow. Similarly, all other quantities are dimensionless. The accumulation ablation rate function $a_S^{\perp}$ has orders of magnitudes of transverse dimensionless ice velocities and is $O(10^{-1})$, or smaller. But it is an important quantity responsible for ice sheet or glacier geometry.

In $(3.4)_1$ the sliding law has been written for temperate ice (ignoring the term involving $a_B^{\perp}$) although attention is focussed on cold ice. The point is that $\mathscr{F}(\cdot)$ may either be the zero function, or else the base may just reach the melting temperature, thus allowing sliding. In the following we shall choose

$$\mathscr{F}(\boldsymbol{\sigma}^* \cdot \boldsymbol{n}) = \mathrm{C}\,[(\boldsymbol{\sigma}^* \cdot \boldsymbol{n})^2]^{(m-1)/2}, \tag{3.6}$$

with $m = (n+1)/2$ for Glen's flow law. This is the Weertmann-type sliding law.

From now on *all* variables will be dimensionless.

4. Simple solutions

In this section we seek exact solutions for the boundary value problem formulated in Setion 3 in the limit $\mathrm{F} = 0$. This is the Stokesian approximation and extremely accurate as $\mathrm{F} \ll 10^{-8}$ so that lead order solutions need not be perturbed.

We focus attention to *plane flow* and a strictly parallel sided ice slab so that base and free surface are defined by $y = y_B(x)$ and $y = y_S(x, t)$, respectively. Field equations and boundary conditions assume the following form:

Field equations:

$$
\begin{aligned}
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\\
&\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau}{\partial y}+\sin\gamma=0,\\
&\frac{\partial \tau}{\partial x}+\frac{\partial \sigma_y}{\partial y}-\cos\gamma=0,\\
&\frac{\partial u}{\partial x}=\tfrac{1}{2}G\exp(A\theta)\mathfrak{f}(\sigma'_{II})(\sigma_x-\sigma_y),\\
&\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=2G\exp(A\theta)\mathfrak{f}(\sigma'_{II})\tau,\\
&\frac{\partial \vartheta}{\partial t}+\frac{\partial \vartheta}{\partial x}u+\frac{\partial \vartheta}{\partial y}v=D\nabla^2\vartheta+2E\exp(A\theta)\mathfrak{f}(\sigma'_{II})\sigma'_{II}
\end{aligned}
\tag{4.1}
$$

in which $\sigma'_{II}=\frac{1}{4}(\sigma_x-\sigma_y)^2+\tau^2$.

Boundary conditions:

$$
\left.
\begin{aligned}
&\tau=0,\qquad \sigma_y=-p^{\mathrm{atm}},\\
&\frac{\partial y_S}{\partial t}+\frac{\partial y_S}{\partial x}u-v=a_S^{\perp},\\
&\vartheta=\vartheta_S,
\end{aligned}
\right\}\quad \text{on}\quad y=y_S=H
\tag{4.2}
$$

and

$$
\left.
\begin{aligned}
&u=C|\tau|^m,\qquad v=0,\\
&\frac{\partial \vartheta}{\partial y}=-Q_{\mathrm{geoth}},
\end{aligned}
\right\}\quad \text{on}\quad y=0.
\tag{4.3}
$$

This boundary value problem must also be complemented by initial conditions. Let us seek a solution of (4.1)–(4.3) for which *all fields are x-independent*. The mechanical part of (4.1)–(4.3) has then the solution

$$
\begin{aligned}
&\tau=\sin\gamma(H-y),\\
&\sigma_y=\sigma_x=-\cos\gamma(H-y)-p^{\mathrm{atm}},\\
&u=C(\sin\gamma H)^m+2G\int_0^y\exp(A\theta)\mathfrak{f}(\tau^2)d\tau,\\
&v=0,\\
&H(t)=\int_0^t a_S^{\perp}(t')dt',
\end{aligned}
\tag{4.4}
$$

and the thermal problem reduces to

$$
\begin{aligned}
&\frac{\partial \vartheta}{\partial t} = \mathrm{D}\nabla^2 \vartheta + 2\mathrm{E}\, \exp(\mathrm{A}\theta)\,\mathfrak{f}(\sigma_{11}')\sigma_{11}', \\
&\vartheta = \vartheta_S, \qquad \text{on} \qquad y = H(t), \qquad (4.5)\\
&\frac{\partial \vartheta}{dy} = -\mathrm{Q}_{\text{geoth}}, \qquad \text{on} \qquad y = 0,
\end{aligned}
$$

with initial condition $\vartheta(y, 0) = \vartheta_0$.

Several remarks are in order. First, within the constitutive class considered here stresses are materially independent. Shear stress is proportional to depth and surface inclination provided that $\sin\gamma \sim \tan\gamma \sim \gamma$. This is a well-known formula in glaciology. Second, forward velocity consists of a *sliding* and a *gliding portion* and both would be known, if the temperature field across depth were known. Third, because shear stress does not change sign and u is expressed in terms of this stress only, the forward velocity can not change sign, restricting this solution to regions far from ice divides. Fourth, transverse velocity must vanish, but in order that this solution is consistent accumulation can at most vary with time. Then thickness will vary with accumulation. Steady state conditions necessarily require $a_S^\perp \equiv 0$.

With known stress distribution, (4.5) is simpy a non-linear *two point boundary value problem* for temperature, the nonlinearity arising from strain heating. In steady state this boundary value problem was solved first by Yuen and Schubert (1979). Once it is solved the u-velocity component can be evaluated.

The easiest way to solve the steady state two-point boundary value problem (4.5) is to assume basal temperature and geothermal heat flow as prescribed quantities. Forward integration then fixes H with the condition $\vartheta = \vartheta_S$. In a final step $u(H)$ from (4.4) may then be calculated. One interesting feature of the function $u(H)$ is that there are thermal conditions for which $u(H)$ is multi-valued (tripple-valued), see Figure 5. A linear instability ana-

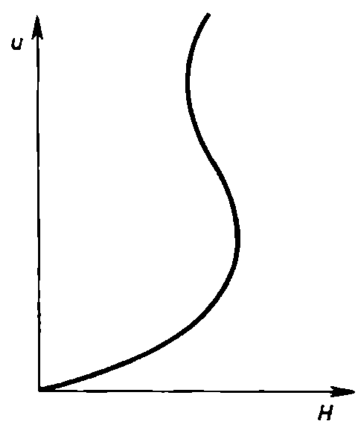

Fig. 5. Plot of forward surface speed against depth H

lysis has not revealed a clue as to which of these solutions would apply, for according to this instability analysis all solutions are stable. A non-linear analysis might resolve this question but is very difficult and has never been attacked so far. Another interesting result of the numerical solution of (4.5) is that for almost all situations strain heating has a very small effect on the temperature distribution across depth. Temperatures according to (4.5) are thus nearly linearly distributed; *this contradicts observation.*

A more general solution of (4.1)–(4.3) should therefore be found. The new assumption is that *stresses and temperature are still x-independent but that velocity components are allowed to vary with x.* For simplicity steady state is assumed. Equations $(4.1)_{2,3}$ still imply

$$\tau = \sin\gamma(H-y), \qquad \sigma_y = -\cos\gamma(H-y) - p^{\text{atm}}, \tag{4.6}$$

but $\sigma_x \neq \sigma_y$ in view of $(4.1)_4$. The velocity field can be determined from equations $(4.1)_{1,4,5}$ which imply

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}=0, \qquad \frac{\partial^2 u}{\partial x^2}=0, \qquad \frac{\partial^2 u}{\partial x \partial y}+\frac{\partial^2 v}{\partial x^2}=0,$$

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \qquad \text{is not a function of } x.$$

From these, together with one basal boundary condition, and the kinematic surface condition

$$u = \frac{a_S^{\perp}}{H}x + h(y), \qquad v = -\frac{a_S^{\perp}}{H}y \tag{4.7}$$

is obtained. Since $\partial u/\partial x = a_S^{\perp}/H$ is non-zero the *flow is extending and compressing,* depending on whether $a_S^{\perp} \gtrless 0$. In a glacier $a_S^{\perp} > 0$ in the upper part, but $a_S^{\perp} < 0$ in the lower portion. Thus, a glacier is extending in the upper accumulating part and compressing in the lower ablating part. This is broadly corroborated.

