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ON THE PRIME NUMBER THEOREM FOR THE COEFFICIENTS OF CERTAIN MODULAR FORMS

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1. Introduction

Since the appearence of the fundamental papers of Rankin [20] and Selberg [22], in which the convolution of Dirichlet series associated with modular forms is introduced, many papers have been written in order to establish analogues of the prime number theorem and related problems for the coefficients of certain modular forms. In fact, the methods of Rankin and Selberg enable one to obtain zero-free regions for the Dirichlet series generating functions of such coefficients. These regions are indeed crucial for the proof of prime number theorems.

Let us recall some results on the subject.

Let Γ denote the modular group $\mathrm{SL}(2, \boldsymbol{Z})$ and let $S_k(\Gamma)$ be the space of cusp forms of weight k for Γ; if $g \in S_k(\Gamma)$ we set

$$g(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}. \tag{1}$$

We will always suppose that g is a normalized eigenfunction for the Hecke operators $T(n)$, i.e.

$$a(1) = 1 \tag{2}$$

and

$$T(n) g = a(n) g \tag{3}$$

for every $n \in \boldsymbol{N}$. We recall that under these assumptions we have $a(n) \in \boldsymbol{R}$ (see for instance the book of Ogg [14]).

If $g \in S_k(\Gamma)$ satisfies (1), (2) and (3) we write $L_g(s) = \sum_{n=1}^{\infty} a(n) n^{-s}$. Then

Goldstein [4] and Moreno [12] obtained some equivalent conditions for the validity of the Riemann hypothesis for $L_g(s)$, namely, that all the non-trivial zeros of $L_g(s)$ lie on the critical line $\sigma = k/2$. These conditions involved the summatory function of the arithmetical functions $\mu(n, g)$ and $\Lambda(n, g)$, the coefficients of the Dirichlet series for $L_g(s)^{-1}$ and $-L_g'/L_g(s)$, respectively.

Unconditional results for the above quantities have been obtained by Moreno [10], [11], Anderson [1], Hahnel [6] and Grupp [5]. Moreover Niebur [13], Rankin [21] and Grupp [5] obtained the prime number theorem for $a(p)^2$.

The aim of this paper is to obtain the following analogue of the Siegel–Walfisz theorem on the uniform distribution of primes in arithmetical progressions.

Let $\psi_{g^2}(x, q; a) = \sum\limits_{\substack{n \leqslant x \\ n \equiv a (\mathrm{mod}\, q)}} a(n)^2 \Lambda(n)$; the result is the following

THEOREM. *Let $g \in S_k(\Gamma)$ satisfy* (1), (2) *and* (3), *N be a real positive number and $(q, a) = 1$. Then there exists a positive constant c, depending only on N, such that*

$$\psi_{g^2}(x, q; a) = x^k/\varphi(q) + O\left(x^k \exp(-c\sqrt{\log x})\right) \tag{4}$$

uniformly for $q \ll (\log x)^N$.

We shall give only a brief sketch of the proof.

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2. The Rankin–Selberg convolution

Let $g \in S_k(\Gamma)$ satisfy (1), (2) and (3); it is well known that

$$L_g(s) = \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\bar{\alpha}_p}{p^s}\right)^{-1}$$

is an entire function, $|\alpha_p| = p^{(k-1)/2}$ and $a(p) = \alpha_p + \bar{\alpha}_p$.

Now let

$$L_{g^2}(s) = \sum_{n=1}^{\infty} a(n)^2 n^{-s}, \tag{5}$$

$$L_{g \otimes g}(s) = \prod_p \left(1 - \frac{\alpha_p^2}{p^s}\right)^{-1} \left(1 - \frac{\alpha_p \bar{\alpha}_p}{p^s}\right)^{-2} \left(1 - \frac{\bar{\alpha}_p^2}{p^s}\right)^{-1}. \tag{6}$$

The relation between (5) and (6) is given by

$$L_{g \otimes g}(s) = \zeta(2s - 2k + 2) L_{g^2}(s),$$

where $\zeta(s)$ is the Riemann zeta function.

In [20] and [22] the analytic continuation and the functional equation of $L_{g\otimes g}(s)$ were established. Several authors studied the analytic properties of the convolution of Dirichlet series associated with cusp forms for various congruence subgroups of Γ; we recall the papers of Ogg [15], Shimura [23], Zagier [24], Li [7], [8], Asai [2], Manin and Panchishkin [9] and Panchishkin [16], [17].

