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L_2-COHOMOLOGY *

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Introduction

For a closed oriented n-dimensional manifold M^n with a triangulation t: $|K| \xrightarrow{\cong} M^n$ the p-th real cohomology group $H^p(M; R)$ is isomorphic to the zero eigenspace of the canonical combinatorical Laplace operator $\Delta = \Delta_p = d\delta + \delta d$: $C^p(K) \to C^p(K)$, $0 \leqslant p \leqslant n$. The isomorphism is given by the harmonic cochains. Thus one has a nice operator-theoretical description of cohomology. When considering open manifolds M^n and infinite simplicial complexes K, we have to develop for the same aim a theory of appropriate functional cohomology spaces, i.e. the theory of L_2-cohomology or, more generally, Sobolev cohomology. A thorough analysis of L_2-cycles and boundaries shows that these are suitable concepts to reflect special features of $|K| = M^n$ at infinity. This is partially done in § 1. A second reason for the introduction and study of L_2-cohomology is the possibility of comparison with analytical theories on M^n, in particular with the analytical L_2-cohomology of an open oriented Riemannian manifold (M^n, g). Finally, several important applications of L_2-cohomology in differential and algebraic geometry a can serve as a third justification. We present here some results concerning complex projective varieties with conical singularities, but they are valid also in a larger class of singular spaces. A further application is the solution of the Hirzebruch conjecture by W. Müller. Sections 2–4 are devoted to these topics.

§ 1. Combinatorical L_2-cohomology of infinite complexes

Let K be an n-dimensional locally finite oriented simplicial complex and $\sigma^q \in K$. We denote $I(\sigma^q) = \# \{\tau^{q+1} \in K \mid \sigma^q < \tau^{q+1}\}$, $I_q(K) = \sup_{\sigma^q \in K} I(\sigma^q)$. The complex K is called uniformly locally finite (u.l.f.) in dimension q if $I_q(K) < \infty$. If this holds for all q, then we call K uniformly locally finite. K is u.l.f. if and only if

* This paper is in final form and no version of it will be submitted for publication elsewhere.

$I_0(K) < \infty$. Let $C^q_2(K) = \{\sum_{\sigma^q} f_\sigma \sigma | \sum_{\sigma^q} f_\sigma^2 < \infty\}$ be the Hilbert space of real square summable q-cochains,

$$\langle f, g\rangle = \langle \sum_{\sigma^q} f_\sigma \sigma, \sum_{\sigma^q} g_\sigma \sigma\rangle = \sum_\sigma f_\sigma \cdot g_\sigma, \qquad d\sigma^q = \sum_{\tau^{q+1}} [\tau^{q+1} : \sigma^q]\tau^{q+1}$$

the usual simplicial coboundary and

$$df = d(\sum_\sigma f_\sigma \sigma) = \sum_{\tau^{q+1}} (\sum_{\sigma^q < \tau^{q+1}} [\tau : \sigma] f_\sigma)\tau^{q+1}$$

the formal linear extension.

LEMMA 1.1. *d is a bounded linear operator from C^q_2 into C^{q+1}_2.*

Proof. We have, for $f = \sum f_\sigma \sigma$,

$$df = \sum_{\tau^{q+1}} (\sum_{\sigma^q < \tau^{q+1}} [\tau : \sigma] f_\sigma),$$

$$\|df\|^2 = \sum_\tau (\sum_{\sigma<\tau} [\tau : \sigma] f_\sigma)^2 \leqslant \binom{q+2}{q+1}^2 \sum_{\tau^{q+1}} (\max_{\sigma<\tau} |f_\sigma|)^2$$

$$\leqslant \binom{q+2}{q+1}^2 \cdot I_q^2 \cdot \sum_\sigma f_\sigma^2 = C^2_{q,q+1} \|f\|^2. \quad \blacksquare$$

$C^*_2(K) = (C^q_2(K), d)_q$ becomes an L_2-complex. We define the L_2-cohomology of K by

$$H^q_2(K) = Z^q_2(K)/B^q_2(K) = \ker(d_q : C^q_2 \to C^{q+1}_2)/\mathrm{im}(d_{q-1} : C^{q-1}_2 \to C^q_2)$$

and the reduced L_2-cohomology by

$$\bar{H}^q_2(K) = Z^q_2(K)/\overline{B^q_2(K)} = \ker d_q/\overline{\mathrm{im}\, d_{q-1}}.$$