The solution (4.7) is still incomplete. We shall not complete it because the sliding condition

$$u(0) = \underbrace{\frac{a_S^{\perp}}{H}x + h(0)}_{\text{function of } x} = \underbrace{C(\tau)^m}_{\text{independent of } x}$$

makes an inconsistency explicit. Nevertheless, glaciologists accept the solution and determine the temperature profile from the boundary value problem

$$\begin{aligned} -\frac{a_S^{\perp}}{H}y\frac{d\vartheta}{dy} &= D\frac{d^2\vartheta}{dy^2} + 2E\exp(A\theta)\,\mathfrak{f}(\sigma'_{\text{II}})\sigma'_{\text{II}}, \\ \vartheta = \vartheta_S, \quad \text{on } y = H, &\qquad \frac{d\vartheta}{dy} = -Q_{\text{geoth}}, \quad \text{on } y = 0, \end{aligned} \tag{4.8}$$

in which the stress deviator invariant can be determined by combining equations $(4.1)_{4,5}$:

$$\left(\frac{a_S^{\perp}}{H}\right)+\exp(2A\theta)\,\mathfrak{f}^2(\sigma'_{\mathrm{II}})\left(\sin\gamma^2(H-y)^2-\sigma'_{\mathrm{II}}\right)=0. \tag{4.9}$$

Equation (4.8) is of *convection-diffusion type*, the convective term being the new essential element as compared to the previous boundary value problem (4.5). It has been mentioned before that strain heating does not affect, in general, the temperature distribution. An approximate solution to (4.8) thus is

$$\vartheta=\vartheta_S+Q_{\mathrm{geoth}}\int_y^H \exp\left(-\operatorname{sgn}(a_S^{\perp})\frac{1}{2}\sqrt{\frac{|a_S^{\perp}|}{HD}}\,\xi^2\right)d\xi, \tag{4.10}$$

and temperature distribution is typically as shown in Figure 6b, c; profiles as shown in Figure 6b are more realistic than solutions of (4.5), but do not yield

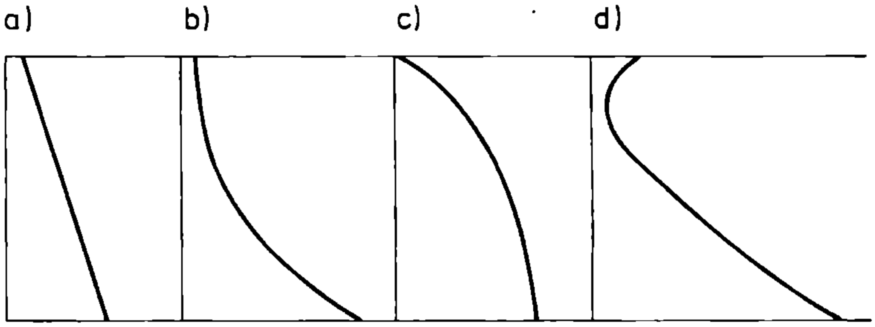

Fig. 6. Sketch of temperature profile: a) when strain heating is ignored according to the boundary value problem (4.5); b) with transverse advection, $a_s^{\perp}>0$ according to (4.10); c) same as b) but $a_S^{\perp}<0$; d) inversion profile (observed frequently) but not predictable by (4.8)

inversion profiles as sketched in Figure 6d, which are frequently observed, see Robin (1955).

Finally we mention that there is a physical situation for which a solution of this type is consistent. This is an ice shelf, for which no sliding law applies. Creep velocities at the lower boundary may then well be x-dependent without violating any boundary condition, see Weertmann (1957), Hutter and Williams (1980).

We close this section by pointing out how the above solutions have been or should be generalized. One physically important application is to geometries of a *nearly parallel-sided ice slab*. Accordingly, the base is assumed to be nearly flat with small amplitudes about a mean bed with inclination γ. The question then arises, how the bottom undulations are transferred to the top. An approximate solution of this problem using small perturbation theory has been given by Hutter et al. (1981). On length scales which are comparable to the ice sheet or glacier lenght the nearly parallel sided ice slab is not realistic, because surface geometry is only determined to within small amplitude deviations from the parallel-sided slab. The solution procedure that must be taken for such larger scales will be explained in the next section. Here it suffices to point out a typical feature of the foregoing solutions. Stresses and velocities were determined first, by assuming that the temperature distribution is known, and the temperature was subsequently determined in a second step. This procedure resulted in a mathematical decoupling of the mechanical and thermal problems. Because observations suggest that stresses and velocities depend on temperature but that exact knowledge of the temperature distribution appears not to be vital, such decoupling of the thermal and the mechanical problems can be achieved by using temperature estimates in the solution of the mechanical boundary value problem and iterating on temperature in the temperature problem. For a detailed explanation of the procedure the reader is referred to Hutter (1983).

5. Singular perturbation solution technique

Slab problems are central zone analyses, not completed to the margins; validity of their solutions is necessarily *local*. Our interest is now in solutions on lenght scales comparable to the lenght of the glacier, and the surface profile will be treated as an unknown. Complete solution of the boundary value problem (4.1)–(4.3) is difficult to find, however, so that approximate solution procedures are sought.

Let L and D be length scales in the longitudinal and transverse directions of the glacier, respectively. For most if not all glaciers and ice sheets the *aspect ratio* D/L is small; furthermore, and except for localized features, basal and surface profiles vary *slowly* with x. These observations suggest that the so called *shallow ice approximation* may be introduced into (4.1)–(4.3) by the *stretching transformations*

$$\xi = \varepsilon x, \quad \bar{t} = \varepsilon t, \quad V = \frac{v}{\varepsilon}, \quad U = u, \quad a_S^* = \frac{a_S^\perp}{\varepsilon}, \tag{5.1}$$

where ε is a small positive number whose value is in the order of the aspect ratio or of a representative surface slope. It will be demonstrated that ε can be related to G. Substituting (5.1) into (4.1) yields the new form of the mechanical field equations

$$
\begin{aligned}
&\frac{\partial U}{\partial \xi}+\frac{\partial V}{\partial y}=0,\\
&\varepsilon\frac{\partial \sigma_x}{\partial \xi}+\frac{\partial \tau}{\partial y}+\sin\gamma=0,\\
(5.2)\qquad &\varepsilon\frac{\partial \tau}{\partial \xi}+\frac{\partial \sigma_y}{\partial y}-\cos\gamma=0,\\
&\varepsilon\frac{\partial U}{\partial \xi}=\tfrac{1}{2}\mathrm{G}\exp(\mathrm{A}\theta)\,\mathfrak{f}(\sigma'_{\mathrm{II}})(\sigma_x-\sigma_y),\\
&\frac{\partial U}{\partial y}+\varepsilon^2\frac{\partial V}{\partial \xi}=2\mathrm{G}\exp(\mathrm{A}\theta)\,\mathfrak{f}(\sigma'_{\mathrm{II}})\,\tau.
\end{aligned}
$$

The corresponding mechanical boundary conditions have the form

$$
\begin{aligned}
&\frac{\partial y_S}{\partial t}+\frac{\partial y_S}{\partial \xi}U-V=a_S^*,\\
(5.3)\qquad &\tau+\varepsilon(\sigma_x-\sigma_y)\frac{\partial y_S}{\partial \xi}+O(\varepsilon^2)=0, \qquad \text{on}\quad y=y_S(\xi,t)\\
&\sigma_y-2\varepsilon\tau\frac{\partial y_S}{\partial \xi}+p^{\mathrm{atm}}+O(\varepsilon^2)=0,
\end{aligned}
$$

and

$$
\begin{aligned}
&U=\zeta C\left(\zeta\tau-\varepsilon(\sigma_x-\sigma_y)\frac{dy_B}{d\xi}+O(\varepsilon^2)\right)^m,\\
(5.4)\qquad & \qquad\qquad\qquad\qquad\qquad\qquad \text{on}\quad y=y_B(\xi),\\
&V=U\frac{dy_B}{d\xi},
\end{aligned}
$$

where $\zeta=\operatorname{sgn}(U+O(\varepsilon^2))$. Frequently the function x^m in $(5.4)_1$ will be replaced by the more general sliding law $\mathscr{F}(x)$. It is this form of the field equations which makes the decoupling of the thermal and mechanical equations mentioned in the last section evident. For, since ε is small, one can consider solutions in the limit as $\varepsilon\to 0$. The form of such asymptotic solutions allows us to differentiate between glaciers and ice sheets. This differentiation hinges upon the order of magnitude of the mean inclination angle. Accordingly we define a glacier as a large ice mass for which $\gamma = O(1)$(1); alternatively an ice sheet has mean bed inclination with small

