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ESSENTIAL MAPPINGS ONTO PRODUCTS OF MANIFOLDS*

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Contents

INTRODUCTION

This paper is a contribution to the theory of essential mappings (in the sense of Hopf [Ho]). We develop the theory for mappings into products of manifolds and give several applications of the results. Our theory unifies and generalizes some old and new ideas worked out by other authors in the cited works.

The essential mappings are usually understood as mappings into finite dimensional cells. Here we adopt more general approach, closer to Hopf's original point of view, considering mappings into arbitrary manifolds or, naturally extending the definition, into (countable) products of manifolds. Instead of manifolds one can consider more general spaces but there is no

* This paper is in final form and no version of it will be submitted for publication elsewhere.

need in this paper for doing that. The author intends to undertake this subject in a subsequent paper. Essential mappings have interesting and long history and numerous applications. There is an extensive literature on this subject from which we list only a few items: [A], [A–P], [Bo], [G–T], [Ho], [Hol_1].

1. Terminology

All *spaces* are assumed to be metrizable. Such spaces are *hereditarily normal* and *countably paracompact*. The former property says that any pair of separated subsets can be enlarged to a pair of disjoint open sets. The latter one says that any countable open covering admits a closed shrinking. Since our spaces are metrizable all cartesian products considered in this paper are countable. By a *manifold* we mean a topological manifold (boundary permitted) which is a continuum. A *cell* is a homeomorph of the unit cube I^n.

2. Contents

Chapter I: Here we present a detailed discussion of basic concepts of this paper. Membranes and separators and other dual families in the sense of Section 5 are introduced and studied. The main result (Th. 6.1) says that if an essential mapping is given by two coordinate maps then any near separator of either of them is a membrane of the other. Archetypes of some results of this chapter are to be found in the papers by Henderson [He], Bing [B_2] and Rubin–Schori–Walsh [R–S–W], where they are expressed in a different language.

Chapter II: Here some applications of the theory developed in Chapter I are given. In Section 1, generalizing a theorem of Borsuk, we prove among others the following: if Y is a subset of $M \times I$, where M is a manifold, interesting each subcontinuum of $M \times I$ joining $M \times (0)$ and $M \times (1)$ then Y is a membrane of the projection $M \times I \to M$. Section 2 contains some results on totally disconnected spaces with other interesting properties. We construct such examples referring once more to an old trick which was so fruitfully applied by other authors: Mazurkiewicz [M], Knaster [K], Lelek [L], Rubin–Schori–Walsh [R–S–W], Pol [P] and Todorov [T_2]. This time we are able to prove certain homotopy properties of the examples (implying their dimensional properties studied by these authors). Here is the trick: Let X be a space, let P be a topological property and let $\mathscr{S}$ be a family of subsets of X such that any $Y \subset X$ which does not satisfy P is disjoint with some member of $\mathscr{S}$. Then any subset of X which meets each member of $\mathscr{S}$ satisfies P. Many other results are derived by using this trivial observation. In Section 3 we prove the following generalization of a theorem of Mazurkiewicz on noncutting subsets of euclidean spaces. Let A be a subset of $\boldsymbol{R}^n$ with $\dim A \leqslant k \leqslant n-2$. Then for each $x, y \in \boldsymbol{R}^n \setminus A$ and each $\varepsilon > 0$ there exist a continuum $Y \subset \boldsymbol{R}^n \setminus A$ containing both x and y and an ε-mapping $f\colon Y \to S^{n-k-1}$ such that Y in an irreducible membrane of f. See [H_1], [H_3] and [T_1] for related results.

Section 4 contains some results on coincidence points. As a corollary we get a result on the set of fixed points of a homotopy defined on a cube. In Section 5 we show that there exists a nondegenerate continuum such that every its subset with dimension > 0 admits an essential map onto every countable product of manifolds. Certain new results on infinite-dimensional Cantor manifolds are contained in Section 6. We prove a theorem on decompositions of strongly infinite-dimensional compacta into countable families of arbitrary sets. As a corollary we obtain a far reaching generalization of the Hurewicz theorem on decompositions of the Hilbert cube into 0-dimensional sets. Also in this section we provide a solution to a problem of R. Pol.

The paper ends with a list of open problems.

3. Notation

By I we denote the unit interval. The letters M and N themself, or with lower case subscripts, will always denote manifolds of dimension $\geqslant 1$, unless the opposite is explicitly stated. These letters with upper case subscripts, as for example M_J, denote cartesian products of manifolds indexed by the set J, i.e. $M_J = \prod_{j \in J} M_j$ (when we use this symbol it is understood that the nature of the manifolds M_j is immaterial). The two symbols for cartesian products will be used interchangeably. By ∂M or $\dot{M}$ we denote the boundary of a manifold M; by $\operatorname{int} M$ or $\mathring{M}$ we denote its interior. More generally, by $\mathring{M}_J$ we denote the product $\prod_{j \in J} \mathring{M}_j$, where M_J is as above. If J is a set then $|J|$ denotes its cardinality. A mapping $f\colon X \to \prod_{j \in J} M_j$ is often denoted by its coordinate mappings, $f = (f_j)$, where $f_j\colon X \to M_j$, $j \in J$. If K is a nonvoid subset of J then $f_K\colon X \to \prod_{j \in K} M_j$ denotes the composition $X \xrightarrow{f} \prod_{j \in J} M_j \xrightarrow{p_K} \prod_{j \in K} M_j$, where the second mapping is the natural projection.

By 2^X we denote the space of all closed subsets of X with the Vietoris topology; the empty set $\emptyset$ is as isolated point of this space.

4. Remark

An earlier version of this paper appeared under the same title in 1984 (Preprint 310, Institute of Mathematics, Polish Academy of Sciences). The present expanded version differs at many essential points from the original.

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Chapter I

THEORY OF ESSENTIAL MAPPINGS. MEMBRANES AND SEPARATORS

1. Essential mappings

Consider a mapping $f\colon X \to M$ into a manifold M. A mapping $g\colon X \to M$ is said to be an *admissible deformation* of f provided there is a homotopy $H\colon (X, f^{-1}(\partial M)) \times I \to (M, \partial M)$ such that $H_0 = f$ and $H_1 = g$. If, in addition, H is a homotopy rel. $f^{-1}(\partial M)$ then g is called a *∂-deformation* of f. If M is closed (i.e. $\partial M = \emptyset$) then the deformations do not differ from ordinary maps homotopic to f.

Analogous notions are defined for mappings into products of manifolds. Namely, let $(f_j)\colon X \to \prod_{j\in J} M_j$. A mapping $(g_j)\colon X \to \prod_{j\in J} M_j$ is said to be an admissible (∂-) deformation of (f_j) provided each g_j, $j \in J$, is an admissible (∂-) deformation of f_j. If all manifolds M_j are closed then (g_j) is simply a mapping homotopic to (f_j).

The following is a basic notion of this paper. A mapping $f\colon X \to \prod_{j\in J} M_j$ is said to be *essential* provided every admissible deformation of f is surjective (comp. [Ho], [A], [Bo], [G–T]). This definition involves the presentation of the target space as a product of manifolds. An independence of essentiality on the presentation is discussed in Section 3.

1.1. PROPOSITION. *Let $f\colon X \to M_J$ be essential. Then we have*

(a) *f is surjective,*

(b) *every admissible deformation of f is essential,*

(c) *$f_K = p_K \circ f$ is essential for every (nonvoid) subset K of J.* ∎

1.2. PROPOSITION. *A mapping $f\colon X \to M_J$ is essential in the following two cases:*

(a) *$f|A$ is essential for some $A \subset X$,*

(b) *X is compact and f_K is essential for every finite subset K of J.* ∎

1.3. LEMMA. *Let $f\colon X \to M$ and let A be a closed subset of X such that $f|A$ is not essential. Then for every $y \in \mathring{M}$, there is a ∂-deformation g of f such that $y \notin g(A)$.*

Proof. Let $h_t\colon (A, A \cap f^{-1}(\partial M)) \to (M, \partial M)$, $t \in I$, be a homotopy connecting $f|A$ and h_1 such that $h_1(A) \neq M$. Let $z \in M \setminus h_1(A)$. There is a homotopy $\varphi_t\colon (M\setminus(z), \partial M\setminus(z)) \to (M\setminus(z), \partial M\setminus(z))$, $t \in I$, such that $\varphi_0 = \mathrm{id}$ and

$\overline{\varphi_1(M\backslash(z))} \neq M$. Let $w \in \mathring{M}\backslash\overline{\varphi_1(M\backslash(z))}$. There is an isotopy ψ: $\mathrm{id}_M \simeq \psi_1$ rel. ∂M on M such that $\psi_1(w) = y$. Let

$$F'\colon (A, A \cap f^{-1}(\partial M)) \times I \to (M, \partial M)$$

be defined by the formula

$$F'(x, t) = \begin{cases} h_{2t}(x) & \text{for } 0 \leqslant t \leqslant 1/2, \\ \psi_{2t-1} \circ \varphi_{2t-1} \circ h_1(x) & \text{for } 1/2 \leqslant t \leqslant 1. \end{cases}$$

