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OPTIMAL STOCHASTIC CONTROL AND HAMILTON-JACOBI-BELLMAN EQUATIONS

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I. Introduction

Let us first describe briefly the optimal stochastic control problem we consider: *the state of the system* we want to control is given by the solution of the following *stochastic differential equation*:

$$(1)\qquad \begin{cases} dy_x(t) = \sigma\big(y_x(t), v(t,\omega)\big)dW_t + b\big(y_x(t), v(t,\omega)\big)dt, \\ y_x(0) = x \in \overline{\mathcal{O}}, \end{cases}$$

where $\overline{\mathcal{O}}$ is a regular domain in $\mathbf{R}^N$; W_t is a Brownian motion in $\mathbf{R}^p$, $\sigma(x, v)$ is a matrix-valued function from $\overline{\mathcal{O}} \times V$; $b(x, v)$ is a vector-valued function from $\overline{\mathcal{O}} \times V$; V is a given closed set in $\mathbf{R}^m$ (for example) and $v(t, \omega)$ is any adapted process taking its values in V. The precise assumptions and meaning of (1) will be given below.

We then introduce a *cost function* $J\big(x, v(\cdot)\big)$ for any control $v(\cdot)$ $(= v(t, \omega) \to V)$:

$$(2)\qquad J\big(x, v(\cdot)\big) = E\Big[\int_0^{\tau_x} f\big(y_x(t), v(t)\big)\exp\Big\{\int_0^t -c\big(y_x(s), v(s)\big)ds\Big\}\Big],$$

where $f(x, v)$, $c(x, v)$ are given real-valued functions from $\overline{\mathcal{O}} \times V$ and τ_x is the first exit time of the process $y_x(t, \omega)$ from $\overline{\mathcal{O}}$.

We want to minimize $J\big(x, (\cdot)\big)$ over all admissible controls, i.e., over all possible adapted processes taking their values in V:

$$(3)\qquad u(x) = \inf_{v(\cdot)} J\big(x, v(\cdot)\big)$$

and u is the *optimal cost function* (or *criterion*) of the problem.

Using the heuristic argument of *dynamical programming* (due to R. Bellman), one expects u to satisfy the following equation (called the *Hamilton–Jacobi–Bellman equation* — HJB in short—):

$$\sup_{v\in V}\{A(v)u(x)-f(x,v)\} = 0 \text{ in } \mathcal{O}, \quad u = 0 \quad \text{on} \quad \partial\mathcal{O} = \Gamma; \tag{4}$$

where $A(v)$ is the 2nd order elliptic operator (possibly degenerate) defined by

$$A(v) = -\sum_{i,j} a_{ij}(x,v)\partial_{ij} - \sum_i b_i(x,v)\partial_i + c(x,v) \tag{5}$$

and where the matrix $a(x,v)$ is given by $a(x,v) = \frac{1}{2}\sigma(x,v)\sigma^T(x,v)$.

If one knows *a priori* that $u(x)$, given by (3), is of class C^2 in $\bar{\mathcal{O}}$, then by using Itô's formula it is not difficult to check that (4) is satisfied (see W. Fleming and R. Rishel [5], or [1], [7]). On the other hand, if there exists a solution $\tilde{u}\in C^2(\bar{\mathcal{O}})$ of (4), then necessarily $u = \tilde{u}$ in $\bar{\mathcal{O}}$ (see [5], [1], for example). Of course, there is no reason why u should be of class C^2 and actually simple examples show that this is in general false.

In this note we want to present very general (and nearly optimal) results obtained by the author concerning (i) the derivation of equation (4) in a convenient sense, (ii) the question whether a given solution of (4) in a convenient sense is the optimal cost function u, (iii) the regularity of u.

The first general results in this direction were obtained by N. V. Krylov (see below for the references) and we want here to extend those results in two aspects: (i) we assume a very weak form of non-degeneracy of the matrices $a(x,v)$, (ii) we introduce a new uniqueness class.

II. Notations and assumptions

We define an admissible system $\mathscr{A} = (\Omega, F, F^t, P, W_t, v(t,\omega), y_x(t))$ as the collection of a probability space (Ω, F, F^t, P) with the usual properties, a Brownian motion in $R^p W_t$ (with respect to F^t), of a progressively measurable stochastic process $v(t,\omega)$ taking its values in V, and of a strong solution $y_x(t)$ of (1) (for $x\in\bar{\mathcal{O}}$).

