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ON HEREDITARILY INDECOMPOSABLE COMPACTA *

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1. Introduction

In this paper we formulate some basic results of hereditarily indecomposable compacta and of piecewise linear mappings of the arc to itself. In particular, we will show that if X is an arbitrary hereditarily indecomposable compactum which is covered by a chain cover $\mathscr{U} = \{U_1, \ldots, U_n\}$ and if $f\colon \{1, 2, \ldots, m\} \twoheadrightarrow \{1, 2, \ldots, n\}$ is a pattern, then there exists a refinement $\mathscr{V}$ of $\mathscr{U}$ covering X such that $\mathscr{V}$ follows pattern f in $\mathscr{U}$. These results generalize and simplify arguments due to Bing and Moise. As a consequence we provide a – hopefully easier – proof of the fact that the pseudo-arc is unique, homogeneous and hereditarily equivalent.

By a *compactum* we mean a compact metric space. A *continuum* is a connected compactum. A continuum X is *decomposable* provided it can be written as the union of two of its proper subcontinua. A continuum is *indecomposable* if it is not decomposable. A compactum is *hereditarily indecomposable* provided all of its subcontinua are indecomposable. A *chain* is a collection of sets $\mathscr{U} = \{U_1, \ldots, U_n\}$ such that

$$U_i \cap U_j \neq \emptyset \Leftrightarrow |i-j| \leqslant 1.$$

A continuum X is *chainable* provided for every $\varepsilon > 0$, there exists an open chain cover $\mathscr{U}$ of X such that $\operatorname{mesh} \mathscr{U} < \varepsilon$. A hereditarily indecomposable chainable continuum P is called a *pseudo-arc*. Knaster [6] has shown that such a continuum exists.

In proving several properties of the pseudo-arc, Bing and Moise (and several other authors) used the fact that the pseudo-arc can be obtained as the intersection of a nested sequence of "crooked" chains. The arguments involved are quite complicated. We will prove several properties of heredita-

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rily indecomposable compacta using new techniques. In constructing covers following an arbitrary pattern in a given cover $\mathscr{U}$ of the hereditarily indecomposable compactum X, we do not use the fact that X is obtained as a nest of crooked chains. Hence X need not be chainable, connected or even 1-dimensional. These results generalize and simplify arguments due to Bing and Moise. Rather than attempting to generalize known results about the pseudo-arc, we have chosen to give a basic proof of some of its most important properties. The results in this paper can be generalized to obtain stronger results on homogeneity of the pseudo-arc and on patterns in hereditarily indecomposable compacta (see for example the remark following Theorem 5). Several results seem to have some interest of their own (see for example [5] and [12] for some applications). Some of the results of this paper are implicit in the work of previous authors. For completeness we have placed in the last section proofs of some theorems which are known (but which we could not find, in the form in which we needed them, in the literature). If $A \subset X$, we will denote by $\mathrm{Bd}(A)$ and $\mathrm{Cl}(A)$ the boundary and closure of A, respectively.

2. Piecewise linear functions

If $m < n$ are positive integers, then we denote by $[m, n]$ the set $\{m, m+1, \ldots, n\}$. A function f: $[1, m] \to [1, n]$ is called a (light) pattern provided $|f(i+1)-f(i)| \leqslant 1$ ($|f(i+1)-f(i)| = 1$, respectively) for $i = 1, \ldots, n-1$. We call 1 and n the extreme points of the range of f and 1 and m the extreme points of the domain of f. We write f: $X \twoheadrightarrow Y$ to indicate that f is a function from X onto Y. Let f: $[1, m] \twoheadrightarrow [1, n]$ be a pattern, where $n \geqslant 2$. We say f is a *simple fold* if there exist integers r_1 and r_2 with $1 \leqslant r_1 < r_2 \leqslant m$ such that f is one to one on each of $[1, r_1]$, $[r_1, r_2]$ and $[r_2, m]$ and, if $1 < r_1 < r_2 < m$, then $f([r_1, r_2]) = f([1, r_1]) \cap f([r_2, 1])$. We say that f is an *end-fold* if either $r_1 = 1$ or $r_2 = m$. If f is a simple fold and f is not an end-fold, then f is said to be an *interior fold*. A pattern F_0: $[1, m] \to [1, p]$ is called *monotone* provided $F_0^{-1}(j) = [a_j, b_j] = \{x \mid a_j \leqslant x \leqslant b_j\}$ for some $a_j, b_j \in [1, m]$ and each $j \in [1, p]$. Let f: $[1, m] \twoheadrightarrow [1, n]$ be a pattern, then $f = f^* \circ F_0$ where F_0 is a monotone pattern and f^* is a light pattern. We will denote by $|A|$ the cardinality of the set A. A proof of the following theorem is given in the last section (see also [13]).