Let d^*_q be the adjoint of d_q with respect to the above $\langle\ ,\ \rangle$, $\langle df, g\rangle = \langle f, d^* g\rangle$ for $f \in C^q_2$, $g \in C^{q+1}_2$. d^* is the usual simplicial boundary,

$$d^* \sigma^p = \sum_{\tau^{p-1} < \sigma^p} [\sigma : \tau]\tau, \quad d^* f = d^*(\sum_\sigma f_\sigma \sigma) = \sum_{\tau^{p-1}} (\sum_{\sigma^p > \tau^{p-1}} [\sigma : \tau] f_\sigma)\tau,$$

where we identified simplicial L_2-cochains and L_2-chains, $C^p_2(K) = C_{p,2}(K)$. We define the L_2-homology of K by

$$H_{p,2}(K) = Z_{p,2}(K)/B_{p,2}(K) = \ker(d^*_{p-1} : C_{p,2} \to C_{p-1,2})/\mathrm{im}(d^*_p : C_{p+1,2} \to C_{p,2})$$

and the reduced L_2-homology of K by

$$\bar{H}_{p,2}(K) = Z_{p,2}(K)/\overline{B_{p,2}(K)} = \ker d^*_{p-1}/\overline{\mathrm{im}\, d^*_p}.$$

Let $\Delta = \Delta^c_p = d_{p-1} d^*_{p-1} + d^*_p d_p : C^p_2(K) \to C^p_2(K)$ be the combinatorical Laplace operator of K and $\mathscr{H}^p(K) = \ker \Delta^c_p$ the Hilbert space of harmonic L_2-cochains = harmonic L_2-chains.

Without proof we state

LEMMA 1.2. (a) *$f \in \mathscr{H}^p$ if and only if $df = d^*f = 0$.*

(b) *There exists an orthogonal decomposition*

$$C_2^p(K) = \overline{dC_2^{p-1}} \oplus \overline{d^* C_2^{p+1}} \oplus \mathscr{H}^p. \ \blacksquare \tag{1.1}$$

COROLLARY 1.3. *There are canonical topological isomorphisms*

$$\bar{H}_2^p(K) \cong \mathscr{H}^p(K) \cong \bar{H}_{p,2}(K).$$

Proof. $Z_2^p = \mathscr{H}^p \oplus \overline{dC_2^{p-1}} = \mathscr{H}^p \oplus \bar{B}_2^p$, $\bar{H}_2^p = Z_2^p/\bar{B}_2^p \cong \mathscr{H}^p$. $\blacksquare$

Let $C_c^p \subset C_2^p$ be real p-cochains with finite supports. Then we have dense inclusions $C_c^p \subset C_2^p$, $B_c^p \subset \bar{B}_2^p$, i.e. $\bar{C}_c^p = C_2^p$, $\bar{B}_c^p = \bar{B}_2^p$.

Thus we get canonical morphisms

$$H_c^p(K) \to H_2^p(K) \to \bar{H}_2^p(K), \qquad H_{p,c}(K) \to H_{p,2}(K) \to \bar{H}_{p,2}(K). \tag{1.2}$$

The main question of combinatorical L_2-theory can be now stated. Which topological – combinatorical properties are reflected by $H_2^*(K)$, $\bar{H}_2^*(K)$, $H_{*2}(K)$, $\bar{H}_{*2}(K)$ and the morphisms (1.2)?

We denote by $P_Z = P_{Z_2^p}$ resp. $P_B = P_{\bar{B}_2^p}$ the orthogonal projection onto Z_2^p, resp. $\bar{B}_2^p$.