Then $F'_0 = f|A$ and $y \notin \overline{F'_1(A)}$. Since $A \cap f^{-1}(\partial M)$ is closed in $f^{-1}(\partial M)$ and $\partial M \in \mathrm{ANR}$, by the homotopy extension theorem there is a homotopy

$$F''\colon f^{-1}(\partial M) \times I \to \partial M$$

such that $F''_0 = f|f^{-1}(\partial M)$ and $F''|(A \cap f^{-1}(\partial M)) \times I = F'|(A \cap f^{-1}(\partial M)) \times I$. The two homotopies define a new homotopy $[A \cup f^{-1}(\partial M)] \times I \to M$. Since the set is closed in X there is a homotopy

$$F\colon X \times I \to M$$

such that $F_0 = f$, $F|A \times I = F'$ and $F|f^{-1}(\partial M) \times I = F''$. It follows that $y \notin \overline{F_1(A)}$. Hence there is a closed cell $Q \subset \mathring{M}\backslash F_1(A)$ such that $y \in \mathring{Q}$ and $\dim Q = \dim M$. Since $F(f^{-1}(\partial M) \times I) \subset \partial M \subset M\backslash Q$ there is an open set U in X such that $f^{-1}(\partial M) \subset U$ and $F(U \times I) \subset M\backslash Q$. Let u: $X \to I$ be a Urysohn map transforming $f^{-1}(\partial M)$ to 0 and $X\backslash U$ to 1. Define the mapping g: $X \to M$ by the formula

$$g(x) = F(x, u(x)).$$

One easily sees that $g(A) \subset M\backslash Q$. On the other hand the homotopy

$$(x, t) \mapsto F(x, u(x) \cdot t), \qquad x \in X,\ t \in I,$$

connects f and g rel $f^{-1}(\partial M)$, which completes the proof. ∎

1.4. COROLLARY. *Let f: $X \to M$ and let $f|Y$ be essential for some $Y \subset X$. Then $f|(Y\backslash \operatorname{int} f^{-1}(\partial M))$ is essential.* ∎

1.5. LEMMA. *Suppose a mapping (h_j): $A \to \prod_{j \in J} M_j$ is not surjective (not essential, resp.). Then there is a closed covering $(A_j)_{j \in J}$ of A such that $h_j(A_j) \neq M_j$ ($h_j|A_j$ is not essential, resp.) for each $j \in J$.*

Proof. Let $(y_j) \notin (h_j)(A)$. Notice that $(A\backslash h_j^{-1}(y_j))_{j \in J}$ is a countable open covering of A. By the countable paracompactness of A there is a closed shrinking $(A_j)_{j \in J}$ of that covering. It satisfies the conclusion. The version in brackets follows similarly. ∎

Now we are ready to prove the following

1.6. THEOREM. *Let f: $X \to \prod_{j \in J} M_j$ be a mapping and let A be a closed* *subset of X such that $f|A$ is not essential. Then for any choice $y_j \in \mathring{M}_j$, $j \in J$, there exist neighborhoods V_j of y_j in M_j and a ∂-deformation g of f such that*

$$(*)\qquad g(A) \cap \prod_{j\in J} V_j = \emptyset.$$

Proof. Let $f = (f_j)$. There is an admissible deformation (h_j) of $f|A$ which is not surjective. By 1.5 there is a closed covering $(A_j)_{j\in J}$ of A such that $h_j(A_j) \neq M_j$ for each $j \in J$. Since $h_j|A_j$ is an admissible deformation of $f_j|A_j$, by 1.3 there is a ∂-deformation g_j of f_j such that $y_j \notin \overline{g_j A_j)}$, $j \in J$. The mapping $g = (g_j)$ and the sets $V_j = M_j \backslash g_j(A_j)$ satisfy $(*)$. ∎

1.7. Corollary. *A mapping $f\colon X \to M_J$ is essential iff every ∂-deformation of f is surjective.* ∎

1.8. Corollary. *Let $X = \bigoplus_{s\in S}$ and let $F\colon X \to M_J$ be essential. Then $f|X_s$ is essential for some $s \in S$.* ∎

1.9. Theorem. *Let $f\colon X \to \prod_{j\in J} M_j$ and for each $j \in J$ let $N_j \subset M_j$ be a manifold with $\dim N_j = \dim M_j$. If f is essential then its restriction $\bar{f}\colon f^{-1}(\prod_{j\in J} N_j) \to \prod_{j\in J} N_j$ is also essential.*

Proof. Let $f = (f_j)$, let $\bar{f} = (\bar{f}_j)$ and let $A = f^{-1}(\prod_{j\in J} N_j) = \bigcap_{j\in J} f_j^{-1}(N_j)$. Then $\bar{f}_j = f_j|A$. Suppose $\bar{f}\colon A \to \prod_{j\in J} N_j$ is not essential. By 1.7 there exist homotopies

$$h_j\colon A \times I \to N_j, \quad j \in J,$$

such that $h_{j0} = f_j|A$, $(h_{j1})(A) \neq \prod_{j\in J} N_j$ and $h_{jt}|\bar{f}_j^{-1}(\partial N_j) = f_j|\bar{f}_j^{-1}(\partial N_j)$ for each $t \in I$. Note that $\bar{f}_j^{-1}(\partial N_j) = A \cap f_j^{-1}(\partial N_j)$ and $A \subset f_j^{-1}(N_j)$. Hence applying the homotopy extension theorem one can extend the homotopies to homotopies

$$H_j\colon f_j^{-1}(N_j) \times I \to N_j, \quad j \in J,$$

such that $H_{j0} = f_j|f_j^{-1}(N_j)$, $H_j|A \times I = h_j$ and $H_{jt}|f_j^{-1}(\partial N_j) = f_j|f_j^{-1}(\partial N_j)$ for each $t \in I$. There exist natural extensions of H_j to

$$\bar{H}_j\colon X \times I \to M_j, \quad j \in J,$$

given by: $\bar{H}_j|f_j^{-1}(N_j) \times I = H_j$ and $H_j(x, t) = f_j(x)$ for $x \notin f_j^{-1}(N_j)$ and each $t \in I$.

Put $g_j = \bar{H}_{j1}\colon X \to M_j$, $j \in J$. It follows that g_j is a ∂-deformation of f_j since $\bar{H}_j\colon f_j \simeq g_j$ rel. $f_j^{-1}(\partial M_j)$. Moreover, $g_j|A = h_{j1}$ and $g_j(X \backslash f_j^{-1}(N_j)) \subset M_j \backslash N_j$.

Let $y = (y_j) \in \prod_{j \in J} N_j \setminus (h_{j1})(A)$ and let $g = (g_j)$. It follows that

$$g^{-1}(y) = \bigcap_{j \in J} g_j^{-1}(y_j) = \bigcap_{j \in J} A \cap g_j^{-1}(y_j) = \bigcap_{j \in J} h_{j1}^{-1}(y_j) = \emptyset,$$

a contradiction since g is an admissible deformation of f. ∎

Now we are going to prove an important property of essential mappings.

1.10. THEOREM. *If $f\colon X \to M_J$ is not essential on a subset $Y \subset X$ then it is not essential on a neighborhood of Y in X.*

Proof. First we consider the special case where $M_J = M$ is a manifold. There is a homotopy $F\colon (Y, Y \cap f^{-1}(\partial M)) \times I \to (M, \partial M)$ such that $F_0 = f|Y$ and $F_1(Y) \neq M$. To complete the proof it suffices to construct a similar homotopy $\bar{F}$ defined on an open neighborhood U of Y in X.

By 1.3 we may assume that $\overline{F_1(Y)} \neq M$. Let V be an open proper subset of M containing $\overline{F_1(Y)} \cup \partial M$. Referring to general properties of ANR's (or the existence of a collar on ∂M) we infer that there exist a deformation $G\colon M \times I \to M$ fixed on M, and an open set $V_0 \subset M$ such that $\partial M \subset V_0 \subset V$, $G_0 = \mathrm{id}_M$, $G_1(V) \subset V$ and $G_1(V_0) \subset \partial M$. Let ω be an open covering of M such that $\mathrm{st}(\partial M, \omega) \subset V_0$ and $\mathrm{st}(F_1(Y), \omega) \subset V$.

Let $X^* = X \times I$, let $Y^* = X \times (0) \cup Y \times I$ and let $X^* \supset Y^* \xrightarrow{f^*} M$ be given by $f^*(x, 0) = f(x)$ for $x \in X$ and $f^*(x, t) = F(x, t)$ for $(x, t) \in Y \times I$. According to Appendix there is a ω-homotopy $H\colon Y^* \times I \to M$ such that $H_0 = f^*$ and H_1 is extendable on a neighborhood U^* of Y^* in X^*. Let $\bar{H}_1$ be such an extension. Since $f^*(Y \times (1)) = F_1(Y)$ and $f^*(Y^* \cap f^{-1}(\partial M) \times I) \subset \partial M$ it follows that

(1) $\bar{H}_1(Y \times (1)) \subset V$,

(2) $H([Y^* \cap f^{-1}(\partial M) \times I] \times I) \subset V_0$.

Combining these with the fact that U^* is a neighborhood of Y^* in X^* we conclude that there is an open neighborhood U of Y in X such that $U \times I \subset U^*$, and

(3) $H([U \cap f^{-1}(\partial M)] \times (0) \times I) \subset V_0$,

(4) $\bar{H}_1([U \cap f^{-1}(\partial M)] \times I) \subset V_0$,

(5) $\bar{H}_1(U \times (1)) \subset V$.

The homotopy $\bar{F}\colon U \times I \to M$ is given by

$$\bar{F}(x, t) = \begin{cases} G(f(x), 3t) & \text{for } 0 \leqslant t \leqslant 1/3, \\ G_1(H(x, 0, 3t-1)) & \text{for } 1/3 \leqslant t \leqslant 2/3, \\ G_1(\bar{H}_1(x, 3t-2)) & \text{for } 2/3 \leqslant t \leqslant 1. \end{cases}$$

One easily checks that $\bar{F}$ is well-defined and maps $[U \cap f^{-1}(\partial M)] \times I$ into ∂M, $\bar{F}_0 = f|U$ and $\bar{F}_1(U) \subset V\ (\neq M)$. This completes the proof of the case where $M_J = M$.