(1) In practice $\gamma=10^{-1}$ may be regarded as $O(1)$.

angle, $\gamma < O(\varepsilon)$. Scrutiny of the scalings (3.2) which are typical in these cases, shows that $G = O(1)$ when $\gamma = O(1)$, but G is large when $\gamma \lesssim O(\varepsilon)$.

a) Steep glaciers. In this case ε is the only small parameter arising in the stretched equations (5.2). A reasonable choice for it is then $\varepsilon = D/L$, since $G = O(1)$. To lowest order, that is in the limit as $\varepsilon \to 0$ the solution of the boundary value problem (5.2)–(5.4) is easily seen to be

$$\begin{aligned} \tau^{(0)} &= \sin \gamma(y_S - y), \\ \sigma_y^{(0)} &= \sigma_x^{(0)} = -\cos \gamma(y_S - y) - p^{\text{atm}}, \\ U^{(0)} &= C(\sin \gamma(y_S - y))^m + 2G \int_{y_B}^{y_S} \exp(A\theta)\, \mathfrak{f}(\tau^2(y))\tau(y)\, dy. \end{aligned} \tag{5.5}$$

With the aid of $(5.2)_1$ and $(5.4)_2$ a formula for $V^{(0)}$ could easily also be deduced. We shall not do it here because an expression for V will not be needed in the sequel. Notice that (5.5) has the same form as $(4.4)_{1,2,3}$ in which all fields were assumed to be x-independent. However, equations (5.5) are based on a much weaker assumption and form the zeroth order terms of a more complete perturbation expansion.

$$(\sigma_x, \ldots, U, V) = \sum_{\nu=0}^{\infty} \varepsilon^{\nu}(\sigma_x^{\nu}, \ldots, U^{\nu}, V^{\nu}), \tag{5.6}$$

which has first been considered by Hutter (1980b, 1981a). In particular y_S may vary with ξ; (5.5) thus generalizes the earlier solutions of the strictly parallel sided ice slab to slowly varying ice geometries and suggests improvements by use of (5.6). Notice that, since $\tau^{(0)}$ does not change sign and because $\mathfrak{f} > 0$ for $\tau \neq 0$ the sliding velocity cannot change sign, implying that *there is no ice divide*. Moreover, $(5.5)_1$ shows that when γ is small, then $\tau \sim \gamma D$, a formula which should be improved. Such corrections can be deduced by substitution of (5.6) into (5.2)–(5.4) and collection of terms of $O(\varepsilon)$. The result is the boundary value problem:

Field equations:

$$\begin{aligned} \frac{\partial \tau^{(1)}}{\partial y} &= -\frac{\partial \sigma_x^{(0)}}{\partial \xi} = \cos \gamma \frac{\partial y_S}{\partial \xi}, \\ \frac{\partial \sigma_y^{(1)}}{\partial y} &= -\sin \gamma \frac{\partial y_S}{\partial \xi}, \\ \sigma_x^{(1)} &= \sigma_y^{(1)} + \frac{2}{G \exp(A\theta)\, \mathfrak{f}(\tau^{(0)2})} \frac{\partial U^{(0)}}{\partial \xi}, \\ \frac{\partial U^{(1)}}{\partial y} &= 2G \exp(A\theta)\, \mathfrak{f}(\tau^{(0)2})\tau^{(1)}, \\ \frac{\partial U^{(1)}}{\partial \xi} &+ \frac{\partial V^{(1)}}{\partial y} = 0. \end{aligned} \tag{5.6}$$

Boundary conditions:

$$\tau^{(1)} = 0, \qquad \sigma_y^{(1)} = -2\tau^{(0)}\frac{\partial y_S}{\partial \xi} = 0 \qquad \text{on} \quad y = y_S(\xi, t),$$

(5.7)

$$U^{(1)} = Cm(\tau^{(0)})^{m-1}\tau^{(1)}, \qquad V^{(1)} = U^{(1)}\frac{\partial y_S}{\partial \xi} \qquad \text{on} \quad y = y_B(\xi).$$

Since the zero-th order solution (5.5) is known, equations (5.6) and (5.7) can be integrated with the following solution for $\tau^{(1)}$, $\sigma_y^{(1)}$ and $U^{(1)}$:

$$\tau^{(1)} = \cos\gamma \frac{\partial y_S}{\partial \xi}(y - y_S),$$

$$\sigma_y^{(1)} = -\sin\gamma \frac{\partial y_S}{\partial \xi}(y - y_S),$$

(5.8)

$$U^{(1)} = -Cm\big(\sin\gamma(y_S - y_B)\big)^{m-1}\cos\gamma\frac{\partial y_S}{\partial \xi}(y_S - y_B) +$$

$$+2G\int_{y_B}^{y_S} \exp(A\theta)\,\mathfrak{f}\big(\sin\gamma^2(y-y_S)^2\big)\cos\gamma\frac{\partial y_S}{\partial \xi}(y - y_B)\,dy \quad ,$$

and with $\sigma_x^{(1)}$ as given by $(5.6)_3$. Again we refrain from explicitly demonstrating the form for $V^{(1)}$.

The above first order solution was constructed in order to explicitly demonstrate the nature of the perturbation solution. As evident the stresses and velocities could be determined by mere quadratures, provided that the geometry $y = y_S$ is known. For the applied glaciologist this is a useful result, for he often knows the geometry from tellurometric measurements, and then can deduce from the above formulas the stress and velocity distributions. Formula $(5.6)_3$ for longitudinal normal stress, however, points at a weakness of our suggested perturbation approach. For when $\mathfrak{f}(0) = 0$, $\sigma_x^{(1)}$ becomes infinitely large, violating the basic assumption of a convergent perturbation series. Such a singularity arises for Glen's (Norton's) flow law at the free surface and can be interpreted physically as an infinitely large apparent viscosity, but can be avoided if the creep response function

(5.9)
$$\mathfrak{f}(x) = \frac{x^{(n-1)/2} + k}{1 + k}$$

is used (Hutter (1980a, 1981a,b)), since for (5.9) $\mathfrak{f}(0) = k/(1+k) \neq 0$. Of course, introducing a finite viscosity law to avoid the perturbation scheme to become invalid is not ideal. A further singular perturbation schemes would therefore be needed. Multiple variable expansions are the appropriate approach, but matched asymptotic expansions have been used (Johnson and Mc Meeking, see Hutter, 1983).

Of the compound solution shear stress and forward velocity component are of certain interest. From (5.5) and (5.8) it is easily seen that these are given by

$$\tag{5.10}\begin{aligned}\tau &= \tau^{(0)}+\varepsilon\tau^{(1)} = \left(\sin\gamma-\varepsilon\cos\gamma\frac{\partial y_S}{\partial\xi}\right)(y_S-y),\\ U_S &= (U^{(0)}+\varepsilon U^{(1)})_{\text{surface}} = U\left(y_S,\ \varepsilon\frac{\partial y_S}{\partial\xi}\right).\end{aligned}$$

Interesting in these formulas is that for small γ basal shear stress is given by $\tau\sim H\alpha_S$, where H is depth and α_S the surface slope (relative to the horizontal). Further, by including $O(\varepsilon)$-terms the surface speed is a function not only of the surface coordinate y_S, but also its derivative $\varepsilon\cdot\partial y_S/\partial\xi$. This observation is important. Improvements of the formulae (5.10) could be obtained by continuing the perturbation scheme to include second or even higher order terms. This has been done by Hutter (1980a). Such calculations show that the surface velocity U_S will have the form $U_S = U(y_S,\ \varepsilon\cdot\partial y_S/\partial\xi,\ \varepsilon^2\cdot\partial^2 y_S/\partial\xi^2,\ldots)$. In other words, with increasing degree of approximation U_S depends on higher and higher derivatives of the function describing the free surface. The mathematical implications of this will be considered in a moment.