LEMMA 1.4. *Every L_2-cocycle appears as an L_2-limit of cocycles from $P_Z(C_c^p)$. The corresponding result holds for L_2-cycles.*

Proof. $P_Z(\overline{C_c^p}) = P_Z(C_2^p) = Z_2^p$ and $P_Z(\overline{C_c^p}) \subseteq \overline{P_Z(C_c^p)} \subseteq Z_2^p$. $\blacksquare$

COROLLARY 1.5. *The following isomorphisms hold*:

$$H_2^p = \overline{P_{Z_2^p}(C_c^p)}/B_2^p, \qquad \bar{H}_2^p = \overline{P_{Z_2^p}(C_c^p)}/\bar{B}_c^p,$$

$$H_{p2} = \overline{P_{Z_{p,2}}(C_{p,c})}/B_{p,2}, \qquad \bar{H}_{p,2} = \overline{P_{Z_{p,2}}(C_{p,c})}/\bar{B}_{p,c}. \ \blacksquare$$

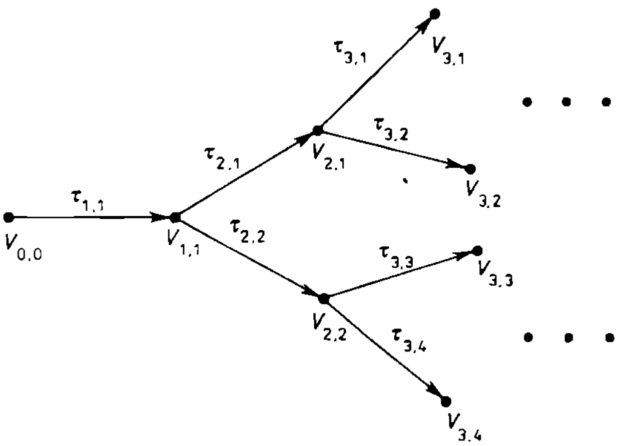
COROLLARY 1.6. *Every converging sequence $(f^{(\nu)})_\nu$ of p-cocycles (p-cycles) with finite support defines a well defined class in H_2^p, resp. $\bar{H}_2^p$ ($H_{p,2}$, resp. $\bar{H}_{p,2}$).* $\blacksquare$

For the concrete computation of L_2-cohomology and homology one has to answer the question: What are the L_2-cocycles, cycles and the elements of B_2^p, $\bar{B}_2^p$, $B_{p,2}$, $\bar{B}_{p,2}$? The study of partial answers exhibits new special features of the L_2-theory of infinite u.l.f. complexes.

LEMMA 1.7. *If $H_{p,2} \neq \bar{H}_{p,2}$, then there exist an infinite number of independent homology classes in $H_{p,2}$ whose image in $\bar{H}_{p,2}$ equals to* 0, i.e. $\dim \ker(H_{p,2} \to \bar{H}_{p,2}) = \infty$. *The same holds for cohomology.* $\blacksquare$

COROLLARY 1.8. *From* $H_{p,2} = (0)$, $H_{p,2} \neq (0)$ *it follows that* $\dim H_{p,2} = \infty$. *The same holds for cohomology.* ∎

GEOMETRICAL EXAMPLES. 1. Consider the tree K



Consider $f = -v_{0,0} \in Z_{0,c} \subset Z_{0,2}$. Then we have $-v_{0,0} \notin B_{0,c}$, $-v_{0,0} \in \overline{B_{0,2}} = \overline{B_{0,c}}$; even $-v_{0,0} \in B_{0,2}$; $f = \lim_{k \to \infty} f^{(k)}$, $f^{(k)} = d^* g^{(k)}$, $g^{(k)} = \sum_{i=1}^{k} 1/2^{i-1} \sum_{l=1}^{2^{i-1}} \tau_{i,l}$, $d^* g^{(k)} = -v_{0,0} + 1/2^{k-1} \sum_{l=1}^{2^{k-1}} v_{k,l}$, $\|f - d^* g^{(k)}\|^2 = 2^{k-1}/(2^{k-1})^2 = 1/2^{k-1} \xrightarrow[k \to \infty]{} 0$, i.e. $f \in \overline{B_{0,2}} = \overline{B_{0,c}}$. Further we have $g^{(k)} \to g = \sum_{k=1}^{\infty} 1/2^{k-1} \sum_{l=1}^{2^{k-1}} \tau_{k,l} \in C_{1,2}$, since $\|g\|^2 = \sum_{k=1}^{\infty} 2^{k-1}/(2^{k-1})^2 = \sum_{k=1}^{\infty} 1/2^{k-1} = 2 < \infty$. Finally, $-v_{0,0} - d^* g = 0$, i.e. $f \in B_{0,2}$.