Let χ be a Dirichlet character (mod q); we are interested in the twisted convolutions

$$L_{g^2}(s, \chi) = \sum_{n=1}^{\infty} a(n)^2 \chi(n) n^{-s}$$

and

$$L_{g\otimes g}(s, \chi) = \prod_p \left(1 - \frac{\alpha_p^2 \chi(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha_p \bar{\alpha}_p \chi(p)}{p^s}\right)^{-2} \left(1 - \frac{\bar{\alpha}_p^2 \chi(p)}{p^s}\right)^{-1},$$

again related by

$$L_{g\otimes g}(s, \chi) = L(2s-2k+2, \chi^2) L_{g^2}(s, \chi).$$

If χ (mod q) is induced from χ_1 (mod q_1) we have the usual relation

$$L_{g\otimes g}(s, \chi) = \prod_{p|q} \left(1 - \frac{\alpha_p^2 \chi_1(p)}{p^s}\right)\left(1 - \frac{\alpha_p \bar{\alpha}_p \chi_1(p)}{p^s}\right)^2 \left(1 - \frac{\bar{\alpha}_p^2 \chi_1(p)}{p^s}\right) L_{g\otimes g}(s, \chi_1).$$

We will need the following lemmas.

LEMMA 1. *Let g be as in the Theorem. Then $L_{g\otimes g}(s, \chi)$ is an entire function, except when χ is the principal character χ_0, and in this case $L_{g\otimes g}(s, \chi_0)$ has a simple pole at $s = k$ with residue*

$$r_q = \zeta(2) k\alpha \prod_{p|q} \left(1 - \frac{\alpha_p^2}{p^k}\right)\left(1 - \frac{\alpha_p \bar{\alpha}_p}{p^k}\right)^2 \left(1 - \frac{\bar{\alpha}_p^2}{p^k}\right)$$

where α is given by (1.4) *of* [20], *part* II.

LEMMA 2. *Let g be as in the Theorem. If χ is primitive* (mod q), *$L_{g\otimes g}(s, \chi)$ has a functional equation of the following form. Set*

$$\Phi_{g\otimes g}(s, \chi) = Q^s \Gamma(s+k-1) \Gamma(s) L_{g\otimes g}(s+k-1, \chi),$$

where Q is a suitable positive real number satisfying $q^2 \ll Q \ll q^2$. Then

$$\Phi_{g\otimes g}(s, \chi) = W_\chi \Phi_{g\otimes g}(1-s, \bar{\chi})$$

where $|W_\chi| = 1$. Moreover, $\Phi_{g\otimes g}(s, \chi)$ is holomorphic over **C**, *except for at most a finite number of simple poles.*

Proof of Lemmas 1 *and* 2. The proofs are contained in the above mentioned papers for most cases, according to the factorization of q into prime powers (recall that the twist of a modular form for Γ belongs to some congruence subgroup of Γ). In the remaining cases one may carry out the proof following the method of [9], with some additional complication in details.

LEMMA 3. *Let g be as in the Theorem. There exists a positive constant c_1 such that $L_{g\otimes g}(s,\chi)\neq 0$ in the region*

$$\sigma > k - c_1/\log\left(q(|t|+2)\right), \qquad s = \sigma + it,$$

except for at most a simple real zero $\beta_0 < k$, which may occur only when χ is a real non-principal character (mod q).

Such a real zero, if it exists, is called "exceptional".

Proof. It is easy to see that for $\sigma > k$ we have

$$-3\frac{L'_{g\otimes g}}{L_{g\otimes g}}(\sigma,\chi_0) - 4\,\mathrm{Re}\,\frac{L'_{g\otimes g}}{L_{g\otimes g}}(\sigma+it,\chi) - \mathrm{Re}\,\frac{L'_{g\otimes g}}{L_{g\otimes g}}(\sigma+2it,\chi^2) \geqslant 0.$$

We use the De la Vallee Poussin–Landau method (see the book of Davenport [3], Ch. 14) and exploit the estimates contained in Perelli [18], expecially Theorem 1 and the lemma in the proof of Theorem 2 (note that all the results in [18] are stated in a normalized form). This yields the classical inequality about real and imaginary parts of the zeros, from which the lemma follows.