In the above the temperature distribution and the surface profile are assumed known. By substituting (5.1) into the thermal equations of (4.1)–(4.3) the thermal boundary value problem is obtained. It reads

$$\tag{5.11}\begin{aligned}&\frac{\partial\vartheta}{\partial\bar t}+U\frac{\partial\vartheta}{\partial\xi}+V\frac{\partial\vartheta}{\partial y} = \frac{1}{\varepsilon}\mathrm{D}\frac{\partial^2\vartheta}{\partial y^2}+\frac{\mathrm{E}}{2\varepsilon}\exp(\mathrm{A}\theta)\,\mathrm{f}^2(\sigma'_{\mathrm{II}})\sigma'_{\mathrm{II}}+\varepsilon\mathrm{D}\frac{\partial^2\vartheta}{\partial\xi^2},\\ &\vartheta = \vartheta_S \qquad \text{on} \quad y = y_S(\xi,\bar t),\\ &\frac{d\vartheta}{dy}-\varepsilon^2\frac{\partial\vartheta}{\partial\xi}\frac{dy_B}{d\xi} = -\mathrm{Q}_{\text{geoth}}\left(1-\frac{\varepsilon^2}{2}\frac{dy_B}{d\xi}\right)^2 \qquad \text{on} \quad y = y_S(\xi,\bar t).\end{aligned}$$

Assuming that stress and velocity estimates have been obtained with a first iterate the two-point-boundary value problem (5.11) allows determination of a corrected temperature distribution. From earlier discussions it is known that steady state temperature profiles obtained by a balance of transverse convected heat with heat conduction match observed temperature profiles reasonably well. Inspection of (5.11) thus suggests a solution procedure which does not strictly follow the usual standard procedures of perturbation schemes. Indeed, in steady state (5.11) is *structurally* different from (4.8) by the inclusion of a longitudinal convective term. This term is partly responsible for *inversion profiles* as illustrated in Figure 6d, see Hutter (1982b). It follows, that an $O(\varepsilon^2)$-approximation of (5.11) is the necessary minimum, if observed features in temperature profiles are to be predicted from (5.11). A complete solution of the boundary value problem has not been given yet, but a numerical scheme has been suggested by Hutter (1983).

There remains determination of the *surface profile*. In this regard scrutiny of the above developments reveals that of the original equations (5.2)–(5.4) all but the kinematic surface equation $(5.3)_1$ were used. One could regard this latter equation as *the* prediction equation for the surface profile, because U and V may be regarded as known functions of y_S and its derivatives. However, simplifications can be introduced leading to the so called *kinematic wave equation*. To derive it, we start from the continuity equation and the basal boundary condition,

$$\frac{\partial U}{\partial \xi}+\frac{\partial V}{\partial y}=0, \qquad V=U\frac{dy_B}{d\xi} \qquad \text{on} \quad y=y_B(\xi).$$

Integrating the first of these from $y=y_B$ to $y=y_S$ and interchanging differentiation and integration with Leibnitz' rule in appropriate terms one obtains

$$\frac{\partial}{\partial \xi}\int_{y_B}^{y_S} U\,dy-U(y_S)\frac{\partial y_S}{\partial \xi}+V(y_B)=0,$$

which, when combined with equation $(5.3)_1$, yields

$$\text{(5.12)} \qquad \frac{\partial y_S}{\partial t}+\frac{\partial Q}{\partial \xi}=a_S^*, \qquad Q\equiv\int_{y_B}^{y_S} U(y)\,dy.$$

This is the kinematic wave equation, see Whitham (1974). In view of the perturbation solutions for U the flux Q may be regarded as a known function of the glacier geometry, $Q=\hat{Q}(y_S, \varepsilon\cdot\partial y_S/\partial\xi, \ldots)$, the number of arguments depending on the order of approximation in the perturbation scheme. Hence

$$\text{(5.13)} \qquad \frac{\partial y_S}{\partial t}+\mathfrak{C}\frac{\partial y_S}{\partial \xi}+\varepsilon\mathfrak{D}\frac{\partial^2 y_S}{\partial \xi^2}+O(\varepsilon^2)=a_S^*$$

with

$$\mathfrak{C}:=\frac{\partial \hat{Q}}{\partial y_S}, \qquad \mathfrak{D}:=\frac{\partial \hat{Q}}{\partial\left(\dfrac{\partial y_S}{\partial \xi}\right)},$$

both coefficients being functions of y_S and $\varepsilon\cdot\partial y_S/\partial\xi\cdot\mathfrak{C}$ is a *wave speed* and $\mathfrak{D}$ a *diffusivity*, and (5.13) is a non-linear (in fact *quasi-linear*) parabolic partial differential equation of the *convection-diffusion type*. When restricting calculations to the zeroth order terms (i.e., terms of $O(\varepsilon)$ are ignored) (5.13) reduces to the simple *forward wave equation*. Choosing for qualitative considerations an uniform temperature distribution ($\exp(A\theta)$ may be set equal to unity) the third of equations (5.5) reveals Q as a function of glacier depth, $H = y_S - y_B$, thus implying

$$\begin{aligned}&\frac{\partial H}{\partial t} + \mathfrak{C}(H)\frac{\partial H}{\partial \xi} = a_S^*(\xi, H, t),\\ &\mathfrak{C}(H) = \underbrace{\mathscr{F}'(\sin\gamma H)\sin\gamma H + \mathscr{F}(\sin\gamma H)}_{\text{sliding}} + \underbrace{2\sin\gamma H^2 \mathfrak{f}(\sin\gamma^2 H^2)}_{\text{gliding}}\end{aligned} \tag{5.14}$$

where $\mathscr{F}$ is defined in (3.6).

$\mathfrak{C}(H)$ is the speed, at which surface bulges travel downglacier; it can be compared with the surface particle speed $U(H)$, $(5.5)_3$. Exploring the two formulas for Glen's flow law and Weertmann type sliding with realistic values of the phenomenological constants then shows that the surface wave speed is about four times the surface particle speed. This compares favorably with observation. In this lowest order approximation (5.14) is also the equation geverning the steady state surface profile. Forward integration from one margin to the other should in this simplified situation determine the profile geometry. For a sliding law with $\mathscr{F}(0) = 0$, $\mathfrak{C}(0) = 0$, (5.14) may be *singular*. The near-margin behavior of the steady state version of (5.14) must therefore by analysed by looking at power series solutions of the form

$$H = k(\xi - \xi_M)^\delta(1 + O(|\xi - \xi_M|)) \quad \text{as} \quad \xi \to \xi_M. \tag{5.15}$$

The free constants k and δ depend on the margin behavior of $\mathfrak{C}$ and a_S^*, but it can easily be shown by substituting (5.15) into (5.14) that, when $\mathscr{F}(0)$ = bounded and $a_S^* \neq 0$ at $\xi = \xi_M$, that $\delta < 1$. *In this case margin slopes are infinite*; the perturbation scheme breaks down as the assumption of slow variation of glacier geometry is violated close to the margin. A separate margin solution must be constructed with the unstretched equations which will *match* the above outer solution.

There is one exceptional case for which such a matching procedure is not necessary. To illustrate it let $\mathscr{F}(\tau) = C\tau^m$ and assume that C is a function of position and becomes singular as the margin is approached as follows:

$$C = C^*|\xi - \xi_M|^{-m} \quad \text{as} \quad \xi \to \xi_M. \tag{5.16}$$

The near margin-power solution has now a finite slope at $\xi = \xi_M$ ($\delta = 1$) and (5.15) becomes

$$H = \operatorname{sgn}(a_S^*)\left(\frac{|a_S^*|}{(m+1)C^*\sin^m\gamma}\right)^{1/(m+1)}|\xi - \xi_M| \quad \text{as} \quad \xi \to \xi_M. \tag{5.17}$$

Integration of (5.14) can be commenced at the left margin, say, with the aid of (5.17) and then continued using the steady state version of (5.14). However a numerical value for C^* should be determined. This is possible as margin velocities can be related to C^* and margin ablation; the reader may show this with the aid of equations (5.5) and (5.16). Measuring margin velocities and margin ablation thus determines C^*.

The above discussion pertains to the steady state version of (5.14), which is valid when $O(\varepsilon)$-terms are ignored. To within terms which are linear in ε, equation (5.13) applies. Here, diffusive terms are included; such terms are known to smooth out processes and might, perhaps, yield regular margin behavior. The problem was analysed in detail by Hutter (1983); he shows that the differential equation is singular in general at the margin so that again a margin solution of the form (5.15) must be sought. Margin slopes turn again out to be singular unless C is singular as indicated in (5.16). Inclusion of diffusive terms has thus not led to profile geometries which would, in general, be uniformly determined from a steady state analysis of (5.13).