Now consider the general case where $f = (f_j)$ and $M_J = \prod_{j\in J} M_j$. By 1.5 there is a covering $(Y_j)_{j\in J}$ of Y such that each map $f_j|Y_j$ is not essential. By the special case above there are open neighborhoods G_j of Y_j in X such that $f_j|G_j$ is not essential, $j\in J$. Then $G = \bigcup_{j\in J} G_j$ is a neighborhood of Y in X. It suffices to show that $f|G = (f_j|G)$ is not essential. Let $(A_j)_{j\in J}$ be a closed (with respect to G) shrinking of the covering (G_j) of G. Since each A_j is closed in G and the restriction of $f_j|G$ to A_j is not essential hence by coordinatewise application of 1.3 we conclude that $(f_j|G)$ is not essential (because (A_j) is a covering of G). ∎

Theorem 1.10 together with the argument in the last paragraph of its proof (where 1.6 is used in place of 1.3) yield the following

1.11. THEOREM. *Let $f = (f_{J_k})$: $X \to \prod_{k\in K} M_{J_k}$ and let $(A_k)_{k\in K}$ be a covering of X such that each restriction $f_{J_k}|A_k$: $A_k \to M_{J_k}$, $k\in K$, is not essential. Then f is not essential.* ∎

Appendix to Section 1

The following fact has been used in the proof of 1.10.

THEOREM. *Let f: $A \to Y$ be a mapping, where A is a subset of a space X and $Y \in$ ANR. Then for any open covering ω of Y there is a neighborhood G of A in X and a mapping g: $G \to Y$ such that f and $g|A$ are ω-homotopic.*

Proof. A similar theorem is proved in [Hu], Theorem 8.1 on p. 146, for the case where A is a closed subset of X (then we can take $G = X$). To obtain a proof in our case modify the argument of [Hu] as follows: (a) for each $U\in\gamma$ pick an open set G_U in X such that $A\cap G_U = f^{-1}(U)$ and for any sequence $U_1, \ldots, U_n$, $U_i\in\gamma$,

$$G_{U_1}\cap\ldots\cap G_{U_n} \neq \emptyset \Rightarrow U_1\cap\ldots\cap U_n \neq \emptyset,$$

(b) define G to be the union of the sets G_U, $U\in\gamma$, (c) instead of π consider the covering $\{G_U\}$ of G. ∎

2. Existence of essential mappings

The following result may be treated as a generalization of the Brouwer fixed point theorem which can be stated as follows: the identity mapping $I^n \to I^n$ is essential.

2.1. THEOREM. *Every homeomorphism h: $X \to \prod_{j\in J} M_j$ is essential.*

Proof. Let $K \subset J$ be a finite nonvoid set. By 1.2 (b) it suffices to show that h_K: $X \to \prod_{j\in K} M_j = M$ is essential. Hence it suffices to show that $p = h_K h^{-1}$: $\prod_{j\in J} M_j \to M$ is essential, since h^{-1} is a homeomorphism. Then p is the projection and M may be considered as a subset of $\prod_{j\in J} M_j$ such that p becomes a retraction. So it is sufficient to prove that id: $M \to M$ is an essential mapping. Suppose it is not true. Then there exist a homotopy H: id $\simeq f$ rel. ∂M and a point $a \in \mathring{M}\setminus f(M)$. Consider the diagram

$$\begin{tikzcd}
(M,\partial M) \arrow[rr, "i"] \arrow[dr, shift left, "\mathrm{id}"] \arrow[dr, shift right, "f"'] \arrow[ddr, "\bar{f}"'] & & (M, M\setminus(a)) \\
& (M,\partial M) \arrow[ur, "i"] & \\
& (M\setminus(a),\ M\setminus(a)) \arrow[uur, "j"'] &
\end{tikzcd}$$

where $\bar{f}(x) = f(x)$ for $x \in M$ and i, j are inclusions. Applying to this diagram the homology functor H_n, $n = \dim M$, with coefficients in Z_2 one easily sees that $H_n(i) = H_n(j)H_n(\bar{f}) = 0$. Since $\mathring{M}$ is Z_2-orientable there is $\alpha \in H_n(M, \partial M)$ (the fundamental class of the Z_2-orientation) such that $H_n(i)(\alpha)$ is a generator of $H_n(M, M\setminus(a)) \approx Z_2$ (see [G]). Hence $H_n(i) \neq 0$, a contradiction. ∎

2.2. PROPOSITION. *If X is compact then f: $X \to I$ is essential iff there is a component of X intersecting both $f^{-1}(0)$ and $f^{-1}(1)$.* ∎

2.3. THEOREM. *If, for each $j \in J$, X_j is compact and f_j: $X_j \to I$ is essential then $\prod_{j\in J} f_j$: $\prod_{j\in J} X_j \to I^J$ is essential.*

Proof. It follows from 1.2(b) that it suffices to prove the result in the case where J is finite. In such case the theorem is proved in [H_2] and [B–K]. ∎

3. Essentiality and the product representations

It is unknown whether or not essentiality of a mapping depends on the presentation of the target space as a product of manifolds. Here we show that in certain cases it does not depend.

3.1. THEOREM. *Let f: $X \to M_J$ and let h: $M_J \to N_K$ be a homeomorphism. In the following cases essentiality of f is equivalent to essentiality of $h \circ f$:*

(a) *M_J is a product of closed manifolds.*

(b) *J is a one-point set, i.e. M_J is a manifold.*

(c) *$M_J = \prod_{j\in J} M_j$, $N_K = \prod_{j\in J}\prod_{i\in I_j} M_{ji}$ (I_j a finite set) and $h = u \circ \prod_{j\in J} h_j$, where h_j: $M_j \to M'_j = \prod_{i\in I_j} M_{ji}$, $j \in J$, is a homeomorphism and u: $\prod_{j\in J} M'_j \to N_K$ is the identification map.*

(d) *M_J and N_K are products of cells.*

Proof in case (a) Recall that mapping into a product of closed manifolds is essential iff every mapping homotopic to it is surjective. Applying this to the identity mapping on M_J we infer by 2.1 that any mapping homotopic to it is surjective. Hence the same is true for the identity mapping on N_K since h is a homeomorphism. It follows that N_K is a product of *closed* manifolds. Now a similar reasoning applied to f and hf yields the conclusion.

Observe that case (b) is a particular case of (c).

Proof in case (c). It follows from 1.6 that f is essential iff $(\prod_{j\in J} h_j)\circ f$ is essential. Hence the proof reduces to the case where $M_j = M_j'$ for each $j\in J$, and $h = u$ is the specific "identity" mapping. The assertion in this case follows from the following two facts the proof of which are left to the reader:

(i) if $u\circ g$ is an admissible deformation of $u\circ f$ then g is an admissible deformation of f,

(ii) if g is a ∂-deformation of f then $u\circ g$ is an admissible deformation of $u\circ f$.

Proof in case (d). Follows from Theorem 4.3 which is proved in the next section.

4. Essentiality and universality of mappings

A mapping $f\colon X\to Y$ is said to be *universal* if for every mapping $g\colon X\to Y$ there is a point $x\in X$ such that $f(x) = g(x)$ [Hol$_2$].

The following is a well-known result (see [Hol$_1$] and also [G–T]).

4.1. Lemma. *A mapping $f\colon X\to I^n$ is essential if and only if it is universal.* ■

Since in this case neither universality nor essentiality depend on topological type of the target space (see 3.1 (b)) the same is true for mappings into arbitrary cells. Hence we have

4.2. Corollary. *Let $f, g\colon X\to Q$ be two mappings into a cell. Then f is not essential on the set*

$$\{x\in X\colon f(x)\neq g(x)\}. \ ■$$

Generalizing 4.1 we have

4.3. Theorem. *Let $f\colon X\to \prod_{j\in J} Q_j$ be a mapping into a product of cells. Then f is essential iff it is universal.*

Proof $\Rightarrow$. Let $f = (f_j)$ be essential and suppose it is not universal. Then there is a mapping $g = (g_j)\colon X\to \prod_{j\in J} Q_j$ such that $f(x)\neq g(x)$ for each $x\in X$. Then the sets

$$A_j = \{x\in X\colon f_j(x)\neq g_j(x)\},\qquad j\in J,$$

constitute a covering of X such that $f_j|A_j$ is not essential, for each $j\in J$, by 4.2. Then by 1.11 it follows that f is not essential, a contradiction.

$\Leftarrow$. Without loss of generality we may assume that each Q_j is the unit ball in the euclidean space $\boldsymbol{R}^{\dim Q_j}$ (see 3.1 (c)).

Assume f is universal and suppose f is not essential. Hence there is a ∂-deformation g of f such that $g^{-1}(0) = \emptyset$, where 0 is the point with all coordinates 0. Therefore

(1) $$g_j | f_j^{-1}(\partial Q_j) = f_j | f_j^{-1}(\partial Q_j), \quad j \in J,$$

and there exist open sets U_j in X such that $g_j^{-1}(0) \subset U_j$ and $\bigcap_{j \in J} U_j = \emptyset$. Let

$$h_j\colon X \to Q_j, \quad j \in J,$$

be an arbitrary extension of the mapping

$$X \setminus U_j \to \partial Q_j, \quad x \to \frac{g_j(x)}{|g_j(x)|}.$$

Note that

(2) $$h_j(X \setminus U_j) \subset \partial Q_j$$

and from (1) it follows that

(3) $$x \in (X \setminus U_j) \cap f_j^{-1}(\partial Q_j) \Rightarrow f_j(x) = h_j(x).$$

We claim that the mapping $(-h_j)$ has no coincidence point with f. For suppose $x \in X$ is such point. Choose $j \in J$ such that $x \in X \setminus U_j$. Then $f_j(x) = -h_j(x)$. On the other hand we infer from (2) and (3) that $f_j(x) = h_j(x)$ a contradiction. It follows that f is not universal, a contradiction. ∎

5. Dual families

In this section we define and discuss several dual families of subsets of a given space. We shall start off with the families of membranes and separators which induce other dual families according to the following abstract scheme.

A. An abstract approach. Let $\mathscr{A}$ be a family of subsets of an arbitrary set X. Put

$$\mathscr{A}^* = \{Y \subset X\colon X \setminus Y \notin \mathscr{A}\}.$$

Note that $\mathscr{A}^{**} = \mathscr{A}$ and therefore $\mathscr{A}$ and $\mathscr{A}^*$ are said to be *dual families* in X.