1. Theorem. *Let f: $[1, m] \twoheadrightarrow [1, n]$ be a pattern. Then there exists an integer k such that $f = F_k \circ \ldots \circ F_1 \circ F_0$ where*

(1) *F_0 is a monotone pattern,*

(2) *each F_i is a simple fold, $1 \leqslant i \leqslant k$,*

(3) *if $f(1)$ (respectively $f(m)$) is an extreme point of the range of f, then $F_i \circ F_{i-1} \circ \ldots \circ F_0(1)$ (respectively $F_i \circ \ldots \circ F_0(m)$) is an extreme point of the range of $F_i \circ \ldots \circ F_0$ for each $i = 0, \ldots, k$.*

3. Crookedness in hereditarily indecomposable compacta

Let $\mathscr{V} = \{V_1, \ldots, V_m\}$ and $\mathscr{U} = \{U_1, \ldots, U_n\}$ be chain covers of a compactum X. Let $f\colon [1, m] \twoheadrightarrow [1, n]$ be a pattern. We say that *$\mathscr{V}$ follows pattern f in $\mathscr{U}$* provided $V_i \subset U_{f(i)}$ for each $i = 1, \ldots, m$. We will call f a pattern on $\mathscr{U}$. One of the most fundamental properties of a hereditarily indecomposable compactum X is that given a chain cover $\mathscr{U}$ of X and a pattern f on $\mathscr{U}$, there exists a chain cover $\mathscr{V}$ of X such that $\mathscr{V}$ follows pattern f in $\mathscr{U}$. The proof of this fact will use Theorem 1 and the following fundamental theorem due to Krasinkiewicz and Minc.

2. THEOREM (Krasinkiewicz and Minc [7] and Krasinkiewicz [8]). *Let X be a compactum; then the following are equivalent:*

(I) *X is hereditarily indecomposable;*

(II) *for every pair of disjoint closed subsets A and B of X and for every open set U intersecting all components of A, there exist closed subsets M and N of X such that:*

(1) $X = M \cup N$,

(2) $A \subset M$, $B \subset N$,

(3) $M \cap N \subset U\backslash A \cup B$;

(III) *for every pair of disjoint closed subsets A and B of X and for every pair of neighbourhoods U of A and V of B, there exist closed subsets X_0, X_1 and X_2 of X such that:*

(1) $X = X_0 \cup X_1 \cup X_2$,

(2) $A \subset X_0$ *and* $B \subset X_2$,

(3) $X_0 \cap X_2 = \emptyset$,

(4) $X_0 \cap X_1 \subset V$ *and* $X_1 \cap X_2 \subset U$.

The above theorem was proved for X a continuum, but the proof is valid for X a compactum.

A cover $\mathscr{U} = \{U_1, \ldots, U_n\}$ of a compactum X is said to be *taut* if and only if $\mathrm{Cl}(U_i) \cap \mathrm{Cl}(U_j) \neq \emptyset$ implies $U_i \cap U_j \neq \emptyset$. If $\mathscr{U}$ is a cover of a compactum and $U \in \mathscr{U}$, then we denote by $i(U, \mathscr{U}) = U\backslash \mathrm{Cl}(\bigcup\{V \in \mathscr{U} \mid V \neq U\})$.

3. THEOREM. *Let X be an hereditarily indecomposable compactum and let $\mathscr{U} = \{U_1, \ldots, U_n\}$ be an open taut chain cover of X such that there exists a continuum $Z \subset X$ such $Z \cap i(U_1, \mathscr{U}) \neq \emptyset \neq Z \cap i(U_n, \mathscr{U})$. Let $f\colon [1, m] \twoheadrightarrow [1, n]$ be a pattern on $\mathscr{U}$. Then there exists an open taut chain cover $\mathscr{V} = \{V_1, \ldots, V_m\}$ of X such that $\mathscr{V}$ follows pattern f in $\mathscr{U}$.*

Moreover, if $x_0 \in i(U_1, \mathscr{U})$ and $f(1) = 1$, we can construct $\mathscr{V}$ such that in addition $x_0 \in i(V_1, \mathscr{V})$.

Proof. By Theorem 1, $f = F_k \circ \ldots \circ F_1 \circ F_0$ where each F_i $(1 \leqslant i \leqslant k)$ is a simple fold and F_0 is monotone. We will inductively construct a sequence $\mathscr{V}_i$ $(i = k+1, \ldots, 0)$ of open taut chain covers of X such that $\mathscr{V}_{k+1} = \mathscr{U}$ and

$\mathscr{V}_i$ follows pattern F_i in $\mathscr{V}_{i+1}$ ($i = 0, \ldots, k$). Then $\mathscr{V}_0$ is the required cover of X.