Consider now the *time dependent problem*. This relates to *waves on glaciers* which manifest themselves in three typical forms, namely *surface waves*, *seasonal waves* and *surges*. Surface waves are undulations of the glacier surface which travel downglacier at small speeds of (typically) three to four times the surface particle speed. Such surface waves were analysed by Blümke and Finsterwalder (1905), Finsterwalder (1907), Weertmann (1958), Nye (1960, 1963a,b), Lick (1970) and Lliboutry (1971). A more careful new analysis is due to Fowler and Larson (1978, 1980b) and Fowler (1980). Seasonal and surge-type waves are very fast waves, which manifest themselves as fluctuations of surface velocities rather than surface bulges. These waves are not yet clearly understood, but one stipulation is that they arise when the sliding function $\mathscr{F}$ is becoming multi-valued (Figure 4b) or its slope is approaching infinity (Figure 4c). Fowler (1980) gives an approximate analysis of the latter situation and the interested reader is referred to his paper.

Surface waves can be studied with the aid of (5.12) or subsequent equations. Associated boundary conditions follow from a ξ-integral of (5.12), see Fowler and Larson (1978), Hutter (1983). Here in this lecture we focus attention to a glacier with non-moving steady head, $\dot{\xi}_M^- = 0$; then the boundary condition is simply $y_S(\xi_M^-) = y_B(\xi_M^-)$ or $H(\xi_M^-) = 0$.

To lowest order, equation (5.14) applies subject to the boundary condition $H(\xi_M^-, \bar{t}) = 0$ and the initial condition $H(\xi_0) = H_0(\xi_0)$ for $\xi_M^- \leqslant \xi_0 \leqslant \xi_M^+$. The *characteristic form* of this initial boundary value problem is

$$\frac{dH}{d\bar{t}} = a_\xi^*, \qquad \frac{d\xi}{d\bar{t}} = \mathfrak{C}(H),$$

$$H(\xi_M^-, \bar{t}) = 0, \qquad t \geqslant 0, \tag{5.18}$$

$$H(\xi_0) = H_0(\xi_0), \qquad \xi_M^- \leqslant \xi_0 \leqslant \xi_M^+.$$

It has been studied in detail by Fowler and Larson (1980a). This study indicates that smooth solutions do exist up-glacier, i.e., from the head downward, but that generally shocks are formed before the head is reached. This can convincingly be seen from (5.14) or (5.18) when these equations are solved for vanishing a_S^*. In this case

$$H = H_0(\xi - \mathfrak{C}(H)\bar{t}) \tag{5.19}$$

is the exact solution defining H implicitly. If $\mathfrak{C}(\cdot)$ is a *monotone increasing* function of its argument, (5.19) implies that a surface bulge will oversteepen with time eventually resulting in a multi-valued function $H(\xi)$(¹). Where this arises, shocks are formed, which are discontinuities of surface bulges. The problem of evaluating the conditions at the shock, can, however, not stem from (5.12) or (5.18) alone because the latter assume differentiability. A clue as to the approximate form of the equation describing conditions across the shock is obtained, if it is recognized that (5.12) is the local form of the volume balance for the glacier as a whole, whose *global* form is

$$\frac{\partial}{\partial t}\int_{\xi_1}^{\xi_2} y_S\, d\xi - (Q(\xi_2) - Q(\xi_1)) = \int_{\xi_1}^{\xi_2} a_S\, d\xi, \tag{5.20}$$

see Figure 7. ξ_1 and ξ_2 are two positions marking an interval which encloses

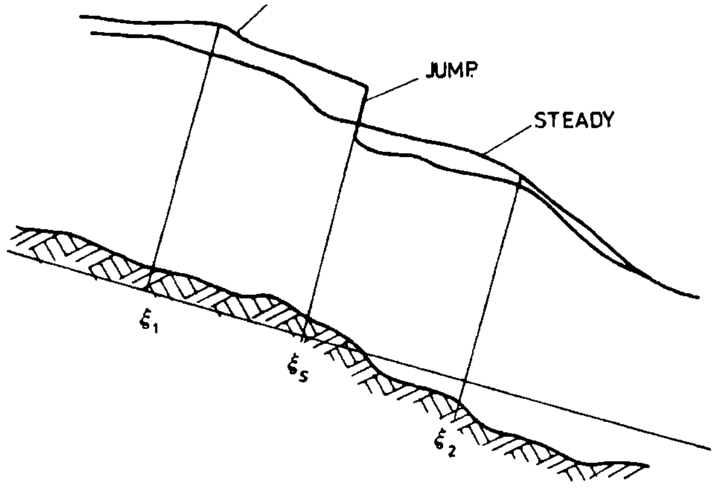

Fig. 7. Explaining the volume balance for the glacier as a whole

the shock at $\xi = \xi_S$. Mathematically, (5.20) is more general than (5.12) because the fields need not be differentiable functions of position and time.

(¹) If $H_1 > H_2$ then $C(H_1) > C(H_2)$. Waves of points with large H are faster than those with small H explaining the over-steepening.

By simply invoking such differentiability assumptions (5.12) is obtained. However, when y_S may suffer a jump at $\xi = \xi_S$, (5.20) implies

$$\int_{\xi_1}^{\xi_2} \frac{\partial y_S}{\partial \bar{t}} d\xi + [\![y_S(\xi_S, \bar{t})]\!] \frac{d\xi_S}{d\bar{t}} - (Q(\xi_2) - Q(\xi_1)) = \int_{\xi_1}^{\xi_2} a_S \, d\xi,$$

or when $\xi_1 = \xi_2 = \xi_S$

$$\frac{d\xi_S}{d\bar{t}} = \frac{[\![Q(\xi_S)]\!]}{[\![y_S(\xi_S)]\!]} \tag{5.21}$$

where $[\![Q]\!]$ and $[\![y_S]\!]$ are the jumps of Q and y_S across the shock. The shock front ξ_S travels with speed (5.21).

It is not our purpose to dwell on the exploration of solutions of (5.18) and (5.21). The analysis parallels usual procedures of weak solutions of *hyperbolic equations that result from conservation laws*, see Dafermos (1974), Courant and Friedrichs (1948). On the other hand, carrying the perturbation solution to $O(\varepsilon)$-terms alters the equation from its hyperbolic form to the parabolic convection-diffusion equation (5.13). This new equation can also be shown to be singular as the diffusivity vanishes at the margins. Although unsteady solutions of this convection-diffusion equation have not been constructed so far, experience with the steady equation indicates that close to the snout the smoothening effect of the diffusive term is probably not suffiently high to override shock formation close to the snout. Moreover, addition of a further spatial derivative supposedly necessitates a further boundary condition. To date, the matter is not clearly understood, but Fowler and Larson (1980) conjecture that in view of the singularity of the equation no further boundary condition will be needed. The resolution of such equations are important ones, as can be seen from (5.13), which suggests that with higher and higher order perturbation terms taken into account higher and higher derivatives will enter the kinematic wave equation. This is corroborated by Hutter (1980a) who shows that when $O(\varepsilon^2)$-terms are accounted for third order derivatives will enter equation (5.13). Effects attributed to these terms are classified as *dispersion*, and Hutter's calculations indicate that they are not negligible.

b) Small bed inclination. We return to equations (5.2)–(5.4) and assume small bed inclinations, $\gamma \leqslant O(\varepsilon)$. At least two terms in $(5.2)_2$ are then $O(\varepsilon)$ and if we assume that τ is $O(1)$ this equation would to lowest order reduce to $\partial\tau/\partial y = 0$, or $\tau \equiv 0$ in view of the boundary condition $(5.3)_2$. This must be wrong, for the flow must be clearly shearing. Hence we assume $\tau = O(\varepsilon)$ so that transverse shear stress gradients can be balanced by longitudinal pressure gradients. From $(5.2)_5$ it then follows, since $\partial U/\partial y = O(1)$ by construction of the scaling,

$$\mathrm{G}\mathfrak{f}(\sigma'_{\mathrm{II}})\tau = O(1).$$

Further, since G is large, $(5.2)_4$ implies to lowest order $\sigma_x = \sigma_y$, thus $\sigma_{\mathrm{II}} \approx \tau$. The last equation must therefore imply

$$\mathrm{G}O(\mathfrak{f}(\tau^2))O(\varepsilon) = O(1), \tag{5.22}$$

and because

$$O(\mathfrak{f}(\tau^2)) = \begin{cases} \mathrm{k} & \text{for polynomial law (2.4),} \\ \tau^{n-1} & \text{for Glen's flow law,} \end{cases}$$

we have

$$\mathrm{G} = \begin{cases} \dfrac{1}{\mathrm{k}\varepsilon} & \text{for polynomial law (2.4),} \\[2ex] \dfrac{1}{\varepsilon^n} & \text{for Glen's flow law.} \end{cases} \tag{5.23}$$