5.1. Proposition. *Every subset of X intersecting every member of $\mathscr{A}$ belongs to $\mathscr{A}^*$* ∎

The family $\mathscr{A}$ is said to be *saturated* provided the following is satisfied

$$A \in \mathscr{A} \ \& \ A \subset B \subset X \Rightarrow B \in \mathscr{A}.$$

5.2. PROPOSITION. *If $\mathcal{A}$ is saturated then $\mathcal{A}^*$ is saturated as well.* ∎

5.3. PROPOSITION. *If $\mathcal{A}$ is a saturated family in X then for any $Y \subset X$ we have*

$$(Y \in \mathcal{A}^*) \Leftrightarrow (Y \text{ meets each member of } \mathcal{A}). \blacksquare$$

Let $\mathcal{A}$ be a family in a *space* X. Then we define

$$\operatorname{near}\mathcal{A} = \{Y \subset X\colon \text{every open neighborhood of } Y \text{ in } X \text{ belongs to } \mathcal{A}\}$$

$$\operatorname{full}\mathcal{A} = \{Y \subset X\colon Y \text{ contains a closed set belonging to } \mathcal{A}\}.$$

5.4. PROPOSITION. *The families* $\operatorname{near}\mathcal{A}$ *and* $\operatorname{full}\mathcal{A}$ *are saturated in X.* ∎

5.5. PROPOSITION. *The families* $\operatorname{near}\mathcal{A}$ *and* $\operatorname{full}\mathcal{A}^*$ *are saturated dual families in X, i.e.* $\operatorname{full}\mathcal{A}^* = (\operatorname{near}\mathcal{A})^*$. ∎

B. Membranes and separators. Let $f\colon X \to M_J$ be a given mapping.

By a *membrane* of f we mean any subset Y of X such that $f|Y\colon Y \to M_J$ is essential. By a *separator* of f is meant any set $S \subset X$ such that $X \backslash S$ is not a membrane of X. In particular, the empty set is a separator of f iff f is not essential.

5.6. PROPOSITION. *Membranes and separators of the mapping f constitute saturated dual families in X.* ∎

From 1.10 it follows that

5.7. COROLLARY. *If A is a subset of X such that every neighborhood of A in X is a membrane of f then A is a membrane of f; equivalently: every separator of f contains a closed separator. In other words: any near membrane is a membrane and any separator is a full separator.* ∎

These results combined with 5.3 yield the following.

5.8. COROLLARY. *A subset of X is a membrane (separator) of f iff it meets each closed separator (membrane) of f.* ∎

Using the new language some other results from Section 1 can be reformulated as follows:

5.9. PROPOSITION. (a) *If g is an admissible deformation of f then every membrane of f (separator of g) is a membrane of g (separator of f)* (see 1.1 (b)).

(b) *If K is a subset of J then every membrane of f (separator of f_K) is a membrane of f_K (separator of f)* (see 1.1 (c)).

(c) *If $M_J = M$ is a manifold and $Y \subset X$ is a membrane (separator) of f then $Y \backslash \operatorname{int} f^{-1}(\partial M)$ is a membrane (separator) of f* (see 1.4).

(d) *If X is a union of two closed subsets A and B neither of which is a membrane of f then $A \cap B$ is a separator of f* (follows from 1.8). ∎

Theorem 1.11 implies the following

5.10. PROPOSITION. *If* $f = (f_{J_k})\colon X \to \prod_{k\in K} M_{J_k}$ *and* S_k *is a separator of* $f_{J_k}\colon X \to M_{J_k}$, *for each* $k \in K$, *then* $\bigcap_{k\in K} S_k$ *is a separator of* f. ∎

A membrane Y of the mapping $f\colon X \to M_J$ is said to be *irreducible* provided no proper closed subset of Y is a membrane of f. In other words: any nonvoid open subset of Y is a separator of $f|Y$. Applying the Kuratowski–Zorn lemma, 5.7 and 5.9 (d) we get

5.11. PROPOSITION. (a) *Any irreducible membrane of* f *is connected.*

(b) *If* X *is compact and* f *is essential then there is a closed irreducible membrane of* f. ∎

C. Full membranes and near separators. Let $f\colon X \to M_J$ be a given mapping.

A subset Y of X is said to be a *full membrane* of f provided it contains a closed membrane of f. A subset S of X is said to be a *near separator* of f provided every (open) neighborhood of S is a separator of f.

By 5.4, 5.5. and 5.6 we obtain

5.12. PROPOSITION. *Full membranes and near separators of* f *are saturated dual families in* X. ∎

5.13. PROPOSITION. *A subset of* X *is a full membrane (near separator) of* f *iff it meets each near separator (closed membrane) of* f. ∎

5.14. PROPOSITION. *Every full membrane (separator) of* f *is a membrane (near separator) of* f. ∎

The converse to 5.14 is not true.

5.15. EXAMPLE. Let $X = L \cup R$ be the standard ($\sin 1/x$)-curve, where L is the limit interval and R is the ray. Let $f\colon X \to I$ be the vertical projection. Then for any point $x \in L$ the set $\{x\} \cup R$ is a membrane which is not a full membrane of f (only X itself is a full membrane of f); $\{x\}$ is a near separator but not a separator of f. ∎

Let us prove the following.

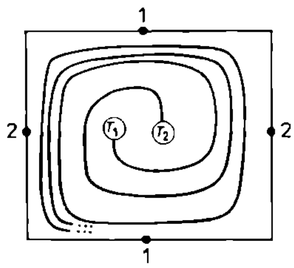
5.16. THEOREM. *If* $f = (f_{J_k})\colon X \to \prod_{k\in K} M_{J_k}$ *and* S_k *is a closed near separator of* f_{J_k}, $k \in K$, *then* $\bigcap_{k\in K} S_k$ *is a near separator of* f.

Proof. Let U be an open neighborhood of $\bigcap_{k\in K} S_k$ in X. It remains to show that U is a separator of f.

Notice that the sets $S_k \setminus U$, $k \in K$, are closed and have empty intersection. By the countable paracompactness of X there are open neighborhoods U_k of $S_k \setminus U$ in X, $k \in K$, with empty intersection. Each $U_k \cup U$ is an open neighborhood of S_k hence it is a separator of f_{J_k}. By 5.10 it follows that $\bigcap_{k\in K} (U_k \cup U) = U$ is a separator of f. ∎

The Theorem fails for non-closed near separators.

5.17. EXAMPLE. In the square $X = I^2$ consider the sets $S_1 = \{1/2\} \times \{0, 1\} \cup T_1$ and $S_2 = \{0, 1\} \times \{1/2\} \cup T_2$, where T_1 and T_2 are the spirals sketched in the figure below. Then S_j is a near separator of the projection p_j: $X \to I$, $j = 1, 2$. Since (p_1, p_2): $X \to I^2$ is essential (it is the identity mapping – hence essential by 2.1) the intersection $S_1 \cap S_2 = \emptyset$ is not a near separator of (p_1, p_2).



D. Near full membranes and full near separators. Let f: $X \to M_J$ be a given mapping.

A set $Y \subset X$ is said to be a *near full membrane* of f provided every neighborhood of Y in X contains a closed membrane. A set $S \subset X$ is said to be a *full near separator* of f provided it contains a closed near separator. By 5.4, 5.5 and 5.12 we obtain:

5.18. PROPOSITION. *Near full membranes and full near separators of f are saturated dual families in X.* ∎

5.19. PROPOSITION. *A subset of X is a near full membrane (full near separator) of f iff it meets each closed near separator (near full membrane) of f.* ∎

With the aid of 5.7 one can show that

5.20. PROPOSITION. (a) *Every full membrane (full near separator) of f is a near full membrane (near separator) of f.*

(b) *Every near full membrane (separator) of f is a membrane (full near separator) of f.* ∎

No converse of these implications is true.

5.21. EXAMPLE. (a) There is a near full membrane (near separator) which is not a full membrane (full near separator): such is the set S_1 for the mapping p_2 (the set S_1 for p_1) in Ex. 5.17.

(b) There is a membrane (full near separator) which is not a near full membrane (separator): such is the set $\{x\} \cup R$ (the set $\{x\}$) in Ex. 5.15. ∎

5.22. *Remark.* One can verify that no new dual families will be produced by further applications of the operators "near" and "full" to the families of membranes and separators.

6. Basic duality relation for essential mappings

The following useful result directly follows from the preceding discussion.

6.1. THEOREM. *Let* (f_J, f_K): $X \to M_J \times M_K$ *be an essential mapping. Then every near separator of* f_J *is a near full membrane of* f_K.

Proof. By 5.7 it suffices to show that every separator S of f_J is a membrane of f_K. Suppose not. Then $X \setminus S$ is a separator of f_K. By 5.10 the set $S \cap (X \setminus S) = \emptyset$ is a separator of (f_J, f_K), a contradiction. ∎

7. Dimension and essential maps

Let A and B be disjoint closed subsets of a space X. By a *partition* of X between A and B we mean a closed subset F of X which separates X between these sets, i.e. there exist two disjoint open subsets U and V of X such that $X \setminus F = U \cup V$, $A \subset U$ and $B \subset V$.

7.1. LEMMA. *Let* f: $X \to I^{n+1}$ *be a mapping and let* X_0 *be a closed subset of* X. *Let* A *be a subset of* X *with* $\dim A \leqslant n$ *and let* t *be an interior point of* I^{n+1}. *Then there exists a* ∂*-deformation* g *of* f *such that* $g|X_0 = f|X_0$ *and* $g^{-1}(t) \cap (A \setminus X_0) = \emptyset$.

Proof. We have $A = A_1 \cup \ldots \cup A_{n+1}$, where $\dim A_j \leqslant 0$ for each $j = 1, \ldots, n+1$ (see [E, p. 259]). Let $t = (t_1, \ldots, t_{n+1})$ and let $f = (f_1, \ldots, f_{n+1})$, where f_j: $X \to I$, $j = 1, \ldots, n+1$. After this presentation it is clear that it suffices to establish our lemma for the case $n = 0$.

