GEOMETRIC AND ALGEBRAIC TOPOLOGY
BANACH CENTER PUBLICATIONS, VOLUME 18
PWN – POLISH SCIENTIFIC PUBLISHERS
WARSZAWA 1986

CELL-LIKE MAPS, DIMENSION AND COHOMOLOGICAL DIMENSION: A SURVEY*

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The terms cell-like set and cell-like mapping were first introduced by R. C. Lacher in 1968 [La]. Briefly, a finite dimensional continuum X is called *cell-like* if for some n, X can be embedded as a cellular subset of $\boldsymbol{R}^n$. This means that $X = \bigcap_{i=1}^{\infty} B_i$, where $B_{i+1} \subset \mathring{B}_i$, and each B_i is an n-cell in $\boldsymbol{R}^n$. Subsequently, Lacher proved that being cell-like implies having finite dimension and the shape of a point. A map is called *cell-like* if it is proper and each point inverse has the shape of a point; as a consequence of this definition the requirement that point inverses be finite dimensional is dropped altogether. I shall use the term cell-like map below to refer to certain results that were proved prior to the introduction of the term. For the purposes of this article, restrict spaces to those that are metrizable except for certain complexes that arise in definitions.

Decompositions of $\boldsymbol{R}^3$ into cell-like sets were studied extensively in the 1950's by R. H. Bing. For these decompositions, the quotient map $q\colon \boldsymbol{R}^3 \to Y$ is a cell-like map. The reader is referred to [Bi1], [Bi2], [Bi3] for some of Bing's work on the subject. We may replace $\boldsymbol{R}^3$ by any manifold M^n and consider $q\colon M \to Y$, a cell-like map. The questions usually asked are

a. Under what conditions are M and Y homeomorphic?

b. Under what conditions are $M \times \boldsymbol{R}^n$ and $Y \times \boldsymbol{R}^n$ homeomorphic?

Already in 1925 (see [Mo]) R. L. Moore proved that if $M = \boldsymbol{R}^2$, then so also is Y. But the examples given by Bing show that no such theorem is true when $M = \boldsymbol{R}^3$.

During the 1960's and 1970's an extensive literature developed about these decomposition problems. Many examples were given of nonmanifolds Y whose product with some $\boldsymbol{R}^n$ (usually $\boldsymbol{R}$) is a manifold. The "double suspension" problem was solved as were many other difficult ones. A survey article [Da1] by R. Daverman should provide the interested reader with

* This paper is in final form and no version of it will be submitted for publication elsewhere.

some background. As a footnote, it should be mentioned that M. Freedman's proof in 1981 of the 4-dimensional Poincaré conjecture made use of the proof techniques that had been developed for the study of decomposition spaces.

In a 1981 paper [Da2], Daverman showed that if the quotient space Y of a cell-like map $q\colon M^n \to Y$ is finite dimensional, then $Y \times \boldsymbol{R}^2 \cong M \times \boldsymbol{R}^2$. It is not known whether $\boldsymbol{R}^2$ can be replaced by $\boldsymbol{R}$; this is a challenging problem. A perhaps more challenging problem is to determine whether the quotient space Y must be finite dimensional, the so-called cell-like map dimension raising problem. I shall return to this shortly (consult also the survey article [Da1]).

Topological dimension theory is an old subject which was apparently mature in 1941 with the first printing of the Hurewicz and Wallman book, *Dimension Theory* [HW]. (Several rivals have since been published.) The experience of this author shows that, although there is a vast literature determining a theory of dimension for finite dimensional (metrizable) spaces, the literature for infinite dimensional spaces cannot be said to determine a theory. This fact came to my attention while working with R. Schori and J. Walsh during the late 1970's on the *cell-like map dimension raising problem*: can a cell-like map of a finite dimensional space have range (such maps are always surjective) which is of higher dimension? From work of George Kozlowski [Ko], if the dimension rises, then it must rise to infinity. It was also known that any space containing subspaces of arbitrarily high finite dimension could not be the target of a cell-like dimension raising map. So naturally the question of whether every infinite dimensional space or compactum must contain subspaces of arbitrarily high finite dimension came to the fore. We were confronted with a paucity of examples and a lack of theory.

From a study of [Bi4], [He], [Za], Schori, Walsh and I developed a systematic approach [RSW] to constructions of (weakly) hereditarily strongly infinite dimensional compacta. For such compacta, every closed subspace is either 0-dimensional or strongly infinite dimensional. Yet there was still the possibility that nonclosed subspaces of arbitrarily large finite dimensions would exist. But then in [Wa1], Walsh showed how to construct compacta whose only finite dimensional subspaces were of dimension 0. This closed off any hope of a quick negative solution to the dimension-raising problem.

Walsh's example of an infinite dimensional space containing "no finite dimensional subspaces" gave rise to more speculation about the nature of infinite dimensional topological spaces, and questions of how they could be classified. A space is called strongly infinite dimensional if it contains an essential family $\{(A_i, B_i) \mid i = 1, 2, \ldots\}$; this means that each (A_i, B_i) is a disjoint pair of closed subsets and if S_i is a closed set separating A_i and B_i, then $\bigcap_{i=1}^{\infty} S_i \neq \emptyset$. It is known that a space having dimension n has an essential family $\{(A_i, B_i) | 1 \leqslant i \leqslant n\}$ and cannot have an essential family of larger cardinality, while infinite dimensional spaces always have essential families of arbitrarily high cardinality. The term weakly infinite dimensional is used for infinite dimensional spaces which are not strongly infinite dimensional. Any countable dimensional space (an infinite dimensional space which can be written as a countable union of 0-dimensional subspaces) is weakly infinite dimensional.