This scaling has first been introduced by Morland and Johnson (1982). Accordingly the stretching parameter is related to the "Glen number" G which for small bed inclinations has been shown to be a large number. With (5.23) and to lowest order we have indeed $\sigma_x = \sigma_y$ proving *a posteriori* that this assumption was correct when constructing (5.23). Subsequent calculations will be performed for Glen's flow law. Introdicing $\sin\gamma = \varepsilon\gamma_0$ and $\cos\gamma \sim 1$, (5.2) have then the form

$$\begin{aligned}
&\frac{\partial U}{\partial \xi} + \frac{\partial U}{\partial y} = 0, \\
&\varepsilon\frac{\partial \sigma_x}{\partial \xi} + \frac{\partial \tau}{\partial y} + \varepsilon\gamma_0 + O(\varepsilon^2) = 0, \\
&\varepsilon\frac{\partial \tau}{\partial \xi} + \frac{\partial \sigma_y}{\partial y} - 1 + O(\varepsilon^2) = 0, \\
&\varepsilon^{n+1}\frac{\partial U}{\partial \xi} = \frac{1}{2}\exp(\mathrm{A}\theta)\mathfrak{f}(\sigma'_{\mathrm{II}})(\sigma_x - \sigma_y), \\
&\frac{\partial U}{\partial y} + \varepsilon^2\frac{\partial V}{\partial \xi} = 2\varepsilon^{-n}\exp(\mathrm{A}\theta)\mathfrak{f}(\sigma'_{\mathrm{II}})\tau.
\end{aligned} \tag{5.24}$$

Boundary conditions (5.3), (5.4) are unchanged. A systematic approximate solution procedure for (5.24), (5.3), (5.4) would be a perturbation expansion of the form (5.6). Here, only lowest order solutions are considered. From $(5.24)_{3,4}$ and $(5.3)_3$ we deduce

(5.25) $$\sigma_x^{(0)} = \sigma_y^{(0)} = (y - y_S) - p^{\text{atm}}.$$

Shear stresses now follow by integrating $(5.24)_2$ subject to boundary condition $(5.3)_2$:

(5.26) $$\tau = \varepsilon\zeta^2\left(\gamma_0 - \frac{\partial y_S}{\partial \xi}\right)(y_S - y), \qquad \zeta = \operatorname{sgn}\left(\gamma_0 - \frac{\partial y_S}{\partial \xi}\right).$$

This formula is important. It shows that τ is indeed $O(\varepsilon)$ and, since $\varepsilon(\gamma_0 - \partial y_S/\partial\xi) \sim \alpha_S$, where α_S is the surface slope measured from a horizontal line, the classical shear stress formula is obtained, namely $\tau = (y_S - y)\alpha_S$. Moreover, τ vanishes at positions $\xi = \xi_d$ where surface slopes are horizontal ($\gamma = \partial y_S/\partial\xi$). For a sliding law with $\mathscr{F}(0) = 0$ these are the positions where the sliding velocity vanishes. It will be shown in a moment that $\xi = \xi_d$ marks the position where the forward velocity vanishes *at all depths*, corresponding to an ice divide. With the shear stress being determined, $(5.24)_5$ implies to lowest order

$$\frac{\partial U}{\partial y} = 2\varepsilon^{-n}\exp(A\theta)\mathfrak{f}(\tau^2(y))\tau(y) = O(1)$$

and after integration, subject to $(5.4)_1$

(5.27) $$U^{(0)} = \zeta C(\zeta\tau(y_B))^m + 2\varepsilon^{-n}\int_{y_B}^{y} \exp(A\theta(y))\mathfrak{f}(\tau^2(y))\tau(y)\,dy$$

in which τ is given by (5.26). It immediately follows that $U^{(0)} = 0$ whenever $\tau = 0$ corroborating that $\xi = \xi_d$ marks positions of *ice divides.* With the continuity equation and the second basal boundary condition an expression for $V^{(0)}$ could equally be derived.

More interesting is to explore the kinematic surface equation; it can also be written in the form of the kinematic wave equation, see (5.12)

(5.28) $$\frac{\partial y_S}{\partial t} + \frac{\partial Q}{\partial \xi} = a_S^*,$$

with

(5.29) $$Q = \zeta C(\zeta\tau(y_B))^m(y_S - y_B) + 2\varepsilon^{-n}\int_{y_B}^{y_S}(y_S - y)\exp(A\theta(y))\mathfrak{f}(\tau^2(y))\tau(y)\,dy.$$

Notice that $Q = Q(y_B, y_S - y_B, \partial y_S/\partial\xi)$. Equation (5.28) is therefore of the convection-diffusion type, as shown in (5.13). Because $y_S - y_B$ and $\partial y_S/\partial\xi$ enter the lead order equations, (5.28), (5.29) yield and equation for y_S rather than the depth variable $H = y_S - y_B$ as it was the case for the lowest order solution of the steep glacier. Thus, the bed profile is not simply superimposed on the geometry of the flat bed. Further, the convection diffusion equation (5.28) (5.29) may be singular at the margins depending on whether the sliding coefficient is bounded or singular at the margin. Local solutions at the margin follow again by assuming power series expansions

$$y_S - y_B = k\,|(\xi - \xi_M)|^{\delta}(1 + O(|\xi - \xi_M|)), \qquad \text{as} \quad \xi \to \xi_M \tag{5.30}$$

in which k and δ can be deterimed by substituting (5.30) into (5.28), (5.29). The *steady state* equation then shows singular behavior at the margin as $\delta < 1$, or $\partial y_S/\partial\xi \to \infty$ unless the sliding coefficient is singular as $C = C^*(y_S - y_B)^{-m}$ in which case margin slope, margin ablation a_S^* and margin velocity $U_M^{(0)}$ are related by

$$U_M^{(0)}\left(\frac{\partial y_S}{\partial\xi} - \frac{\partial y_B}{\partial\xi}\right) = a_S^* \qquad \text{at} \quad \xi = \xi_M . \tag{5.31}$$

In other words, measuring margin ablation, margin velocity and bottom slope will yield the surface slope $\partial y_S/\partial\xi$.

To integrate the steady convection-diffusion equation two cases must thus be differentiated. In the first $\delta < 1$, and the small slope assumption breaks down close to the margin as the conditions of the shallow ice approximation are no longer fulfilled. A separate margin solution of the full equations must be constructed. At a small distance from the margin this solution provides the initial conditions, namely values for y_S and $\partial y_S/\partial\xi$ for the integration of the convection diffusion equation. The second margin subsequently follows from the condition $y_S = y_B$. This matched-asymptotic expansion procedure has not been analysed in detail to date. In the second case, namely when (5.31) holds at the margin integration is started at the margin by (5.31)

$$y_S - y_B = \left(\frac{a_S^*}{U^{(0)}}\right)_M (\xi - \xi_M), \qquad \text{as} \quad \xi \to \xi_M,$$

yielding the initial condition for the subsequent forward integration. This numerical integration has been performed by Morland and Johnson (1982). For details the reader is referred to that paper. They show that with integration from the upper margin a flat horizontal surface far distant from the margin is obtained; on the other hand, when integration is commenced at the lower margin (the snout!) a surface profile results which realistically models ice sheet geometries. However close to the other, upper margin $\partial y_S/\partial\xi$ turns out to be large, thus again violating the assumptions of a shallow ice approximation. In the neighborhood of the head again an independent solution that is matched to the outer solution should be found. This has not been done yet. For further results the reader is referred to Morland and Johnson (1981, 1982) and to Johnson (1981).

6. Three-dimensional flow effects in ice sheets

Ice sheets spread in more than just one direction, and flow is only approximately planar. Here in this final section we investigate the inferences that can be drawn from the application of the shallow ice approximation in the full three-dimensional flow problem. To simplify matters the ice sheets will be assumed to be wholly *grounded*, and a temperature dependence of the creep law will be ignored.