So, let f: $X \to I$, let $t \in \mathring{I}$ and let $\dim A \leqslant 0$. It follows from the separation theorem [E, Th. 4.1.13] that there is a partition L of X between $f^{-1}(0)$ and $f^{-1}(1)$ such that $L \cap X_0 = (f|X_0)^{-1}(t)$ and $L \cap (A \setminus X_0) = \emptyset$. Now it is not difficult to construct a ∂-deformation g of f such that $g|X_0 = f|X_0$ and $g^{-1}(t) = L$. Such a mapping g satisfies the conclusion. ∎

A family (A_j, B_j), $j \in J$, is said to be *essential* in X if each (A_j, B_j) is a pair of closed disjoint sets in X such that $\bigcap_{j \in J} P_j \neq \emptyset$ for every choice of partitions P_j of X between A_j and B_j. It is well-known that $\dim X \geqslant n$ iff there is an essential family in X consisting of n elements. Natural infinite-dimensional counterparts of finite dimensional spaces are defined as follows. A space X is said to be *strongly infinite-dimensional*, SID, provided there is an infinite essential family in X. If X is not SID then we call it *weakly infinite-dimensional*, WID. One easily shows that a mapping f: $X \to I^J$ is essential iff the family $(f_j^{-1}(0), f_j^{-1}(1))$, $j \in J$, is essential in X. So, in terms of mappings, we have (see for instance [Hol_1]):

7.2. THEOREM. *For any space* X *we have* $\dim X \geqslant n$ *iff* X *admits an essential mapping onto the n-cube* I^n. ∎

7.3. THEOREM. [A–P, p. 531]. *A space X is* SID *iff X admits an essential map onto the Hilbert cube I^{∞}.* ∎

7.4. THEOREM. *Let $f\colon X \to M_J$ be an essential map.*

(i) *If J is finite then* $\dim X \geqslant \dim M_J$.

(ii) *If J is infinite then X is* SID.

Proof. For each $j \in J$ let $Q_j \subset M_j$ be a closed cell with $\dim Q_j = \dim M_j$. Let $n = \sum_j \dim M_j$ ($n = \infty$ if J is infinite). Denote $Q_J = \prod_{j \in J} Q_j$ and $X_0 = f^{-1}(Q_J)$. By 1.9 the mapping $X_0 \to Q_J$ determined by f is essential. Since Q_J is homeomorphic to I^n we infer by 3.1 (d) that there exists an essential mapping $X_0 \to I^n$. Thus the conclusion follows from 7.2 and 7.3 (in case (ii) we use the fact that X_0 is a closed subset of X). ∎

Chapter II

APPLICATIONS

1. On a lemma of Borsuk

A classical theorem of Borsuk states that a compact subset A of $\boldsymbol{R}^{n+1}$ separates this space if and only if A admits an essential mapping into the sphere S^n. As an illustration of the ideas discussed in Chapter I we present a short proof of the necessity part of the theorem.

So, assume A is a compact subset of $\boldsymbol{R}^{n+1}$ separating this space. We shall show that A admits an essential mapping into S^n. Without loss of generality we may assume that A separates $\boldsymbol{R}^{n+1}$ between 0 and ∞ and

$$A \subset X = \{x \in \boldsymbol{R}^{n+1}\colon\ 1 \leqslant |x| \leqslant 2\},$$

where $|\cdot|$ is the standard norm on $\boldsymbol{R}^{n+1}$. For $x \in X$ let $f(x) = x/|x|$ and let $g(x) = |x|-1$. Then the mapping

$$(f, g)\colon\ X \to S^n \times I$$

is essential being a homeomorphism, see I.2.1. We are going to apply I.5.8. So, consider a closed separator S of $f\colon X \to S^n$. By I.6.1 it follows that S is a compact membrane of $g\colon X \to I$. Then I.2.2 implies that S contains a subcontinuum joining $g^{-1}(0)$ and $g^{-1}(1)$. But our set A meets each such a continuum. Hence A meets S. By I.5.8 we conclude that $f|A\colon A \to S^n$ is essential, which completes the argument.

The same technique yields much stronger results.

1.1. THEOREM. *Let $(f_J, f_K)\colon X \to M_J \times M_K$ be an essential mapping. Let Y be a subset of X which meets each closed membrane of $f_K\colon X \to M_K$. Then Y is a membrane of $f_J\colon X \to M_J$. Moreover, every neighborhood of Y in X contains a closed membrane of f_J.*

1.2. *Remark.* If X is compact it suffices to assume that Y meets each compact connected membrane of f_K, see I.5.11.

Proof. By I.5.20 it suffices to prove the second conclusion. By I.6.1 every closed near separator of f_J is a closed membrane of f_K. Thus Y meets each closed near separator of f_J. The conclusion follows from I.5.19. ∎

Setting $(f_J, f_K) = \mathrm{id}$ and referring to I.2.1 we get the following

1.3. COROLLARY. *Let Y be a subset of $M_J \times M_K$ intersecting each compact connected membrane of the projection p_K: $M_J \times M_K \to M_K$. Then Y is a membrane of the second projection p_J: $M_J \times M_K \to M_J$. Moreover, every neighborhood of Y in $M_J \times M_K$ contains a closed membrane of p_J.* ∎

Applying I.2.2 we get also

1.4. COROLLARY. *Let (f, g): $X \to M_J \times I$ be an essential mapping, where X is compact. If $Y \subset X$ meets each subcontinuum of X joining $g^{-1}(0)$ and $g^{-1}(1)$ then Y is a membrane of f: $X \to M_J$. Moreover, every neighborhood of Y in X contains a closed membrane of f.* ∎

1.5. COROLLARY. *Let Y be a subset of $M_J \times I$ intersecting each subcontinuum of $M_J \times I$ joining $M_J \times (0)$ and $M_J \times (1)$. Then Y is a membrane of the projection p: $M \times I \to M_J$. Moreover, every neighborhood of Y in $M_J \times I$ contains a closed membrane of p.* ∎

Restricting the class of metrizable spaces X we can choose the sets Y to be highly disconnected subspaces of X. This is done in the following section.

2. On totally disconnected spaces

All spaces in this section are assumed to be metrizable and separable

A space Y is said to be *totally disconnected* provided its every quasi-component is degenerate, i.e. Y is not connected between any two points. It is known that Y is totally disconnected if and only if it admits an injective mapping into the Cantor set [K, p. 148]. This very convenient characterization enables us to construct totally disconnected spaces with strong homotopy properties, as in the preceding section. In particular, with large dimension. Examples with the latter property are well known (see [M], [L], [R–S–W], [T_2]). What we do is essentially a repetition of the same trick (see the Introduction) which was used in all these papers. It is also used in [P]. As a result of our theory we are able to prove their additional properties.

To construct such spaces which are simultaneously topologically complete we make use of the following known result: if f: $X \to Z$ is a mapping from a compactum into a 0-dimensional space then there exists a G_δ-selector Y of f, i.e. Y is a G_δ-subset of X, $f(Y) = f(X)$ and $f|Y$: $Y \to Z$ is injective [M].

2.1. *The construction.* Consider an essential mapping (f_J, f_K): $X \to M_J$

$\times M_K$. Applying the results of the preceding section we construct a few examples of singular membranes of f_J: $X \to M_J$. Note that the construction is applicable to the particular case where $X = M_J \times M_K$ and f_J, f_K are the projections.

Let $\mathscr{A}$ be the family of all closed membranes of f_K: $X \to M_K$. Then card $\mathscr{A} \leqslant \mathfrak{c}$ since X is separable. Let Z be a subset of M_K with card $Z = \mathfrak{c}$. Then there is a surjection α: $Z \to \mathscr{A}$. Define

$$X_0 = \bigcup_{z \in Z} \alpha(z) \cap f_K^{-1}(z).$$

Note that each summand is nonvoid since each essential mapping is surjective. Therefore $f_K(X_0) = Z$. Consider a selector $Y \subset X_0$ of the mapping $X_0 \to Z$ determined by f_K. Then Y meets each closed membrane of f_K. Thus by 1.1 we have

(1) Y is a near full membrane of f_J: $X \to M_J$.

Under some extra assumptions about Z and α we can choose Y with additional properties. Namely,

Case (a). *Z is 0-dimensional.* Then Y is totally disconnected since Z embeds into the Cantor set. For the case where f_J and f_K are the projections we have also $\dim Y = \dim M_J$ since $Y \subset M_J \times Z$.

Case (b). *Z contains no Cantor set.* Then the same is true for Y since $Y \xrightarrow{f_K} Z$ is injective. Thus every compact subset of Y is countable.

Case (c). *X and Z are compact and Z is 0-dimensional.* Then we may assume that α: $Z \to \mathscr{A}$ is a continuous surjection ($\mathscr{A}$ with the Vietoris topology) since $\mathscr{A}$ is a compact subset of 2^X in this case. According to the introductory remark we can choose Y to be a G_δ-subset of X.

Here is another method of constructing totally disconnected spaces based on a theorem of R. H. Bing.

2.2. THEOREM [B_1]. *Let A and B be two closed disjoint subsets of a continuum X. Then there exists a partition P of X between A and B such that every continuum in X joining A and B contains a component of P.* ∎

A compact subset of X with the latter property will be called a *Bing set* between A and B. There exist Bing sets such that each of their components is hereditarily indecomposable.

2.3. THEOREM. *Let P be a Bing set in $M_J \times I$ between $M_J \times (0)$ and $M_J \times (1)$. Then every selector Y of components of P is a near full membrane of the projection $M_J \times I \to M_J$. Moreover, Y is totally disconnected,* $\dim Y = \dim M_J$ *and it can be chosen to be a G_δ-subset of X.*

Proof. The first conclusion follows from 2.2 and 1.5. Since P is compact the space P^* of components of P is 0-dimensional. The remaining conclusion follows from the fact that the quotient map $P \to P^*$ is continuous. ∎

We close this section by stating two results from [Kr] which give yet another method of producing totally disconnected spaces with large dimension.

Let C denote a Cantor set in I and let $\dim X \geqslant \infty$ mean that X is SID.