Case 1. F_k is an interior fold. Put $g = F_k$ and let domain $g = [1, m_k]$. Hence there exist integers r_1 and r_2, $1 < r_1 < r_2 < m_k$, such that g is one to one on each of $[1, r_1]$, $[r_1, r_2]$ and $[r_2, m_k]$ but not on the union of any two of these sets. Since g is an interior fold, we have either $g(1) = 1$ and $g(m_k) = n$ or $g(1) = n$ and $g(m_k) = 1$. Without loss of generality we may assume $g(1) = 1$ and $g(m_k) = n$. Put

$$A = \mathrm{Cl}[U_1 \cup \ldots \cup U_{g(r_2)-1} \cup i(U_{g(r_2)}, \mathscr{U})] \quad \text{and}$$
$$B = \mathrm{Cl}[i(U_{g(r_1)}, \mathscr{U}) \cup U_{g(r_1)+1} \cup \ldots \cup U_n].$$

Then A and B are disjoint closed sets. Put $U = \bigcup_{i \leq g(r_2)} U_i$ and $V = \bigcup_{i \geq g(r_1)} U_i$. By III of Theorem 2, there exist closed subsets X_0, X_1 and X_2 of X such that $X = X_0 \cup X_1 \cup X_2$, $A \subset X_0$, $B \subset X_2$, $X_0 \cap X_2 = \emptyset$, $X_0 \cap X_1 \subset V$ and $X_1 \cap X_2 \subset U$. Let H and K be open sets such that

$$X_0 \cap X_1 \subset H \subset \mathrm{Cl}(H) \subset U_{g(r_1)} \cap U_{g(r_1)-1},$$
$$X_1 \cap X_2 \subset K \subset \mathrm{Cl}(K) \subset U_{g(r_2)} \cap U_{g(r_2)+1}$$

and

$$\mathrm{Cl}(H) \cap X_2 = \mathrm{Cl}(K) \cap X_0 = \mathrm{Cl}(H) \cap \mathrm{Cl}(K) = \emptyset.$$

Define a chain cover $\mathscr{V}_k = \{V_1, \ldots, V_{m_k}\}$ by

$$V_j = \begin{cases} U_j \cap X_0 \backslash \mathrm{Cl}(H) & j < r_1, \\ [X_0 \cup X_1 \backslash \mathrm{Cl}(K)] \cap U_{r_1} & j = r_1, \\ U_{2r_1 - j} \cap X_1 & r_1 < j < r_2, \\ [X_2 \cup X_1 \backslash \mathrm{Cl}(H)] \cap U_{g(r_2)} & j = r_2, \\ U_{j-2(r_2 - r_1)} \cap X_2 \backslash \mathrm{Cl}(K) & j > r_2. \end{cases}$$

Case 2. F_k is an end fold, put $F_k = g$. There is nothing to prove if g is one to one. Let $[1, m_k]$ be the domain of g. Hence there exists an integer r, $1 < r < m_k$, such that g is one to one on each of $[1, r]$ and $[r, m_k]$ but g is not one to one on their union. Since g is an end fold, we have either $g(1) \in \{1, n\}$ or $g(m_k) \in \{1, n\}$. Without loss of generality we may assume $g(1) = 1$. Define $A = \mathrm{Cl}[U_1 \cup \ldots \cup U_{g(m_k)-1} \cup i(U_{g(m_k)}, \mathscr{U})]$ and $P = \mathrm{Cl}[i(U_n, \mathscr{U})]$. Let Y be a continuum in Z such that

$$Y \cap i(U_n, \mathscr{U}) \neq \emptyset \neq Y \cap U_{g(m_k)}, \qquad Y \cap \mathrm{Cl}\, i(U_{g(m_k)}, \mathscr{U}) = \emptyset$$

and let $B = Y \cup P$. Then A and B are disjoint closed sets. Let $U = U_n$. By II of Theorem 2, there exist closed sets M and N such that $A \subset M$, $B \subset N$, $X = M \cup N$ and $M \cap N \subset U \backslash A \cup B$. The rest of the proof of Case 2 is similar to that of case 1 and is omitted.

Case 3. F_k is monotone. Using the fact that X is normal, it is easy to construct the required taut open chain cover $\mathscr{V}_k$.

In either case we can construct a taut open chain cover $\mathscr{V}_k$ such that $\mathscr{V}_k$ follows pattern F_k in $\mathscr{V}_{k+1} = \mathscr{U}$. Replacing $\mathscr{U}$ by $\mathscr{V}_k$ and F_k by F_{k-1} we can construct a taut open chain cover $\mathscr{V}_{k-1}$ of X such that $\mathscr{V}_{k-1}$ follows pattern F_{k-1} in $\mathscr{V}_k$. If $x_0 \in i(U_1, \mathscr{U})$, then it follows from the construction that $x_0 \in i(V_1^j, \mathscr{V}_j) \cup i(V_{m_j}^j, \mathscr{V}_j)$ where V_1^j and $V_{m_j}^j$ are the end links of $\mathscr{V}_j$ for each $j = k, k-1, \ldots, 0$. Moreover, if $f(1) = 1$, $x_0 \in i(V_1^0, \mathscr{V}_0)$.