We assume that $\sigma(x,v)$, $b(x,v)$, $f(x,v)$, $c(x,v)$ satisfy the following conditions:

$$\begin{cases} \varphi(x,v)\in W^{2,\infty}(\mathcal{O}) \quad \forall v\in V \quad \text{and} \quad \sup_{v\in V}\|\varphi\|_{W^{2,\infty}(\mathcal{O})} < +\infty, \\ \varphi(x,v) \text{ is continuous in } v\in V \text{ for } x\in\bar{\mathcal{O}} \end{cases} \tag{6}$$

for $\varphi = \sigma_{ij}$ $(1\leqslant i\leqslant N, 1\leqslant j\leqslant p)$, b_i $(1\leqslant i\leqslant N)$, c, f;

$$\lambda = \inf_{\substack{x\in\bar{\mathcal{O}}\\ v\in V}} c(x,v) > 0. \tag{7}$$

Finally the infimum in (3) has to be understood as the infimum over all admissible systems.

Remark II.1. Let us notice that the probability space and the Brownian motion are not fixed above. But actually it is proved in [18] that if we fix (Ω, F, F^t, P, W_t), then the corresponding optimal cost function is equal to u given by (3) — in particular, it is independent of the chosen probability space.

III. Main results

Our first result concerns the derivation of (4):

THEOREM III.1. *Under assumption* (6) *and if the following is satisfied:*

$$(8)\qquad \exists\nu > 0,\ \forall x \in \Gamma,\ \forall v \in V,\ \forall\xi \in R^N\colon\ \sum_{i,j} a_{ij}(x, v)\,\xi_i\xi_j \geqslant \nu|\xi|^2,$$

then there exists λ_0 (*depending explicitly on* $\|D^\alpha\varphi\|_{L^\infty}$ *for* $|\alpha| = 1, 2$; $\varphi = \sigma, b$) *such that, if* $\lambda > \lambda_0$, *then we have* $u \in W^{1,\infty}(\mathcal{O})$ *and*

$$(9)\qquad A(v)u \in L^\infty(\mathcal{O})\ \ \forall v \in V \quad \textit{and} \quad \sup_{v\in V} \|A(v)u\|_{L^\infty(\mathcal{O})} < +\infty,$$

$$(4')\qquad \sup_{v\in V}\{A(v)u(x) - f(x, v)\} = 0 \text{ a.e.} \quad \textit{in} \quad \mathcal{O}, \quad u = 0 \quad \textit{on} \quad \Gamma.$$

In addition, u is semi-concave, i.e.,

$$(10)\qquad \exists C > 0,\ \frac{\partial^2 u}{\partial\xi^2} \leqslant C \quad \textit{in} \quad \mathcal{D}'(\mathcal{O}), \quad \forall\xi \in R^N,\ |\xi| = 1.$$

Remark III.1. If $\mathcal{O} = \boldsymbol{R}^N$, assumption (8) disappears. If σ and b do not depend on x, then $\lambda_0 = 0$.

Remark III.2. This result extends previous results due to N. V. Krylov [7]–[11], M. Nisio [22]; H. Brézis and L. C. Evans [2], P.-L. Lions [12], L. C. Evans and A. Friedman [3], P.-L. Lions and J. L. Menaldi [20], [21]; M. V. Safonov [23]–[25].

The result is proved in P.-L. Lions [18] (see also [17]) by a combination of methods of partial differential equations and of probability theory: a pure analytical proof of Theorem III.1 in the case where the matrices $a(x, v)$ are non-degenerate can be found in P.-L. Lions [13] (see also [14]); a pure probabilistic proof of Theorem III.1 in the case where $\mathcal{O} = \boldsymbol{R}^N$ can be found in P.-L. Lions [15] (see also [16]).

Remark III.3. Since we want a solution u which satisfies $u = 0$ on $\Gamma = \partial\mathcal{O}$, it is natural to assume (8) (variants are possible, see [18]). Of course, this assumption is not satisfied in the case of deterministic control (i.e., $\sigma(x, v) \equiv 0$): in this case, however, similar techniques may be used to obtain very general and optimal results (see P.-L. Lions [19]). Let us also remark that if $\mathcal{O} = \boldsymbol{R}^N$, Theorem III.1 contains the deterministic case.