2.4. THEOREM. *Let X be a space with $\dim X \geqslant n+1$, $n = 1, 2, \ldots, \infty$. Then there exists a mapping $f\colon X \to I$ and a closed membrane A of f such that*

$$(*) \qquad Y \subset A \,\&\, f(Y) \supset C \Rightarrow \dim Y \geqslant n. \ \blacksquare$$

2.5. COROLLARY. *Let $p\colon I^{n+1} \to I$ be the projection onto the first factor. Then there exists a continuum $A \subset I^{n+1}$ such that*

(i) $p(A) = I$

(ii) $\dim A = n$

(iii) $Y \subset A \,\&\, p(Y) \supset C \Rightarrow \dim Y = n.$ ■

3. On a theorem of Mazurkiewicz

In this section we generalize the following well-known theorem of Mazurkiewicz: If A is a subset of $\boldsymbol{R}^n$ with dimension $\leqslant n-2$ then $\boldsymbol{R}^n \backslash A$ is a semicontinuum (see [E, Th. 1.8.19]).

Some other related generalizations are to be found in [H_1], [H_3] and [T_1].

Let M be a manifold in a space X. Then M is said to be *k-flat* in X if there exists an embedding $h\colon M \times I^k \to X$ such that $M = h(M \times (t))$ for some interior point t of I^k.

3.1. THEOREM. *Let A be a subset of a space X with $\dim A \leqslant k$ $(\geqslant 0)$. Let M be a $(k+1)$-flat manifold in X and let $B \subset M \backslash A$ be a compact set. Then for each $\varepsilon > 0$ there exist a continuum $Y \subset X \backslash A$ and a mapping $f\colon Y \to M$ such that*

(i) *Y is an irreducible membrane of f,*

(ii) *$f^{-1}(B) = B$ and $f(x) = x$ for each $x \in B$,*

(iii) *$\varrho(x, f(x)) < \varepsilon$ for each $x \in Y$.*

3.2. *Remark*. Let A be a subset of $\boldsymbol{R}^n$ with $\dim A \leqslant k$, $k \leqslant n-2$. Let $B = \{x, y\} \subset \boldsymbol{R}^n \backslash A$. Consider any $(k+1)$-flat manifold $M \subset \boldsymbol{R}^n$ containing B (any geometric sphere or convex cell with dimension $\leqslant n-k-1$ containing B is such a manifold). Then our Theorem says that there exist a continuum $Y \subset \boldsymbol{R}^n \backslash A$ joining x and y, and a mapping $f\colon Y \to M$ which satisfy conditions (i)–(iii).

Therefore we get the following generalization of the Mazurkiewicz's theorem: for each pair $x, y \in \boldsymbol{R}^n \backslash A$ and each $\varepsilon > 0$ there exist a continuum $Y \subset \boldsymbol{R}^n \backslash A$ joining x and y and an ε-mapping $f\colon Y \to S^{n-k-1}$ such that Y is an irreducible membrane of f. ■

Theorem 3.1 directly follows from the following

3.3. LEMMA. *Let A be a subset of $M \times I^{k+1}$ with* $\dim A \leqslant k$, $k \geqslant 0$. *Let t be an interior point of I^{k+1} and let B be a closed subset of M such that $B \times (t) \subset M \times (t) \backslash A$. Then there exists a continuum $Y \subset M \times I^{k+1} \backslash A$ such that*

(i) *Y is an irreducible membrane of the projection $p\colon M \times I^{k+1} \to M$,*

(ii) $Y \cap p^{-1}(B) = B \times (t)$.

Proof. Let $q\colon M \times I^{k+1} \to I^{k+1}$ be the other projection. By I.7.1 there is a ∂-deformation $\bar{q}$ of q such that $\bar{q}|p^{-1}(B) = q|p^{-1}(B)$ and $\bar{q}^{-1}(t) \cap (A \backslash p^{-1}(B)) = \emptyset$. Then $\bar{q}^{-1}(t)$ is a separator of q, and it follows that it is a compact membrane of p. Hence there is a closed irreducible membrane $Y \subset \bar{q}^{-1}(t)$ of p. One easily verifies that Y satisfies the conclusion. ∎

4. On coincidence points

Let $f, g\colon X \to Y$ be two mappings. A point $x \in X$ such that $f(x) = g(x)$ is called a *coincidence point* of the mappings.

The following is a direct corollary to I.4.3.

4.1. LEMMA. *Let $f\colon X \to Q$ be a mapping into a cube (the Hilbert cube included). Then for any mapping $g\colon X \to Q$ the set of coincidence points of f and g is a closed separator of f.* ∎

The main observation on coincidence points is given by the following

4.2. THEOREM. *Let $(f_M, f_Q)\colon X \to M_J \times Q$ be an essential mapping, where Q is a cube (the Hilbert cube included). Then for any mapping $g\colon X \to Q$ the set*

$$\{x \in X\colon g(x) = f_Q(x)\}$$

is a closed membrane of f_M.

Proof. Directly follows from 4.1 together with I.6.1. ∎

By the Theorem and the fact that the identity mapping on any product of manifolds is essential (see I.2.1) we get

4.3. COROLLARY. *Let $g\colon M_J \times Q \to Q$ be a mapping, where Q is a cube (the Hilbert cube included). Then*

$$\{x \in M_J \times Q\colon g(x) = p_Q(x)\}$$

is a closed membrane of p_M, where p_M and p_Q denote respective projections of $M_J \times Q$ onto the factors. ∎

4.4. COROLLARY. *Let $h_t\colon Q \to Q$, $t \in I$, be a homotopy, where Q is a cube (the Hilbert cube indeed). Then the set*

$$A = \{(t, x) \in I \times Q\colon h_t(x) = x\}$$

contains a continuum joining the faces $(0) \times Q$ and $(1) \times Q$ of $I \times Q$.

Proof. We present two proofs. (I) Let $h\colon I \times Q \to Q$ be given by $h(t, x)$

$= h_t(x)$. By 4.3 the set A is a compact membrane of the projection $I \times Q \to I$. The conclusion follows from I.2.2

(II) (a direct argument given by R. Pol). Suppose the conclusion fails. Then there is a mapping $g\colon I \times Q \to I$ such that $g((0) \times Q) = (1)$, $g((1) \times Q) = (0)$ and $g(A) \subset \partial I$. Define a mapping $f\colon I \times Q \to I \times Q$ by the formula

$$f(t, x) = (g(t, x), h_t(x)).$$

Note that there is no fixed point for f, a contradiction.

The second argument works equally well for any compact space Q such that $Q \times I$ has the fixed point property. ∎

5. Constructing special subsets in compacta which essentially map onto infinite products

The main result of this section is Theorem 5.5. Theorem 5.1 which opens this section is weaker than Theorem 5.5, but its proof is so simple that we have decided to present it independently.

The basic ideas generalize those used in the papers: [He], [B_2], [R–S–W] and [W].

5.1. THEOREM. *Let X be a compactum and let $f\colon X \to \prod_{j \in J} M_j$ be essential. Then for every infinite sequence $J_0, J_1, \ldots$ of nonvoid disjoint subsets of J such that $J = J_0 \cup J_1, \ldots$, there exists a continuum $Y \subset X$ such that*

(i) *Y is a membrane of f_{J_0},*

(ii) *each nondegenerate subcontinuum of Y is a membrane for one of the mappings f_{J_n}, $n \geqslant 1$.*

5.2. *Remark*. We shall construct Y satisfying the following condition a bit stronger than (ii):

(ii)′ for every $\varepsilon > 0$ there exists $n(\varepsilon) \geqslant 1$ such that each nondegenerate subcontinuum of Y whose diameter is $\geqslant \varepsilon$ is a membrane for one of the mappings $f_{J_1}, \ldots, f_{J_{n(\varepsilon)}}$.

Furthermore, it will be clear from the proof that one can prove more: every nondegenerate subcontinuum of Y is a membrane for infinitely many mappings f_{J_n}.

Letting $J_0, J_1, \ldots$ to be infinite sets one obtains a hereditarily (with respect to closed subsets) strongly infinite-dimensional continuum. The first example with such a property was constructed by D. Henderson [He]. Other examples were given in [B_2], [R–S–W] and [Z]. ∎

5.3. LEMMA. *Let X be a compactum and let $f\colon X \to M_J$. If A and B are disjoint closed subsets of X, then there exists a separator Y of f such that each subcontinuum of Y joining A and B is a membrane of f.*

Proof. (comp. [R–S–W, 6.1]). Let $G_1, G_2, \ldots, G_j \subset X\setminus(A \cup B)$, be a sequence of open disjoint subsets of X such that each G_n separates X between A and B. Let $Z_1, Z_2, \ldots$ be a sequence of closed separators of f dense in the space of all closed separators of f (considered as a subspace of 2^X). Put

$$Y = (X\setminus\bigcup_n G_n) \cup \bigcup_n (G_n \cap Z_n).$$

It follows from I.1.8 and I.5.8 that Y is a desired set. ∎

Proof of Theorem 5.1. It is easy to construct a sequence of pairs $(A_1, B_1), (A_2, B_2), \ldots$ of disjoint closed subsets of X such that

(1) for every $\varepsilon > 0$ there exists $n(\varepsilon) \geqslant 1$ such that every subcontinuum of X whose diameter is $\geqslant \varepsilon$ intersects both sets A_i and B_i for some $1 \leqslant i \leqslant n(\varepsilon)$.

By Lemma 5.3 there exists a closed separator Y_1 of $f_{J_1}: X \to \prod_{j\in J_1} M_j$ such that each subcontinuum of Y_1 joining A_1 and B_1 is a membrane of f_{J_1}. By I.6.1 the compactum Y_1 is a membrane of $f_{J\setminus J_1}$. Hence $f_{J\setminus J_1}|Y_1: Y_1 \to \prod_{j\in J\setminus J_1} M_j$ is essential.