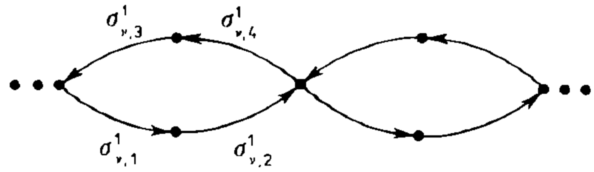
2. Rotation of k from example 1 around an axis parallel to $\tau_{1,1}$ and disjoint to $\tau_{1,1}$, i.e. replacing the vertices $v_{i,j}$ by triangulated S^1 and the edges by cylinders, produces a 2-dimensional branched complex K and an $f \in Z_{1,c} \subset Z_{1,2}$, $f \notin B_{1,c}$, $f \in \overline{B_{1,2}}$, even $f \in B_{1,2}$.

3. Replacing the triangulated S^1 by a closed triangulated oriented manifold M^q gives an $f \in Z_{q,c}$, $f \notin B_{q,c}$, $f \in \overline{B_{q,2}}$, even $f \in B_{q,2}$.

4. COROLLARY. *The homomorphisms* $H_{q,c} \to H_{q,2}$, $H_{q,c} \to \bar{H}_{q,2}$ *are in general not injective.*

5. Now consider a sequence $(S^1_\nu)_{\nu \in Z}$ of triangulated 1-spheres, each with 4 vertices $e_{\nu,1}$, $e_{\nu,2}$, $e_{\nu,-1}$, $e_{\nu,-2}$ and 4 oriented 1-simplexes. We form the complex

$$\ldots \vee S^1_{-1} \underset{e_{-1,1} = e_{0,1}}{\vee} S^1_0 \underset{e_{0,1} = e_{1,-1}}{\vee} S^1_1 \vee \ldots$$



and build up over each S^1_ν the tree of the second example. Then

$$f = \sum_{\nu=-\infty}^{+\infty} 1/2^{|\nu|} \sum_{\mu=1}^{4} \sigma^1_{\nu\mu} \in Z_{1,2}, \quad f \notin Z_{1,c}, \quad f \in B_{1,2} \subset \overline{B_{1,2}},$$

i.e. the compactness of the base cycle (as in 1–3) is not necessary.

6. The manifolds in examples 1–5 are not triangulated, but taking thickenings $|K_{\text{th}}| = M^n$ and appropriate triangulations K_{th} one gets corresponding cycles $\in Z_{q,2}$ inside the thickenings with the desired properties of 1–5.

We now have to carry out some computations. We concentrate attention on L_2-homology, since homology is better for geometrical imagination.

THEOREM 1.9. *Let K be infinite, u.l.f. and connected. Then $\bar{H}_{0,2}(K) = \bar{H}^0_2(K) = (0)$. If $H_{0,2}(K) \neq (0)$, then* $\dim H_{0,2}(K) = \infty$.

Proof. The first assertion comes from the following propositions.

(a) Every vertex of K is an element of $\overline{B_{0,2}}$.

(b) Every 0-chain with finite support is an element of $\overline{B_{0,2}}$.

(c) Every 0-chain $f \in C_{0,2} = Z_{0,2}$ is an L_2-limit of a convergent sequence of 0-chains with finite support. $\overline{B_{0,2}}$ being closed, we get $f \in \overline{B_{0,2}}$, $Z_{0,2} = \overline{B_{0,2}}$. It is clear that (a) implies (b); (c) is trivial. We need only show (a). Let $v_0 \in K^0$ be a vertex. From the assumptions we obtain a sequence of 1-simplexes $\sigma^1_i = [v_i, v_{i+1}]$, $i = 0, 1, 2, \ldots$, with $v_i \neq v_j$ for $i \neq j$, i.e. an infinite edge path starting at v_0 without cycles. Let $f^{(\nu)} = -\sum_{i=0}^{10^\nu - 1} (1 - 1/10^\nu)\sigma^1_i$. Then $d^* f^{(\nu)} = v_0 - \sum_{i=1}^{10^\nu - 1} 1/10^\nu v_i$, $\|v_0 - d^* f^{(\nu)}\|^2 = (10^\nu - 1)/10^{2\nu} \leqslant 1/10^\nu \xrightarrow[\nu\to\infty]{} 0$, $v_0 = \lim_{\nu\to\infty} d^* f^{(\nu)}$. The second assertion of the theorem follows from Corollary 1.8. ∎

Let $g = \sum_{\sigma^p \in K} g_\sigma \sigma$ be a p-chain. We define by $\text{cl}\{\sigma^p \in K \mid g_\sigma \neq 0\}$ the support complex of g denoted by $\text{supp}\, g$. Using this concept one easily gets the following generalizations of 1.9.