LEMMA 4. *Let g be as in the Theorem. There exists a positive constant c_2 such that if $z \geqslant 3$ there is at most one real primitive character χ* (mod q), *with $q \leqslant z$, for which $L_{g\otimes g}(s,\chi)$ has a simple real zero β_0 satisfying*

$$\beta_0 > k - c_2/\log z.$$

Proof. The proof is deduced from the proof of Lemma 1 in analogy with the argument given by Landau and Page in the classical case of Dirichlet L-functions (compare [3], Ch. 14).

LEMMA 5. *Let g be as in the Theorem and χ a real non-principal character* (mod q). *Then for any $\varepsilon > 0$ there exists a positive number $c(\varepsilon)$ such that $L_{g\otimes g}(s,\chi)\neq 0$ for $s > k - c(\varepsilon)/q^{\varepsilon}$.*

Proof. This lemma follows at once from Theorem 2 of Perelli and Puglisi [19]. In that paper the real zeros of a class of Dirichlet series are investigated (again in normalized form). Moreover the theorems in [19] are in the present case unconditional, since Hypothesis S–T of [18] is not needed in the proof of "complex" zero-free regions, i.e. in Lemma 1.

3. Proof of the Theorem

From (4) we have, for $\sigma > k$,

$$-\frac{L'_{g\otimes g}}{L_{g\otimes g}}(s,\chi) = \sum_{n=1}^{\infty} \Lambda(n, g\otimes g)\chi(n)n^{-s}$$

where

$$\Lambda(n, g\otimes g) = \begin{cases} (\alpha_p^m + \bar{\alpha}_p^m)^2 \log p & \text{if } n = p^m, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\psi_{g\otimes g}(x,\chi) = \sum_{n\leqslant x} \Lambda(n, g\otimes g)\chi(n);$$

then from $|\alpha_p| = p^{(k-1)/2}$ we get

$$\psi_{g^2}(x, q; a) = \frac{1}{\varphi(q)} \sum_{\chi(\mathrm{mod}\, q)} \bar{\chi}(a)\psi_{g\otimes g}(x,\chi) + O(x^{k-1/2}). \tag{7}$$

We now derive the explicit formula for $\psi_{g\otimes g}(x,\chi)$ when $\chi \neq \chi_0$.

As usual we express $\psi_{g\otimes g}(x,\chi)$ in terms of $-\dfrac{L'_{g\otimes g}}{L_{g\otimes g}}(s,\chi)$ by means of the truncated form of Perron's formula. Shifting the path of integration to the left, we compute the residue at the poles of $-\dfrac{L'_{g\otimes g}}{L_{g\otimes g}}\dfrac{x^s}{s}$ and we estimate the integral on the path of integration using for instance the results of [18]. We thus obtain the "finite" explicit formula with a remainder, which contains some constants depending on χ. Such constants are then estimated by using the logarithmic derivative of the Weierstrass product for $\Phi_{g\otimes g}(s,\chi)$ and Lemma 3. The above reasoning holds when χ is primitive, but the transition to non-primitive χ is made in the usual way. We thus obtain the following formula: let $2 \leqslant T \leqslant x$ and $\chi \neq \chi_0$; then

$$\psi_{g\otimes g}(x,\chi) = -\frac{x^{\beta_0}}{\beta_0} - \sum_{|\gamma|\leqslant T}{}' \frac{x^{\varrho}}{\varrho} + O\left(\frac{x^k \log^2 qx}{T} + x^{k-3/4}\log qx\right) \tag{8}$$

uniformly in q, where β_0 is the exceptional zero and $'$ means that β_0 and $2k-1-\beta_0$ are excluded from the summation.

The final step of the proof is to sum (8) over non-principal characters and to use the estimates of Lemmas 3–5 and the estimates for the number of zeros in the critical strip given in [18]. The contribution of this sum is absorbed into the error term of (4), while the main term of (4) is obtained by applying the same techniques to $\psi_{g\otimes g}(x,\chi_0)$. Note that all the ingredients needed for the estimation of $\psi_{g\otimes g}(x,\chi_0)$ are obtained analogously to, but more simply than those for $\psi_{g\otimes g}(s,\chi)$.

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