4. LEMMA. *Let X be a hereditarily indecomposable compactum and let $\mathscr{U} = \{U_1, \ldots, U_n\}$ be a taut open chain cover of X. Suppose $x_0 \in i(U_p, \mathscr{U})$ for some p, $1 \leqslant p \leqslant n$ and there exists a continuum $H \subset X$ such that $x_0 \in H$ and $H \cap [i(U_1, \mathscr{U}) \cup i(U_n, \mathscr{U})] \neq \emptyset$. Then there exists a taut open chain cover $\mathscr{V} = \{V_1, \ldots, V_n\}$ of X such that $V_i \subset U_i$ for each $i = 1, \ldots, n$, an integer m, a function $f\colon [1, m] \to [1, n]$ and a sequence X_i, $0 \leqslant i \leqslant m$ of subcontinua of X such that*

(1) $X_0 = \emptyset$, $x_0 \in X_1 \subset X_2 \subset \ldots \subset X_m$,
(2) $X_i \subset \bigcup_{j \leqslant i} V_{f(j)}$,
(3) $X_i \cap i(V_{f(i)}, \mathscr{V}) \neq \emptyset$ *and* $X_{i-1} \cap \mathrm{Cl}[i(V_{f(i)}, \mathscr{V})] = \emptyset$ $i \leqslant m$,
(4) $f(1) = p$,
(5) $f(m) \in \{1, n\}$ *and* $f(j) \notin \{1, m\}$ *for each* $j < m$.

Proof. We will inductively construct a sequence of taut open chain covers $\mathscr{V}_j = \{V_1^j, \ldots, V_n^j\}$ of X such that $V_t^{j+1} \subset V_t^j$, $1 \leqslant t \leqslant n$, functions $f_j\colon [1, j] \to [1, n]$ such that $f_j | [1, j-1] = f_{j-1}$ and continua X_j satisfying (1)–(4). Put $\mathscr{V}_1 = \mathscr{U}$, $X_1 = \{x_0\}$ and define $f_1(1) = p$, then (1)–(4) are satisfied. Suppose that $\mathscr{V}_j$, f_j and X_j have been constructed satisfying (1)–(4) for $j \leqslant u$ and $f_u(j) \notin \{1, n\}$, $1 \leqslant j \leqslant u$. Put $a = \min\{f_u([1, u])\}$ and $b = \max\{f_u([1, u])\}$. If there exists a continuum Z such that $X_u \subset Z$, $Z \cap i(V_{a-1}^u, \mathscr{V}_u) \neq \emptyset$ $(Z \cap i(V_{b+1}^u, \mathscr{V}_u) \neq \emptyset)$ and $Z \subset \bigcup_{j=a-1}^{b} V_j^u$ $(Z \subset \bigcup_{j=a}^{b+1} V_j^u)$, put $\mathscr{V}_{u+1} = \mathscr{V}_u$, $X_{u+1} = Z$ and define $f_{u+1}\colon [1, u+1] \to [1, n]$ by $f_{u+1} | [1, u] = f_u$ and $f_{u+1}(u+1) = a-1$ $(f_{u+1}(u+1) = b+1$, respectively). Then (1)–(4) are satisfied. Hence, suppose that such a continuum Z does not exist. Put $P = \mathrm{Bd}(V_a^u) \cap V_{a-1}^u$ and $Q = \mathrm{Bd}(V_b^u) \cap V_{b+1}^u$. Let Y be a subcontinuum of X which is irreducible with respect to intersecting X_u and $P \cup Q$, then $X_u \subset Y$. Without loss of generality we may assume that $Y \cap P \neq \emptyset$, choose $y \in Y \cap P$. Let ε be one half of the minimum of the Lebesque number for the cover $\mathscr{V}_u$ and the distance from y to X_u. Define $U = S(y, \varepsilon/2) \subset \mathrm{Cl}(U) \subset V_{a-1}^u \backslash X_u$ and $\mathscr{V}_{u+1} = \{V_1^{u+1}, \ldots, V_n^{u+1}\}$ by $V_j^{u+1} = V_j^u$ if $j \neq a$, $V_a^{u+1} = V_a^u \backslash \mathrm{Cl}(U)$, $f_{u+1}\colon [1, u+1] \to [1, n]$ by $f_{u+1} | [1, u] = f_u$ and $f_{u+1}(u+1) = a-1$. Since Y is irreducible with respect to intersecting $P \cup Q$ and X_u, there exists a proper subcontinuum $Z \subset Y$ such that $X_u \subset Z$, $Z \cap U \neq \emptyset$ and $Z \cap [P \cup Q] = \emptyset$. Put $Z = X_{u+1}$; then (1)–(4) are satisfied. By induction we can continue this construction until (5) is satisfied.