Repeating the same argument, taking $f_{J\setminus J_1}|Y_1, J_2$ and $(A_2 \cap Y_1, B_2 \cap Y_1)$ in place of f, J_1 and (A_1, B_1), respectively, one infers that there exists a compactum $Y_2 \subset Y_1$ such that Y_2 is a membrane of $f_{J\setminus(J_1\cup J_2)}$ and each subcontinuum of Y_2 joining A_2 and B_2 is a membrane of f_{J_2}.

Repeating this procedure one constructs a decreasing sequence $Y_1 \supset Y_2 \supset \ldots$ of compacta in X such that for each $n \geqslant 1$ we have

(2) Y_n is a membrane of $f_{J\setminus(J_1\cup\ldots\cup J_n)}$,

(3) each subcontinuum of Y_n joining A_n and B_n is a membrane of f_{J_n}.

From (2) it follows that Y_n is a membrane of f_{J_0}. By I.5.7 the intersection $\bigcap_n Y_n$ is also a membrane of f_{J_0}. Define Y to be an irreducible membrane of f_{J_0} contained in $\bigcap_n Y_n$. Hence Y satisfies (i) and by I.5.11 it is a continuum. From (1) and (3) it follows that Y satisfies the condition (ii)′ as well. ∎

5.4. COROLLARY. *There exists a nondegenerate continuum Y such that each nondegenerate subcontinuum of Y admits an essential mapping onto every countable product of manifolds.*

Proof. There exists a countable set $\mathcal{M}$ of manifolds such that every manifold is homeomorphic to an element of $\mathcal{M}$ [C–K]. Let $J_0, J_1 \ldots$ be mutually disjoint countable infinite sets and let $J = J_0 \cup J_1 \cup \ldots$ Let $J \ni j \to M_j \in \mathcal{M}$ be an indexing of $\mathcal{M}$ such that every element of $\mathcal{M}$ is indexed by infinitely many elements of each set J_n. Let $X = \prod_{j\in J} M_j$ and let $f: X \to \prod_{j\in J} M_j$ be the identity. By I.2.1 it follows that f is essential. Now, let $Y \subset X$ be the continuum given by Theorem 5.1. Using I.5.9 (b) one easily checks that Y satisfies the desired conditions. ∎

5.5. THEOREM. *Let $f\colon X \to \prod_{j\in J} M_j$ be an essential mapping, where X is a compactum and J is infinite. Then for every decomposition $J = J_0 \cup J_1 \cup \ldots$ into an infinite collection of nonvoid mutually disjoint sets there exists a continuum $Y \subset X$ such that*

(i) *Y is a membrane of f_{J_0},*

(ii) *every subset of Y with dimension > 0 is a near full membrane for one of the mappings f_{J_n}, $n \geqslant 1$.*

5.6. *Remarks.* (a) One can modify the proof so that one gets more: every subset of Y with dimension > 0 is a near full membrane for infinitely many f_{J_n}'s.

(b) Letting $J_0, J_1, \ldots$ to be infinite sets one obtains by I.7.4 a hereditarily (with respect to arbitrary subsets) strongly infinite dimensional subcontinuum Y of X. It follows from I.7.3 that every strongly infinite dimensional compactum contains such a subcontinuum. For more general results in this direction the reader is referred to the papers by L. R. Rubin [R] and R. Pol [P]. ∎

The proof is preceded by two lemmas. The first one is used only in the proof of the second, which is the crucial point in the argument. The proof uses some ideas of J. J. Walsh [W].

5.7. LEMMA. *Let X be a compactum, let $f\colon X \to M_J$, let $g\colon X \to I$ and let C be a Cantor set in I. Then there exists a separator Y of f such that for each continuum $D \subset Y$ joining two distinct fibers $g^{-1}(t)$, $t \in C$, there exists $t_0 \in C$ such that $D \cap g^{-1}(t_0)$ is a membrane of f.*

Proof. Let $(a_1, b_1), (a_2, b_2), \ldots, a_i < b_i$, be the sequence (without repetitions) of all open intervals in $I \setminus C$ whose endpoints belong to C. Let $A_i = g^{-1}([0, a_i])$ and $B_i = g^{-1}([b_i, 1])$, $i \geqslant 1$. For each $i \geqslant 1$ let $G_{i1}, G_{i2}, \ldots, G_{ij} \subset X \setminus (A_i \cup B_i)$, be a sequence of disjoint open subsets of X such that each G_{ij} separates X between A_i and B_i and each neighborhood of $g^{-1}(a_i)$ contains almost all elements of this sequences.

Let $\mathcal{S}$ be a countable set of closed separators of f dense in the space of all closed separators of f.

Let us arrange the elements of $\mathcal{S}$ in a sequence $S_1, S_2, \ldots$ in such a way that each element of $\mathcal{S}$ occurs in the sequence infinitely many times.

Now define: $Y = (X \setminus \bigcup_{i,j} G_{ij}) \cup \bigcup_{i,j} (G_{ij} \cap S_j)$. One easily sees that Y is a separator of f. Consider a continuum $D \subset Y$ joining two distinct fibers $g^{-1}(t)$, $t \in C$. Then $a_i, b_i \in g(D)$ for some $i \geqslant 1$. It follows from the construction that $D \cap g^{-1}(a_i)$ intersects each element $S \in \mathcal{S}$ since D intersects each set $G_{ij} \cap S_j$, $j \geqslant 1$. By I.5.8 this completes the proof. ∎

5.8. LEMMA. *Let X be a compactum, let $(f_{J_1}, f_{J_2})\colon X \to M_{J_1} \times M_{J_2}$, let $g\colon X \to I$ and let C be a Cantor set in I. Then there exists a closed near separator Y of (f_{J_1}, f_{J_2}) such that every subset of Y intersecting each fibre $g^{-1}(t)$, $t \in C$, is a near full membrane of either f_{J_1} or f_{J_2}.*

Proof. Let $f = (f_{J_1}, f_{J_2})$. Let $\mathscr{S}_i$ denote the space of all closed near separators of f_{J_i}, $i = 1, 2$. Since each $\mathscr{S}_i$ is a compactum, there is a continuous surjection $(S_1, S_2)\colon C \to \mathscr{S}_1 \times \mathscr{S}_2$. We may assume that it is surjective on the irrational part of C (in fact, it suffices to take a surjection with uncountable point-inverses).

Keeping the notation from the proof of Lemma 5.7 define

$$Y = \bigcup_{t \in C} g^{-1}(t) \cap (S_1(t) \cup S_2(t)) \cup \bigcup_{i \geqslant 1} g^{-1}([a_i, b_i]) \cap S_2(a_i) \cup g^{-1}(I \setminus (\bar{0}, \bar{1})),$$

where $(\bar{0}, \bar{1})$ is the open interval with end points $\bar{0} = \inf C$ and $\bar{1} = \sup C$. Then Y is a closed subset of X. It remains to show that Y satisfies the conclusion of the lemma.

First we shall show that Y is a near separator of f. So, let E be a closed membrane of f. We must prove that (see I.5.13)

$$E \cap Y \neq \emptyset. \tag{1}$$

To this end consider a closed separator Y_1 of f_{J_1} satisfying the conclusion of Lemma 5.7. Hence we have

(2) for every continuum $D \subset Y_1$ joining two distinct fibers $g^{-1}(t)$, $t \in C$, there is $t_0 \in C$ such that $D \cap g^{-1}(t_0)$ is a membrane of f_{J_1}.

The set $E \cap Y_1$ is a membrane od f_{J_2}, being a separator of $f_{J_1}|E$ (see I.6.1). Hence there is a membrane $D \subset E \cap Y_1$ of f_{J_2} which is continuum. The proof of (1) will be completed once we prove that

$$D \cap Y \neq \emptyset. \tag{3}$$

The proof of (3) is divided into four special cases:

(a) $g(D) \not\subset (\bar{0}, \bar{1})$. Then $D \cap Y \supset D \cap g^{-1}(I \setminus (\bar{0}, \bar{1})) \neq \emptyset$,

(b) $g(D) \subset [a_i, b_i]$ *for some* $i \geqslant 1$. Then $D \cap Y \supset D \cap S_2(a_i)$ and the latter set is nonvoid because D is a closed membrane of f_{J_2}.

(c) $g(D) = (t)$ *for some* $t \in C$. Then $D \cap Y \supset D \cap S_2(t) \neq \emptyset$ for the same reason as in (b).

(d) $g(D) \cap C$ *contains at least two points*. By (2) there is $t_0 \in C$ such that $D \cap g^{-1}(t_0)$ is a membrane of f_{J_1}. Then we have $D \cap Y \supset D \cap g^{-1}(t_0) \cap S_1(t_0) \neq \emptyset$.

These cases exhaust all the possibilities, hence (3) is satisfied. This completes the proof that Y is a near separator of f.

Now, consider a set $A \subset Y$ intersecting each fibre $g^{-1}(t)$, $t \in C$. We shall show that A is a near full membrane of either f_{J_1} or f_{J_2}. Suppose it is not true. Then there is a closed near separator S_i' of f_{J_i} such that $A \cap S_i' = \emptyset$, $i = 1, 2$, see I.5.19. Let t be an irrational point of C such that $(S_1(t), S_2(t)) = (S_1', S_2')$. Since $Y \cap g^{-1}(t) \subset S_1(t) \cup S_2(t)$ we infer that $A \cap g^{-1}(t) = \emptyset$, a contradiction completing the proof. ∎

Proof of Theorem 5.5. Let $g_k\colon X \to I$, $k = 1, 2, \ldots$, be a sequence of mappings separating points of X (to the effect that the diagonal map $g = (g_k)\colon X \to I \times I \times \ldots$ is an embedding; it suffices to take a sequence dense in the space of all mappings from X to I). Let $h_l\colon C \to I$, $l = 1, 2, \ldots$, be a sequence of embeddings of the Cantor set C into I dense in the space of all such embeddings. Let us arrange all the pairs (g_k, h_l) in a sequence $\alpha_1, \alpha_2, \ldots$. We shall be applying Lemma 5.8 to the mappings $(f_{J_{2n-1}}, f_{J_{2n}})\colon X \to \prod_{j \in J_{2n-1}} M_j \times \prod_{j \in J_{2n}} M_j$, $n \geqslant 1$. It follows from this lemma that for each $n \geqslant 1$ there is Y_n such that

(1) Y_n is a closed near separator of $(f_{J_{2n-1}}, f_{J_{2n}})$,

(2) if $\alpha_n = (g_k, h_l)$, then every set $A \subset Y_n$ such that $h_l(C) \subset g_k(A)$ is a near full membrane of either $f_{J_{2n-1}}$ or $f_{J_{2n}}$.