Remark III.4. The assumption that λ is large is in general necessary, as is shown in the example of Genis and Krylov [6]. Nevertheless, in the special case where the matrices $a(x, v)$ are non-degenerate on $\bar{\mathcal{O}}$, we may just assume $\lambda \geqslant 0$ (this is proved in L. C. Evans and P.-L. Lions [4]).

Remark III.5. It is not difficult to build various examples which show that the regularity obtained in Theorem III.1 on u is optimal.

We now turn to a uniqueness result:

THEOREM III.2. *Under assumptions* (6), (7) *and* (8), *we have:*

(i) *If* $w \in W^{1,\infty}_{\mathrm{loc}}(\mathcal{O}) \cap C_b(\bar{\mathcal{O}})$ *and if* w *satisfies*

$$(11) \qquad A(v)w \leqslant f(v) \quad \text{in} \quad \mathcal{D}'(\mathcal{O}) \quad \forall v \in V; \quad w \leqslant 0 \quad \text{on} \quad \Gamma,$$

then $w(x) \leqslant u(x)$ *in* $\bar{\mathcal{O}}$.

(ii) *If* $w \in W^{1,\infty}(\mathcal{O})$ *and satisfies* (9), (4′) *and*

$$(12) \qquad \Delta w \leqslant g \quad \text{in} \quad \mathcal{D}'(\mathcal{O}) \quad \text{with} \quad g \in L^{\infty}_{\mathrm{loc}}(\mathcal{O}),$$

then $w(x) = u(x)$ *in* $\bar{\mathcal{O}}$.

Again it is not difficult to show, by various examples, that this result is optimal. Functions w satisfying (12) are called *semi-super harmonic* (SSH in short).

Remark III.6. Of course, under the assumptions of Theorem III.1, u is the maximum subsolution (i.e., satisfying (11)) of (4); and u is the unique SSH solution of (4′).

Remark III.7. It is possible to replace (12) by

$$(12') \qquad Aw \leqslant g \quad \text{in} \quad \mathcal{D}'(\mathcal{O}) \quad \text{with} \quad g \in L^{N}_{\mathrm{loc}}(\mathcal{O}),$$

where A is any 2nd order, uniformly elliptic operator with smooth coefficients; then the conclusion of Theorem III.2 is preserved.

We finally mention a regularity result:

THEOREM III.3. *Under the assumptions of Theorem* III.1, *and if in addition we assume that there exist an open set* $I \subset \mathcal{O}$, $k \in \{1, \ldots, N\}$, $\nu > 0$

such that

$$(13)\qquad \begin{cases} \forall x \in I,\ \exists n \geqslant 1,\ \exists (v_1, \ldots, v_n) \in V^n, \exists (\theta_1, \ldots, \theta_n) \in]0,1[^n, \\ \hfill \sum_{i=1}^{n} \theta_i = +1, \\ \sum_{i,j,l} \theta_l a_{ij}(x, v_l)\,\xi_i \xi_j \geqslant \nu \sum_{i=1}^{k} |\xi_i|^2 \quad \forall \xi \in \mathbf{R}^N, \end{cases}$$

then we have

$$(14)\qquad \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^\infty(I) \quad \forall 1 \leqslant i, j \leqslant k.$$

EXAMPLE III.1. Take $V = \{1, \ldots, N\}$, $p = N$, $\sigma_{ij}(x, v) = \sqrt{2}\delta_{ik}\delta_{jk}$, $b(x, v) = 0$, $c(x, v) = \lambda > 0$. Then (4) becomes

$$(15)\qquad \max_{1 \leqslant m \leqslant N} \left\{ -\frac{\partial^2 u}{\partial x_m^2} + \lambda u - f(x, m) \right\} = 0 \text{ a.e.} \quad \text{in} \quad \mathbf{R}^N.$$

Then the combination of Theorems III.1–III.3 shows that the corresponding cost function $u(x)$ is the unique solution in $W^{2,\infty}(\mathbf{R}^N)$ of (15), as soon as $f(x, m) \in W^{2,\infty}(\mathcal{O})$ $(\forall 1 \leqslant m \leqslant N)$.

Theorems III.2 and III.3 are proved in P.-L. Lions [18] (see also [17], [8], [11]).

Remark III.8. All the results presented here are easily adapted to the case of time-dependent stochastic integrals, of stopping time problems (see [18]), and also of impulse problems and jump diffusion processes (works in preparation).

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