5. THEOREM. *Let X be an hereditarily indecomposable compactum and let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be a taut open chain cover of X. Suppose that $x_0 \in i(U_j, \mathcal{U})$ for some $j \in \{1, \ldots, n\}$ and there exists a continuum $Y \subset X$ such that $x_0 \in Y$ and $Y \cap [i(U_1, \mathcal{U}) \cup i(U_n, \mathcal{U})] \neq \emptyset$. Then there exists a taut open chain cover $\mathcal{V} = \{V_1, \ldots, V_m\}$ of X such that $\mathcal{V}$ refines $\mathcal{U}$ and $x_0 \in i(V_1, \mathcal{V}) \cup i(V_m, \mathcal{V})$.*

Proof. By Lemma 4 (and considering a refinement of $\mathcal{U}$ when necessary) we may assume that there exist an integer m, a function f: $[1, m] \to [1, n]$ and an increasing sequence of continua X_i ($i = 0, \ldots, m$) such that

(1) $X_0 = \emptyset$, $x_0 \in X_1 \subset X_2 \subset \ldots \subset X_m$,

(2) $X_t \subset \bigcup_{i \leq t} U_{f(i)}$, for each $t \in [1, m]$,

(3) $X_t \cap i(U_{f(t)}, \mathcal{U}) \neq \emptyset$ and $X_{t-1} \cap \mathrm{Cl}[i(U_{f(t)}, \mathcal{U})] = \emptyset$, for each $t \in [1, m]$,

(4) $f(1) = j$,

(5) $f(m) \in \{1, n\}$ and $f(t) \notin \{1, n\}$ if $t < m$.

If $\mathcal{V} = \{V_1, \ldots, V_l\}$ is a taut open chain cover of X, define $d(\mathcal{V})$ to be the length of the shortest subchain $\mathcal{W}$ of $\mathcal{V}$ such that $\mathcal{W}$ covers a continuum Z with $x_0 \in Z$ and

$$Z \cap [i(V_1, \mathcal{V}) \cup i(V_l, \mathcal{V})] \neq \emptyset.$$

Note that by (3) f is one to one, and hence by (5) $d(\mathcal{U}) = m$. If there exists a taut open chain cover $\mathcal{V}$ of X such that $\mathcal{V}$ refines $\mathcal{U}$ and $d(\mathcal{V}) = 1$, then it is easy to see that there exists a refinement $\mathcal{V}'$ of $\mathcal{V}$ which satisfies the conclusion of the Theorem. Hence it suffices to construct a sequence of taut open chain covers $\mathcal{V}_p$ of X such that $\mathcal{V}_{p+1}$ refines $\mathcal{V}_p$ and $d(\mathcal{V}_{p+1}) < d(\mathcal{V}_p)$. Put $\mathcal{U} = \mathcal{V}_1$. Since $f(m) \in \{1, n\}$, we may assume without loss of generality that $f(m) = n$. Since f is one to one,

$$|\text{Range } f| = |\text{domain } f| = m = d(\mathcal{V}_1).$$

Put $a = \min\{f([1, m])\}$; then $m = d(\mathcal{V}_1) = n - a + 1$ and $a > 1$. Put

$$B = \mathrm{Cl}\Big[\bigcup_{t \leq a-2} U_t \cup i(U_{a-1}, \mathcal{U})\Big] \quad \text{and} \quad U = i(U_n, \mathcal{U}).$$

Then B and X_m are disjoint closed subsets of X and $U \cap X_m \neq \emptyset$. Hence, by II of Theorem 2, there exist closed sets M and N such that $X = M \cup N$, $X_m \subset M$, $B \subset N$ and $M \cap N \subset U$. Define $\mathcal{V}_2 = \{V_1^2, \ldots, V_{2n-a}^2\}$ by

$$V_s^2 = \begin{cases} U_s \cap N & s < n, \\ U_n & s = n, \\ U_{2n-s} \cap M & s > n. \end{cases}$$

Then $\mathscr{V}_2$ is a taut open chain cover of X and $\mathscr{V}_2$ refines $\mathscr{V}$. Moreover, X_{m-1} is a subcontinuum of X such that $x_0 \in X_{m-1}$, $X_{m-1} \subset \bigcup_{s=n+1}^{2n-a} V_s^2$ and $X_{m-1} \cap i(V_{2n-a}^2, \mathscr{V}_2) \neq \emptyset$.

Hence $d(\mathscr{V}_2) \leqslant 2n-a-(n+1)+1 = n-a < n-a+1 = d(\mathscr{V}_1)$. The theorem follows now easily by induction.