The ice sheet is assumed to be embedded in an Euclidean 3-space with Cartesian coordinates x, y, z; x and z horizontal, and y is vertical. Thus, $y = y_S(x, z, t)$ and $y = y_B(x, z)$ denote, respectively the free surface and the rock-bed. Governing equations are equations (3.3). When restricting attention to the mechanical equations and when invoking the shallow ice approximation

$$\xi = \varepsilon x,\ \zeta = \varepsilon z,\ \bar{t} = \varepsilon t, \qquad V = \frac{v}{\varepsilon},\ U = u, \qquad W = w,\ y = \eta,\ a_S^\perp = \frac{a_S^*}{\varepsilon} \tag{6.1}$$

these equations read

$$\begin{aligned}
-\varepsilon\frac{\partial p}{\partial \xi} + \varepsilon\frac{\partial \sigma_x'}{\partial \xi} + \frac{\partial \tau_{xy}'}{\partial \eta} + \frac{\partial \tau_{xz}'}{\partial \zeta} + \varepsilon g_1 &= 0,\\
\varepsilon\frac{\partial \tau_{xy}'}{\partial \xi} - \frac{\partial p}{\partial \eta} + \frac{\partial \sigma_y'}{\partial \eta} + \frac{\partial \tau_{yz}'}{\partial \zeta} - g_2 &= 0,\\
\varepsilon\frac{\partial \tau_{xz}'}{\partial \xi} + \frac{\partial \tau_{yz}'}{\partial \eta} - \varepsilon\frac{\partial p}{\partial \zeta} + \varepsilon\frac{\partial \sigma_z'}{\partial \zeta} + \varepsilon g_3 &= 0,\\
\frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \eta} + \frac{\partial W}{\partial \zeta} &= 0,\\
\varepsilon\frac{\partial U}{\partial \xi} &= G\mathfrak{f}(\tau_{II})\sigma_x',\\
\frac{\partial U}{\partial \eta} + \varepsilon^2\frac{\partial V}{\partial \xi} &= 2G\mathfrak{f}(\tau_{II})\tau_{xy}',\\
\varepsilon\frac{\partial U}{\partial \zeta} + \varepsilon\frac{\partial W}{\partial \xi} &= 2G\mathfrak{f}(\tau_{II})\tau_{xz}',\\
\varepsilon\frac{\partial V}{\partial \eta} &= G\mathfrak{f}(\tau_{II})\sigma_y',\\
\varepsilon^2\frac{\partial V}{\partial \zeta} + \frac{\partial W}{\partial \eta} &= 2G\mathfrak{f}(\tau_{II})\tau_{yz}',\\
\varepsilon\frac{\partial W}{\partial \zeta} &= G\mathfrak{f}(\tau_{II})\sigma_z',
\end{aligned} \tag{6.2}$$

in which

$$\tau_{II} = \tfrac{1}{2}(\sigma_x'^2 + \sigma_y'^2 + \sigma_z'^2) + \tau_{xy}'^2 + \tau_{yz}'^2 + \tau_{zx}'^2 \tag{6.3}$$

is the second stress deviator invariant, and g_1, g_2 and g_3 are components of the gravity force; for a horizontal bed $g_1 = g_3 = 0$, $g_2 = 1$. Alternatively, the boundary conditions at the free surface and the base, (3.5) and (3.6), can, respectively be shown to have the form:

At the free surface: $y = y_S(\xi, \zeta, \bar{t})$;

$$\frac{\partial y_S}{\partial t} + \frac{\partial y_S}{\partial \xi} U + \frac{\partial y_S}{\partial \zeta} W - V = a_S^*,$$

$$\begin{aligned} -\varepsilon(-p + \sigma_x')\frac{\partial y_S}{\partial \xi} + \tau_{xy}' - \varepsilon\tau_{xz}'\frac{\partial y_S}{\partial \zeta} &= \varepsilon p^{\text{atm}}\frac{\partial y_S}{\partial \xi}, \\ -\varepsilon\tau_{xy}'\frac{\partial y_S}{\partial \xi} + (-p + \sigma_y') - \varepsilon\tau_{yz}'\frac{\partial y_S}{\partial \zeta} &= -p^{\text{atm}}, \\ -\varepsilon\tau_{xz}'\frac{\partial y_S}{\partial \xi} + \tau_{yz}' - (-p + \sigma_z')\frac{\partial y_S}{\partial \zeta} &= \varepsilon p^{\text{atm}}\frac{\partial y_S}{\partial \zeta}. \end{aligned} \tag{6.4}$$

At the bottom surface: $y = y_B(\xi, \zeta)$;

$$\frac{\partial y_B}{\partial \xi} U + \frac{\partial y_B}{\partial \zeta} W - V = 0,$$

$$\begin{aligned} U &= -\frac{\mathscr{F}}{\Psi}\left(\sigma_x^* \varepsilon\frac{\partial y_B}{\partial \xi} - \tau_{xy}^* + \tau_{xz}^* \varepsilon\frac{\partial y_B}{\partial \zeta}\right), \\ \varepsilon V &= -\frac{\mathscr{F}}{\Psi}\left(\tau_{xy}^* \varepsilon\frac{\partial y_B}{\partial \xi} - \sigma_y^* + \tau_{yz}^* \varepsilon\frac{\partial y_B}{\partial \zeta}\right), \\ W &= -\frac{\mathscr{F}}{\Psi}\left(\tau_{xz}^* \varepsilon\frac{\partial y_B}{\partial \xi} - \tau_{yz}^* + \sigma_z^* \varepsilon\frac{\partial y_B}{\partial \zeta}\right), \end{aligned} \tag{6.5}$$

where $\mathscr{F}$ is a function of $(\boldsymbol{\sigma}^* \cdot \boldsymbol{n})^2$ and

$$\Psi = \sqrt{1 + \left(\frac{\partial y_B}{\partial x}\right)^2 + \left(\frac{\partial y_B}{\partial z}\right)^2}.$$

Equations (6.2) to (6.4) are now in a form suitable for the lowest order approximation to be treated here. Before such an approximate treatment can be given, the parameters must be expressed in terms of G. To find this relationship, let us return to equations $(6.2)_{1,3}$. For vanishing g_1 a lowest order force balance in the ξ-direction requires the transverse stress gradient $\partial\tau_{xy}/\partial\eta$ to vanish unless τ_{xy} is $O(\varepsilon)$. In this case transverse shear stress gradients can be balanced by the horizontal pressure gradient $\partial p/\partial\xi$.

Similarly in equation $(6.2)_3$; here transverse shear stress gradients $\partial\tau_{yz}/\partial\eta$ can balance the horizontal pressure gradient $\partial p/\partial\zeta$ only provided that $\tau_{yz} = O(\varepsilon)$. *We assume such order of magnitude relationships hold*; they are plausible from the basic approximate shear stress formula according to which the shear stresses are proportional to the surface inclination angle, which is small after all. It then follows from the sixth and the ninth of equations (6.2), since $\partial U/\partial\eta$ and $\partial W/\partial\eta$ are $O(1)$ that

$$\text{(6.6)} \qquad \mathrm{G}O(\mathfrak{f}(\cdot)\tau_{xy}) = O(1), \qquad \mathrm{G}O(\mathfrak{f}(\cdot)\tau_{yz}) = O(1).$$

Assuming further (and momentarily) σ_x, σ_y, σ_z and τ_{xz} to be $O(\varepsilon)$ or smaller it follows that $\mathfrak{f}(\cdot)$ is $O(\mathrm{k})$ for a finite voscosity law, but $\mathfrak{f}(\cdot) = O(\varepsilon^{n-1})$ for a power flow law with exponent n. Hence (6.6) implies

$$\mathrm{G} = \begin{cases} \dfrac{1}{\mathrm{k}\varepsilon} & \text{for a finite viscosity law,} \\[2ex] \dfrac{1}{\varepsilon^n} & \text{for Glen's flow law.} \end{cases}$$

Considerating this relationship between G and ε we shall now simplify the boundary value problem by ignoring all terms of order higher than and equal to ε. From $(6.2)_{5,7,8,10}$ it then follows that σ_x, σ_y, σ_z and τ_{xz} are all $O(\varepsilon^2)$ thereby corroborating our earlier assumption, so that *to lowest order* we have

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xz} = 0.$$

Therefore, the field equations reduce to

$$\text{(6.7)} \qquad \begin{aligned} -\frac{\partial p}{\partial\xi} + \frac{\partial T_x}{\partial\eta} \qquad\quad + g_1 &= 0, \\ -\frac{\partial p}{\partial\eta} \qquad\quad - g_2 &= 0, \\ \frac{\partial T_z}{\partial\eta} - \frac{\partial p}{\partial\zeta} + g_3 &= 0, \\ \frac{\partial U}{\partial\xi} + \frac{\partial V}{\partial\eta} + \frac{\partial W}{\partial\zeta} &= 0, \\ \frac{\partial U}{\partial\eta} &= 2\mathrm{F}(T_x^2 + T_z^2)\,T_x, \\ \frac{\partial W}{\partial\zeta} &= 2\mathrm{F}(T_x^2 + T_z^2)\,T_z, \end{aligned}$$

in which $T_x = \tau_{xy}/\varepsilon$ and $T_z = \tau_{yz}/\varepsilon$ are $O(1)$ and F is defined by

$$\text{(6.8)} \qquad \mathrm{F}(T_x^2 + T_z^2) = \varepsilon^{1-n}\mathfrak{f}(\varepsilon^2(T_x^2 + T_z^2)) \quad [= O(1)].$$