THEOREM 1.10. *Assume that, for a cycle $z \in Z_{p,2}(K)$ there exists a subcomplex $L \subset K$ such that L is simplicial isomorphic to the canonical* *triangulation (or a subdivision of bounded degree) of* $|\operatorname{supp} z| \times R_+$; *then* $z \in \overline{B_{p,2}}$. ∎

As usual, K^p denotes the p-dimensional skeleton of K.

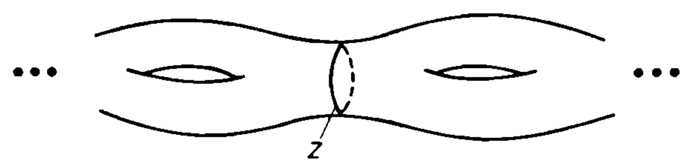
THEOREM 1.11. *Let* $z = z_0 \in Z_{p,2}$. *Assume there exists a sequence* $z_1, z_2, \ldots \in Z_{p,2}$ *such that* $\|z_i\| \leqslant c$, $z_i \sim z_{i+1} \mod \overline{B_{p,2}}$, $z_{i+1} - z_i = d^* c_i$, $c_i \in C_{p+1,2}(K)$, $(\operatorname{supp} c_i)^{p+1} \cap (\operatorname{supp} c_k)^{p+1} = \emptyset$, $k \neq i, i+1$ *and* $(\operatorname{supp} c_i)^{p+1} \cap (\operatorname{supp} c_{i+1})^{p+1} = \operatorname{supp} z_{i+1}$, $i = 0, 1, \ldots$ *Then* $z = z_0 \in \overline{B_{p,2}(K)}$.

Proof. We have to replace in the proof of 1.9 v_0 by z_0, v_i by z_i and σ_i^1 by c_i. We set $g^{(\nu)} = -\sum_{i=0}^{10^\nu - 1} (1 - 1/10^\nu) c_i$. Then

$$d^* f^{(\nu)} = z_0 - \sum_{i=1}^{10^\nu - 1} 1/10^\nu z_i,$$

$$\|z_0 - d^* f^{(\nu)}\|^2 = \Big\| \sum_{i=1}^{10^\nu - 1} 1/10^\nu z_i \Big\|^2 \leqslant (10^\nu - 1)/10^{2\nu} \cdot c^2 \xrightarrow[\nu \to \infty]{} 0. \; ∎$$

EXAMPLES. Consider the infinite ladder with a translation invariant (and therefore u.l.f.) triangulation and the cycle z.



Theorem 1.11 implies $z \in \overline{B_{1,2}}$. From the following remark it becomes clear that $z \notin B_{1,2}$.

Remark. Let $z \in Z_{p,2}$ and suppose that $\operatorname{supp} z$ is a pseudomanifold. Then

(a) $z \in Z_{p,c}$.

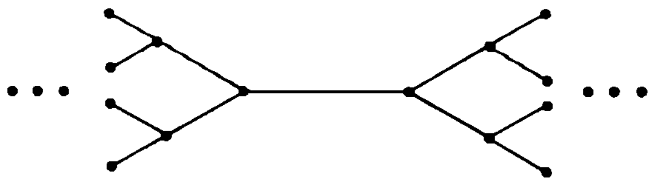
(b) There exists no $c \in C_{p+1,2} \setminus C_{p+1,c}$ with $d^* c = z$ and such that $\operatorname{supp} c$ is a pseudomanifold with boundary.

The above z becomes an element of $B_{1,2}$ if the ladder "branches" sufficiently often.

Starting from this ladder one easily constructs higher-dimensional examples.

We call a p-chain $c \in C_{p,2}$ totally branched if $|\operatorname{supp} c|$ has an infinite number of ends an no end is isolated. In particular, we consider totally branched L_2-cycles.