Remark. The notion of a simple fold (on an arc) as defined in Section 2 may be extended to simple folds on more general spaces. For example, a map $f\colon G_2 \twoheadrightarrow G_1$ from a graph G_2 onto a graph G_1 is an interior fold provided $G_2 = X_0 \cup X_1 \cup X_2$ where

(1) X_i is a subgraph of G_2, $i = 1, 2, 3$,

(2) $X_i \cap X_j$ is finite if $i \neq j$,

(3) $f|X_i$ is one to one $i = 1, 2, 3$,

(4) $f(X_1) = f(X_0) \cap f(X_2)$,

(5) $f(X_1)$ separates $\mathrm{Cl}(f(X_0)\setminus f(X_1))$ from $\mathrm{Cl}(f(X_2)\setminus f(X_1))$ in G_1.

End folds can be defined in a similar way. If we replace $\mathscr{U}$ by a taut open cover $\mathscr{C}$ whose nerve is a graph G_1 and f by a composition of interior folds from a graph G_2 onto the graph G_1, then Theorem 3 remains valid (i.e., there exists a cover $\mathscr{D}$ of X such that $\mathscr{D}$ refines $\mathscr{C}$, the nerve of $\mathscr{D}$ is G_2 and f is a pattern of $\mathscr{D}$ in $\mathscr{C}$ (some of the elements of the cover $\mathscr{D}$ may be empty). Other extensions of results in this section using these more general folds are also possible. See [14] for some related results.

4. Chainable hereditarily indecomposable continua

In this section we will show that the pseudo-arc, P, is unique, homogeneous and hereditarily equivalent. Given a chain cover $\mathscr{U} = \{U_1, \ldots, U_n\}$ of a compactum X we denote by $F(\mathscr{U})$ the set U_1. We will use the following well-known theorem (cf. [1]).

6. Theorem. *Let $\mathscr{U}_i$ and $\mathscr{V}_i$ be sequences of taut open chain covers of continua X and Y, respectively and let $x_0 \in X$ and $y_0 \in Y$ be points such that:*

(1) $|\mathscr{U}_i| = |\mathscr{V}_i|$,

(2) $\lim_i \mathrm{mesh}(\mathscr{U}_i) = \lim_i \mathrm{mesh}(\mathscr{V}_i) = 0$,

(3) *both $\mathscr{U}_{i+1}$ and $\mathscr{V}_{i+1}$ follow a pattern f_i in $\mathscr{U}_i$ and $\mathscr{V}_i$, respectively,*

(4) *$x_0 \in i(F(\mathscr{U}_i), \mathscr{U}_i)$ and $y_0 \in i(F(\mathscr{V}_i), \mathscr{V}_i)$.*

Then there exists a homeomorphism $h\colon X \twoheadrightarrow Y$ such that $h(x_0) = y_0$.

7. Theorem. *Let X and Y be hereditarily indecomposable chainable continua and let $x_0 \in X$ and $y_0 \in Y$. Then there exists a homeomorphism $h\colon X \twoheadrightarrow Y$ such that $h(x_0) = y_0$.*

Proof. By Theorem 5 there exist sequences $\mathscr{U}_i$ and $\mathscr{V}_i$ of taut open chain covers of X and Y, respectively, such that for $i = 1, 2, \ldots$

(1) $\mathscr{U}_{i+1}$ refines $\mathscr{U}_i$ and $\mathscr{V}_{i+1}$ refines $\mathscr{V}_i$,
(2) $x_0 \in i(F(\mathscr{U}_i), \mathscr{U}_i)$ and $y_0 \in i(F(\mathscr{V}_i), \mathscr{V}_i)$,
(3) $\operatorname{mesh}(\mathscr{U}_{i+1}) < \frac{1}{i}$ and $\operatorname{mesh}(\mathscr{V}_{i+1}) < \frac{1}{i}$.

Moreover, we may assume that $|\mathscr{U}_1| = |\mathscr{V}_1|$. Let p_1 be a pattern such that $\mathscr{U}_2$ follows pattern p_1 in $\mathscr{U}_1$ and $p_1(1) = 1$. By Theorem 3, there exists a taut open chain cover $\mathscr{O}_1$ of Y such that $\mathscr{O}_1$ follows pattern p_1 in $\mathscr{V}_1$ and $y_0 \in i(F(\mathscr{O}_1), \mathscr{O}_1)$. Let n_1 be an index such that $\mathscr{V}_{n_1}$ refines $\mathscr{O}_1$ and let q_1 be a pattern such that $\mathscr{V}_{n_1}$ follows pattern q_1 in $\mathscr{O}_1$ and $q_1(1) = 1$. By Theorem 3 there exists a taut open chain cover $\mathscr{W}_1$ of X such that $x_0 \in i(F(\mathscr{W}_1), \mathscr{W}_1)$ and $\mathscr{W}_1$ follows pattern q_1 in $\mathscr{U}_2$. Let n_2 be an index such that $\mathscr{U}_{n_2}$ refines $\mathscr{W}_1$ and let p_2 be a pattern such that $\mathscr{U}_{n_2}$ follows pattern p_2 in $\mathscr{W}_1$ and $p_2(1) = 1$.