A similar reduction is possible with the boundary conditions. Deleting for the moment the kinematic surface condition $(6.4)_1$ the boundary conditions of stress at the free surface reduce to

(6.9)

$$p = p^{\mathrm{atm}}, \qquad T_x = -p^{\mathrm{atm}}\frac{\partial y_S}{\partial \xi}, \qquad T_z = -p^{\mathrm{atm}}\frac{\partial y_S}{\partial \zeta} \qquad \text{on} \quad y = y_S(\xi, \zeta, t),$$

and those at the base become

$$\begin{aligned} V &= U\frac{\partial y_B}{\partial \xi} + W\frac{\partial y_B}{\partial \zeta}, \\ U &= C'[T_x^2 + T_z^2]^{(m-1)/2}\, T_x, \\ W &= C'[T_x^2 + T_z^2]^{(m-1)/2}\, T_z, \end{aligned} \tag{6.10}$$

where $C' = \varepsilon^m C$ is $O(1)$ for viscous sliding to be significant.

In order to explicitly demonstate the integrability of equations (6.8)–(6.10) let us look at the simplified case in which $g_1 = g_3 = 0$, $g_2 = 1$. We shall also ignore the influence of the atmospheric pressure which is negligibly small anyhow. It then follows from $(6.7)_2$ and $(6.9)_1$

$$p = (y_S - \eta). \tag{6.11}$$

With this result $(6.7)_{1,3}$ can now be integrated subject to the boundary conditions $(6.9)_{2,3}$. This yields:

$$T_x = -\frac{\partial y_S}{\partial \xi}(y_S - \eta), \qquad T_z = -\frac{\partial y_S}{\partial \zeta}(y_S - \eta). \tag{6.12}$$

By considering the equilibrium equations and associated boundary conditions of stress it has therefore been possible to determine the complete stress distribution. Equation (6.11) gives the overburden pressure, which increases linearly with depth and (6.12) provide formulas for the horizontal shear stresses. These formulas are very important ones as they generalize the famous shear stress formula for two dimensions according to which shear stress is proportional to surface inclination and depth. Our nondimensionalization and the stretching transformations have been such that to lowest order these dependencies are recovered. Indeed, (6.12) states that the two horizontal shear stress components are proportional to depth and surface gradient *in the direction of the stress component.* Moreover, T_x and T_z may change sign at positions where $\partial y_S/\partial \xi$ and $\partial y_S/\partial \zeta$ go through zero. (6.10) then implies that the corresponding component of the sliding velocity will vanish. Later on it will be shown that U and V vanish at all depths in positions where $\partial y_S/\partial \xi = 0$ and $\partial y_S/\partial \zeta = 0$, respectively. This result is of very practical significance, for it implies the following important facts which can be tested by observation:

(i) At any given position the horizontal velocity vector does not change direction with depth, or the ratio of the two horizontal velocity components are only functions of ξ, ζ but not η.

(ii) At any position the direction of the velocity vector is that of the steepest descent of the surface profile.

(iii) A dome or a trough is the location of zero velocity.

The proof of these statements is straightforward.

In order to determine the velocity field the results (6.11) and (6.12) are used in $(6.7)_{5,6}$, which can be written as

$$(6.13)\qquad \begin{aligned} \frac{\partial U}{\partial \eta} &= -2\mathrm{F}\left(\mathscr{T}^2(\eta)\right)\frac{\partial \bar{y}_S}{\partial \xi}(y_S-\eta) \equiv g_\xi(\eta), \\ \frac{\partial W}{\partial \eta} &= -2\mathrm{F}\left(\mathscr{T}^2(\eta)\right)\frac{\partial \bar{y}_S}{\partial \zeta}(y_S-\eta) \equiv g_\zeta(\eta), \end{aligned}$$

where

$$(6.14)\qquad \mathscr{T}^2(\eta) = (y_S-\eta)^2\left(\left(\frac{\partial y_S}{\partial \xi}\right)^2+\left(\frac{\partial y_S}{\partial \zeta}\right)^2\right).$$

The right hand sides of (6.13) are known functions of η (and ξ, ζ and $\bar{t}$ whose dependence is not indicated). Defining

$$(6.15)\qquad g_\xi^{(1)} = \int_{y_B}^{\eta} g_\xi(\bar{\eta})\,d\bar{\eta}, \qquad g_\zeta^{(1)} = \int_{y_B}^{\eta} g_\zeta(\bar{\eta})\,d\bar{\eta},$$

integration of (6.13), subject to the boundary conditions $(6.10)_{2,3}$ reveals that

$$(6.16)\qquad \begin{aligned} U(\xi,\zeta,\eta,t) &= -g_\xi^{(1)}(\eta)-\mathrm{C}'|\mathscr{T}(y_B)|^{m-1}\frac{\partial y_S}{\partial \xi}(y_S-y_B), \\ W(\xi,\zeta,\eta,t) &= -g_\zeta^{(1)}(\eta)-\mathrm{C}'|\mathscr{T}(y_B)|^{m-1}\frac{\partial y_S}{\partial \zeta}(y_S-y_B), \end{aligned}$$

proving that U and W vanish whenever $\partial y/\partial\xi$ and $\partial y/\partial\zeta$ are zero. By, finally, integrating the continuity equation, the transverse velocity component V could also be determined.

We conclude by deriving the differential equation for the surface profile. To this end the two dimensional analogue of the kinematic wave equation (5.13) is needed. We leave it as an excercise to the reader to show that the kinematic surface equation

$$\frac{\partial y_S}{\partial \bar{t}}+\frac{\partial y_S}{\partial \xi}U+\frac{\partial y_S}{\partial \zeta}W-V = a_S^*,$$

together with the boundary condition $(6.10)_1$ allows deduction of the two-dimensional kinematic wave equation

$$\frac{\partial \bar{y}_S}{\partial t}+\frac{\partial Q_\xi}{\partial \xi}+\frac{\partial Q_\zeta}{\partial \zeta}=a_S^*, \tag{6.17}$$

with

$$Q_\xi := \int_{y_B}^{y_S} U\, d\eta, \qquad Q_\zeta := \int_{y_B}^{y_S} W\, d\eta.$$

As seen from (6.16) this equation is of the convection-diffusion type. Defining

$$g_\xi^{(2)} = \int_{\bar{y}_B}^{\bar{y}_S} g_\xi^{(1)}(\eta)\, d\eta, \qquad g_\zeta^{(2)} = \int_{\bar{y}_B}^{\bar{y}_S} g_\zeta^{(1)}(\eta)\, d\eta,$$

its steady state analogue has the form

$$\begin{aligned}\frac{\partial}{\partial \xi}\left(g^{(2)}\left(y_S, \frac{\partial y_S}{\partial \xi}, \frac{\partial y_S}{\partial \zeta}\right)+C'|\mathscr{T}(y_B)|^{m-1}(y_S-y_B)^2\frac{\partial y_S}{\partial \xi}\right)+\\ +\frac{\partial}{\partial \zeta}\left(g^{(2)}\left(y_S, \frac{\partial y_S}{\partial \xi}, \frac{\partial y_S}{\partial \zeta}\right)+C'|\mathscr{T}(y_B)|^{m-1}(y_S-y_B)^2\frac{\partial y_S}{\partial \zeta}\right)=-\alpha_S^*\end{aligned} \tag{6.18}$$

in which the structure of a second order equation becomes evident. It involves y_S and y_B separately and can not be written as an equation for the difference (y_S-y_B). So, the bed profile is not simply superimposed on a corresponding flat bed surface profile. However, (6.18) is singular at the margin unless C' is singular as $(\bar{y}_S-\bar{y}_B)^{-m}$ as the margin is approached. When solving (6.18) a procedure should therefore be known how (6.18) is handled close to the margins. This is not easy, since the problem is not well posed. Boundary conditions can only be presented along part of the closed boundary from which integration proceeds into the ice. The remaining portion of the boundary is then obtained from the condition $y_S = y_B$. The method, how this integration is best performed is still unknown and awaits its resolution.

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