By (1) and I.5.16 the set $\bigcap_n Y_n$ is a near separator of $f_{J \setminus J_0}$. Hence by I.6.1 it is a membrane of f_{J_0}. Define Y to be a compact connected membrane of f_{J_0} contained in $\bigcap_n Y_n$.

Consider a set $A \subset Y$ with $\dim A > 0$. To complete the proof it suffices to show that A is a near full membrane for one of the mappings f_{J_n}, $n \geqslant 1$. Let us note that

(3) $g_k(A)$ is not 0-dimensional for some $k \geqslant 1$.

In fact, otherwise $g(A) \subset \prod_k g_k(A)$ and the latter set is 0-dimensional; hence A is such since g is an embedding.

It follows from (3) that $h_l(C) \subset g_k(A)$ for some $l \geqslant 1$. Then $(g_k, h_l) = \alpha_n$ for some $n \geqslant 1$. Since $A \subset Y \subset Y_n$ it follows from (2) that A satisfies the desired condition. ∎

Theorem 5.5 implies the following corollary. Its proof is similar to that of Cor. 5.4 and is omitted.

5.9. Corollary. *There exists a nondegenerate continuum Y such that every subset of Y with dimension > 0 admits an essential mapping onto every countable product of manifolds.* ∎

6. On infinite-dimensional Cantor manifolds

A compact space X is said to be an *infinite-dimensional Cantor manifold* if it contains at least two points and no WID space separates X (see [S]). We abbreviate this term to ID Cantor manifold. Each such a manifold is a continuum.

The purpose of this section is: (a) to establish criteria for detecting ID Cantor manifolds, (b) to prove a generalization of a result from [S] on the existence of ID Cantor manifolds, and (c) to prove some theorems on countable collections of subsets in SID compacta.

In particular, we obtain an affirmative solution to the following problem posed by R. Pol in a conversation with the author: given a decomposition $Q = X_1 \cup X_2 \cup \ldots$, where Q is the Hilbert cube and X_i's are arbitrary subsets of Q, must one of the X_i's contain a nondegenerate connected set?

A. Detecting ID Cantor manifolds.

6.1. THEOREM. *Let X be a compactum and let $f\colon X \to M_J$, where J is an infinite set. If Y is a closed irreducible membrane of f, then Y is an* ID *Cantor manifold.*

Proof. Suppose a WID set separates Y. It follows that there exist two closed proper subsets A and B of Y such that $Y = A \cup B$ and $A \cap B$ is WID. From the assumption it follows that neither A nor B is a membrane of f. By 1.1.2 there is a finite set $K \subset J$ such that neither A nor B is a membrane of f_K. By 1.5.9 (d) and 1.6.1 we infer that $A \cap B$ is a membrane of $f_{J\setminus K}$. Since $J \setminus K$ is infinite, 1.7.4 implies that $A \cap B$ is SID, a contradiction. ∎

The above theorem combined with 1.7.3 and 1.5.11 (b) implies the following

6.2. THEOREM [S]. *Every* SID *compactum contains an* ID *Cantor manifold.* ∎

B. A generalization of Skljarenko's theorem. A space Z is said to be a *generalized infinite-dimensional Cantor manifold* if it contains at least two points and no weakly infinite-dimensional subset of Z separates Z. We abbreviate this term to GID Cantor manifold.

The lack of compactness in the present definition is the only difference between this notion and the notion of an ID Cantor manifold.

Every GID Cantor manifold is a SID space.

6.3. THEOREM. *Let X be a* SID *compactum. Then every subset of X intersecting each* ID *Cantor manifold contained in X contains a* GID *Cantor manifold.*

We shall prove two lemmas. The theorem is a direct consequence of 6.2 and the second lemma containing some extra information.

6.4. LEMMA. *Let X be a* SID *compactum. Then every subset Y of X intersecting each* ID *Cantor manifold contained in X is* SID.

Proof. By I.7.3 there exists an essential mapping form X to the Hilbert cube $\prod_{j\in N} I_j$. Let J and K be two disjoint infinite subsets of N such that $N = J \cup K$. Let us present f in the form (f_J, f_K): $X \to \prod_{j\in J} I_j \times \prod_{j\in K} I_j$. Now suppose Y is WID. Then by I.5.8 and I.7.4 there exists a closed separator S of f_J disjoint from Y. By I.6.1 the space S is a membrane of f_K. By I.7.4 we infer that S is SID. By 6.2 the set S contains an ID Cantor manifold C; then $C \cap Y = \emptyset$, a contradiction. ∎

6.5. LEMMA. *Let X be an* ID *Cantor manifold. Then every subset Y of X intersecting each* ID *Cantor manifold contained in X is a* GID *Cantor manifold dense in X.*

Proof. Clearly, Y contains at least two points. Suppose a WID subset P of Y separates Y. Then there exist a closed subset P^* of X separating X such that $P^* \cap Y \subset P$. By our assumption P^* is a SID compactum, and $P^* \cap Y$ is a WID subset of P^* (being a closed subset of the WID space P). It follows from the preceding lemma that $P^* \backslash (P^* \cap Y) \subset X \backslash Y$ contains an ID Cantor manifold, a contradiction.

It remains to show that Y is dense in X. Suppose this is false. Hence there is a point $x \in X \backslash \bar{Y}$ and a closed set S in X separating X between x and Y. Then S is a SID compactum. Again by 6.2 S contains an ID Cantor manifold C; then $C \cap Y = \emptyset$, a contradiction completing the proof. ∎

C. Decompositions of SID compacta. The following result answers the question of R. Pol.

6.6. THEOREM. *Let X be a* SID *compactum and let $X_1, X_2, \ldots$ be arbitrary subsets of X. If $X = \bigcup_n X_n$, then there exists an index n such that X_n contains a* GID *Cantor manifold closed in X_n.*

Proof. Suppose the conclusion is false. Let C_0 be an ID Cantor manifold in X. Then the set $C_0 \cap X_1$, being a closed subset of X_1, is not a GID Cantor manifold. By 6.5 there exists an ID Cantor manifold $C_1 \subset C_0 \backslash (C_0 \cap X_1) = C_0 \backslash X_1$. By the same argument there exists an ID Cantor manifold $C_2 \subset C_1 \backslash X_2$. Repeating this procedure we construct a decreasing sequence of compacta $C_0 \supset C_1 \supset \ldots$ such that $C_n \cap X_n = \emptyset$ for each $n \geqslant 1$. Hence $\bigcap_n C_n$ is a nonvoid subset of X disjoint with each X_n, a contradiction completing the proof. ∎

6.7. THEOREM. *Let X be an* ID *Cantor manifold and let $X_1, X_2, \ldots$ be subsets of X. If no set X_n contains a* GID *Cantor manifold closed in X_n then the complement $X \backslash \bigcup_n X_n$ is a* GID *Cantor manifold dense in X.*

6.8. *Remark*. No WID space contains a GID Cantor manifold as a *closed* subset, hence 6.7 is applicable to the case where X_n's are WID subsets of X. ■

Proof. The set $X \backslash \bigcup_n X_n$ contains at least two points by 6.6. Suppose it is separated by a WID subset P. Then there exists a closed subset P^* of X separating X such that $P^* \cap (X \backslash \bigcup_n X_n) \subset P$. If follows that P^* contains an ID Cantor manifold C. Then

$$C = (C \cap X_1) \cup (C \cap X_2) \cup \ldots \cup C \cap (X \backslash \bigcup_n X_n).$$

From our assumptions and 6.6 it follows that $C \cap (X \backslash \bigcup_n X_n)$ contains a GID Cantor manifold D closed in this set. It is easy to check that D is a closed subset of P. It follows that D is WID, a contradiction.

Suppose there is a point $x \in X \backslash \overline{(X \backslash \bigcup_n X_n)}$. Let S be a closed subset of X separating X between x and $X \backslash \bigcup_n X_n$. Then S is a SID compactum. Moreover, $S = (S \cap X_1) \cup (S \cap X_2) \cup \ldots$ and no of the sets $S \cap X_n$ contains a GID Cantor manifold closed in $S \cap X_n$. This contradicts 6.6 and completes the proof. ■

Problems

1. Let $f\colon X \to \prod_{j \in J} M_j$ be an essential mapping, where $|J| = \infty$, and let $h\colon \prod_{j \in J} M_j \to \prod_{k \in K} N_k$ be a homeomorphism. Must $h \circ f$ be essential?

2. Let $f\colon X \to M$ be essential and let N be a submanifold of M. Is the mapping $f^{-1}(N) \to N$ determined by f essential?

3. Let $f_j\colon X_j \to I$, $j \in J$, be essential mappings. Must the product $\prod_{j \in J} f_j\colon \prod_{j \in J} X_j \to I^J$ be essential?

4. Characterize the class of manifolds M such that, for any mapping $f\colon X \to M$, f is essential if and only if f is universal.

5. Does there exist an hereditarily strongly infinite-dimensional continuum with trivial shape?

6. Does there exist a nondegenerate continuum X such that every nondegenerate subcontinuum of X is an infinite dimensional Cantor manifold?

7. Let X be a weakly infinite-dimensional space. Is it possible to separate X by an hereditarily weakly infinite-dimensional subspace?

Added in proof. K. Lorentz has informed the author that the answer to 3 is negative – even for connected spaces. There exists a connected space X such that $\dim X \times X = 1$.

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