$$\begin{tikzcd}
\mathscr{U}_1 \arrow[d, leftrightarrow] & \mathscr{U}_2 \arrow[l, "p_1"'] \arrow[d, leftrightarrow] & \mathscr{W}_1 \arrow[l, "q_1"'] \arrow[d, leftrightarrow] & \mathscr{U}_{n_2} \arrow[l, "p_2"'] \arrow[d, leftrightarrow] & \cdots \arrow[l] & X \arrow[d, leftrightarrow, "h"] \\
\mathscr{V}_1 & \mathscr{O}_1 \arrow[l, "p_1"'] & \mathscr{V}_{n_1} \arrow[l, "q_1"'] & \mathscr{O}_2 \arrow[l, "p_2"'] & \cdots \arrow[l] & Y
\end{tikzcd}$$

By induction we can construct sequences $\mathscr{U}_1, \mathscr{U}_2, \mathscr{W}_1, \mathscr{U}_{n_2}, \mathscr{W}_2, \mathscr{U}_{n_3}, \ldots$ and $\mathscr{V}_1, \mathscr{O}_1, \mathscr{V}_{n_1}, \mathscr{O}_2, \mathscr{V}_{n_2}, \ldots$ of taut open chains covers of X and Y respectively satisfying the conditions of Theorem 6.

8. Corollary. *Let P be a hereditarily indecomposable chainable continuum, then P is unique, homogeneous and hereditarily equivalent.*

5. Compositions of simple folds

In this section we will prove Theorem 1. Without being stated explicitly, this theorem has been used by previous authors. Since every pattern is the composition of a monotone pattern and a light pattern, it suffices to prove the theorem for light patterns.

Proof of Theorem 1. The proof is by a double induction, first on n and then on m starting with $n = 2$.

Let $n = 2$. Clearly the theorem is true for $m = 2$. Suppose the theorem is true for all light patterns $g\colon [1, s] \twoheadrightarrow [1, 2]$ where $2 \leqslant s < m$. Let $f\colon [1, m] \twoheadrightarrow [1, 2]$ be a light pattern.

If m is even, then without loss of generality we may assume $f(1) = 1$ and $f(m) = 2$. By induction $f' = f|[1, m-2]\colon [1, m-2] \twoheadrightarrow [1, 2]$ may be written as a composition of simple folds, $f' = F'_k \circ \ldots \circ F'_1$, where $F'_i \circ \ldots \circ F'_1(1)$ and $F'_i \circ \ldots \circ F'_1(m-2)$ are (distinct) extreme points of the range of $F_i \circ \ldots \circ F_1$ for $i = 1, \ldots, k$. Without loss of generality $F'_i \circ \ldots \circ F'_1(1) = 1$ for $i = 1, \ldots, k$. Let $m_i = |\text{domain } F'_i|$ and $n_i = |\text{Range } F'_i|$ for $i = 1, \ldots, k$. Then $f = F_{k+1} \circ F_k \circ \ldots \circ F_1$ where $F_i\colon [1, m_i+2] \twoheadrightarrow [1, n_i+2]$ is defined by

$$F_i(j) = \begin{cases} F'_i(j) & \text{for} \quad j \leqslant m_i \text{ if } 1 \leqslant i \leqslant k, \\ n_i + j - m_i & \text{for} \quad j \geqslant m_i \end{cases}$$

and $F_{k+1}\colon [1, 4] \twoheadrightarrow [1, 2]$ is defined by

$$F_{k+1}(j) = \begin{cases} 1 & \text{if} \quad j = 1, 3, \\ 2 & \text{if} \quad j = 2, 4. \end{cases}$$

If m is odd, then without loss of generality $f(1) = f(m) = 1$ and $f(m-1) = 2$. By induction $f' = f|[1, m-1] \twoheadrightarrow [1, 2]$ may be written as a composition of simple folds $f' = F'_k \circ \ldots \circ F'_1$ where (without loss of generality) $F'_i \circ \ldots \circ F'_1(1) = 1$ and $F'_i \circ \ldots \circ F'_1(m-1) = \max\{\text{Range } F'_i \circ \ldots \circ F'_1\}$ for $i = 1, \ldots, k$. Let $m_i = |\text{domain } F'_i|$ and $n_i = |\text{Range } F'_i|$ for $i = 1, \ldots, k$. Then $f = F_{k+1} \circ F_k \circ \ldots \circ F_1$ where $F_i\colon [1, m_i+1] \twoheadrightarrow [1, n_i+1]$ is defined by

$$F_i(j) = \begin{cases} F'_i(j) & \text{for} \quad j \leqslant m_i \quad \text{if} \quad 1 \leqslant i \leqslant k, \\ n_i + 1 & \text{for} \quad j = m_i + 1 \end{cases}$$

and $F_{k+1}\colon [1, 3] \to [1, 2]$ is given by

$$F_{k+1}(j) = \begin{cases} 1 & \text{if} \quad j = 1, 3, \\ 2 & \text{if} \quad j = 2. \end{cases}$$