The Hilbert cube Q is strongly infinite dimensional, while any subspace X of Q which is a countable union say of simplexes of arbitrarily high dimensions is countable dimensional. There are examples of strongly infinite dimensional subspaces of Q which are totally disconnected. Nevertheless, according to [Ru1], [Ru2], each strongly infinite dimensional space X contains a hereditarily strongly infinite dimensional closed subspace Y. This means that each subspace of Y is either 0-dimensional or strongly infinite dimensional. The results [Ru1], [Ru2] therefore go much further than that of [Wa1] and apply to noncompact spaces as well. Recently, J. Krasinkiewicz (see this volume, pp. 357–404) has generalized the results of [Ru1] although, as yet, his techniques do not apply to noncompact spaces as in [Ru2].

As stated earlier, there is not a theory for infinite dimensional spaces. Until the publication of Roman Pol's example [Po] in 1982, the existence of an infinite dimensional compactum which was neither countable dimensional nor strongly infinite dimensional was not known. Pol's example remains a singularity. There is yet no effective method for obtaining different types of examples and therefore no clue about how to classify spaces of infinite dimensions in a dimension-theoretic way. It should be mentioned that Pol's example does satisfy Property C which was introduced by W. Haver, and is the subject of study by Addis and Gresham [AG]. Also, there has been some classification of countable dimensional spaces (see for example, [EP], [En]).

There is a type of dimension, cohomological dimension, which cannot increase under cell-like mappings. Cohomological dimension (c-dim) may be defined in terms of an Eilenberg–MacLane complex $K(\mathbf{Z}, n) = K_n$. This CW-complex can be obtained by starting with an n-sphere S^n, and attaching an infinite collection of cells so that all homotopy groups of dimension greater than n are "killed off", while for $k \leqslant n$, $\pi_k(K_n) \approx \pi_k(S^n)$. Then say that $c\text{-dim}\, X \leqslant n$ if for each closed subset $A \subset X$ and each map $f\colon A \to K_n$, there exists a map $F\colon X \to K_n$ extending f. By definition, $c\text{-dim}\, X = \min\{k | c\text{-dim}\, X \leqslant k\}$ ($c\text{-dim}\, X = \infty$ if there are no such k). It is known that if $\dim X < \infty$, then $c\text{-dim}\, X = \dim X$.

In [Wa3], John Walsh presents a theory which relates the study of cohomological dimension to that of cell-like maps. There he proves the beautiful Edwards–Vietoris theorem: There exists a compactum of infinite dimension and finite cohomological dimension iff there exists a cell-like map of a compactum of finite dimension onto a compactum of infinite dimension. This result has been improved by myself and P. J. Schapiro in that we have removed compactness, but we do require our spaces to be separable [RS]. These results show the strong relationship between a homotopy-theoretic question concerning cohomological dimension and a geometric-topological question involving cell-like maps. We further were able to show as a result of this connection that each separable space of cohomological dimension n embeds in a topologically complete space of the same cohomological dimension.

Some progress has been made in determining that under certain conditions a cell-like map cannot raise dimension. Notably in [KW], [KRW] it is proved that a cell-like map of a subset of a 3-manifold or of a 3-dimensional polyhedron with 1-dimensional fibers does not raise dimension. Also, it was already known that cell-like maps of 1-dimensional spaces or of 2-dimensional ANR's could not raise dimension. In [Wa4], Walsh proved that every integral homology 3-manifold has dimension 3. Apart from results such as these, current thinking is to look for a counterexample. Because of the results mentioned in the preceding paragraph, instead of trying to find dimension raising maps, one could search for a space of finite cohomological dimension but of infinite (topological) dimension.

The search for counterexamples for the cell-like dimension raising problem is to a large extent thwarted by our ignorance of the nature of infinite dimensional spaces. For example, we do not know whether all strongly infinite dimensional compacta have infinite cohomological dimension. In [Wa2], we find calculations showing that many "known" examples have infinite cohomological dimension. I carried these calculations further in [Ru3], [Ru4]; there I considered both compact and noncompact (totally disconnected) spaces. For some of the examples, I could prove the cohomological dimension to be infinite. For others, both compact and noncompact, the cohomological dimension is not known although one would suspect it to be infinite.

Such a program of selecting certain types of examples for consideration is unlikely to succeed by itself. Certainly it can help eliminate contenders and it can provide some direction or insight into which way to approach the problem. A study of the classification of infinite dimensional spaces is in order. To this time, dimension has typically been defined in terms of mappings into certain objects and the ability to extend such maps, or in terms of coverings. Property C is defined in terms of coverings and perhaps provides a clue for new ways to make definitions via covers. On the other hand, an approach based on mappings, if the right class of target objects could be found, might yield the desired results in a systematic way. (The reader might consult [An1], [An2] for information about the relation between cell-like maps, proper hereditary shape equivalences and Property C.)

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Presented to the Topology Semester
April 3 – June 29, 1984