Suppose the theorem is proved for all light patterns with cardinality of the range $< n$. There is nothing to prove if $m = n$. Suppose, therefore, that the theorem has been proved for all light patterns $g\colon [1, s] \twoheadrightarrow [1, p]$ where $2 \leqslant p \leqslant n$ and $p \leqslant s < m$. Let $f\colon [1, m] \twoheadrightarrow [1, n]$ be a light pattern. There are essentially four cases to consider. We prove in detail only the first two cases. The other cases are similar.

Case 1. $f(1) = 1$, $f(m) = n$ and $1 < f(j) < n$ for $1 < j < m$. Then $f(m-1) = n-1$. By induction $f' = f|[1, m-1]\colon [1, m-1] \twoheadrightarrow [1, n-1]$ is a composition of simple folds $f' = F'_k \circ \ldots \circ F'_1$ where $F'_i \circ \ldots \circ F'_1(1) = 1$ and $F'_i \circ \ldots \circ F'_1(m-1) = \max\{\text{range } F'_i \circ \ldots \circ F'_1\}$ for $i = 1, \ldots, k$. Let $m_i = |\text{domain } F'_i|$ and $n_i = |\text{Range } F'_i|$. Then $f = F_k \circ \ldots \circ F_1$ where $F_i\colon [1, m_i + 1] \twoheadrightarrow [1, n_{i+1}]$ is given by

$$F_i(j) = \begin{cases} F'_i(j) & \text{if} \quad j \leqslant m_i, \\ n_i + 1 & \text{if} \quad j = m_i + 1. \end{cases}$$

Case 2. $f(1) = f(m) = 1$. Let r be an integer such that $f(r) = n$. Then $f' = f|[1, r]\colon [1, r] \twoheadrightarrow [1, n]$ is a composition of simple folds $f' = F'_k \circ \ldots \circ F'_1$

where $F_i' \circ \ldots \circ F_1'(1) = 1$ and $F_i' \circ \ldots \circ F_1'(r) = \max\{\text{Range}\, F_i' \circ \ldots \circ F_1'\}$ for $i = 1, \ldots, k$. Let

$$m_i = |\text{domain}\, F_i'| \quad \text{and} \quad n_i = |\text{Range}\, F_i'|.$$

Define $g\colon [1, m-r+1] \twoheadrightarrow [1, n]$ by $g(j) = f(j+r-1)$. Then $g = G_p' \circ \ldots \circ G_1'$ is a composition of simple folds such that $G_i' \circ \ldots \circ G_1'(1) = \max\{\text{Range}\, G_i' \circ \ldots \circ G_1'\}$ and $G_i' \circ \ldots \circ G_1'(m-r+1) = 1$ for $i = 1, \ldots, p$. Let $r_i = |\text{domain}\, G_i'|$ and $s_i = |\text{Range}\, G_i'|$ for $i = 1, \ldots, p$. Then $f = H \circ F_k \circ \ldots \circ F_1 \circ G_p \circ \ldots \circ G_1$ where $G_i\colon [1, r+r_i-1] \twoheadrightarrow [1, r+s_i-1]$ is defined by

$$G_i(j) = \begin{cases} j & \text{for } j \leqslant r \\ r+s_i-G_i'(j-r+1) & \text{for } j \geqslant r \end{cases} \quad \text{for } i = 1, \ldots, p;$$

$F_i\colon [1, m_i+n-1] \twoheadrightarrow [1, n_i+n-1]$ is given by

$$F_i(j) = \begin{cases} F_i'(j) & j \leqslant m_i \\ n_i+j-m_i & j \geqslant m_i \end{cases} \quad \text{for } i = 1, \ldots, k$$

and $H\colon [1, n_k+n-1] \twoheadrightarrow [1, n]$ is given by

$$H(j) = \begin{cases} j & \text{if } \; j \leqslant n_k = n, \\ 2n-j & \text{if } \; j \geqslant n_k. \end{cases}$$

Case 3. $f(1) = 1$, $f(m) = n$ and $f(r) = 1$ for some r with $1 < r$. Let $1 < s < r$ such that $f(s) = \max\{\text{Range}\, f|[1, r]\}$. The proof uses the inductive hypothesis on the intervals $[1, s]$, $[s, r]$ and $[r, m]$.

Case 4. $f(m)$ is not an extreme point of the range f. Let r be the smallest integer such that $f([1, r]) = [1, n]$. The proof uses induction on the intervals $[1, r]$ and $[r, m]$.

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