EXAMPLES. 1. Consider the tree



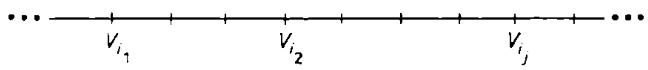
with coefficients as in the first example after Corollary 1.8.

2. 1-spheres glued with step by step branching cylinders and the same splitting of coefficients produces a 2-dimensional example. Here we might also replace the cylinders by bordisms between two S^1 with a bounded number of 2-simplexes.

3. Higher-dimensional examples are constructed immediately along the line of the second example.

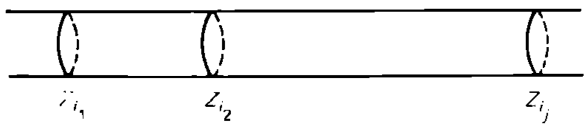
We call a p-cycle $z \in Z_{p,2}$ a limit cycle if it is the L_2-limit of a converging sequence of cycles with compact support. By a standard argument we may assume that every limit cycle is a converging sum of cycles with compact support.

EXAMPLES. 1. K = triangulated line



$$f^{(1)} = v_{i_1}, f^{(2)} = v_{i_1} + 1/2\, v_{i_2}, \dots, f^{(\nu)} \to f = \sum_{j=1}^{\infty} \frac{1}{j} v_{i_j} \in Z_{0,2}.$$

2. K = infinite cylinder with a translation invariant triangulation



$$f = \sum_{j=1} \frac{1}{j} z_{i_j}.$$

In this way we have at least two rather different classes of L_2-cycles. It would be nice and it would be a big step forward in understanding L_2-theory of infinite complexes if this gave a full description. The answer being unknown for the time, we pose the following

CONJECTURE 1. *Totally branched and limit cycles generate all L_2-cycles, i.e. every L_2-cycle equals to an L_2-converging sum of such cycles.*

CONJECTURE 2. *Let $z \in Z_{p,c}$ be a cycle with compact support and $z \notin B_{p,c}$.*

Then there exists no $c \in C_{p+1,2}$ *with* $d^* c = z$, $\operatorname{sup} c$ *being strongly connected and* $|\operatorname{supp} c|$ *having exactly one end.*

THEOREM 1.12. *The validity of Conjecture 2 for all u.l.f. complexes implies the validity of Conjecture 1.*

Proof. Let $z \in Z_{p+1,2}$. We consider the strong components $L_1, L_2, \ldots$ of $\operatorname{supp} z$. Then $z|_{L_i} = z_i$ is again a cycle and $z = \sum_i z|_{L_i} = \sum_i z_i$. Let L be one of the L_i. There are several possibilities. If $|L|$ is compact, i.e. L is finite, then $z|_L$ is a limit cycle, in fact a cycle with compact support. If $|L|$ is noncompact, i.e. L infinite, then there are two possibilities. In the first case there exists a decomposition $z|_L = \sum_i z_i'$, all z_i' with finite support. Then $z|_L$ is again a limit cycle. In the second case there exists no such decomposition. In every decomposition $z|_L = \sum z_i'$ there exists at least one z_i' with infinite support which is not further decomposable. We consider such a z_i' and denote it by z', $L = \operatorname{supp} z'$. We have to show that z' is totally branched. Assume z' is not totally branched. Then there exists an isolated end $\mathscr{E}$ of $|\operatorname{supp} z'|$. Let $|L_1| \supset |L_2| \supset \ldots$ be a base of $\mathscr{E}$. Set $c = z'|_{L_\nu}$, $\nu \geqslant 2$ fixed, and $z'' = d^* c \neq 0$. Thus we have a u.l.f. simplicial complex L_ν and a p-cycle $z'' \in Z_{p,c}(L_\nu)$ such that $z'' \notin B_{p,c}(L)$, $z'' = d^* c$, $c \in C_{p+1,2}(L)$, $\operatorname{supp} c$ is strong connected and $|\operatorname{supp} c|$ has exactly one end. This contradicts the assumption of validity of Conjecture 2 for all u.l.f. complexes. The above z' must be totally branched and Conjecture 1 has to be valid. ∎

If Conjecture 2 were valid, we would have a nice and in a certain sense complete description of all L_2-cycles. It seems that a proof of conjecture 2 will not be simple. The next natural question concerns those L_2-cycles which are not boundaries mod $B_{p,2}$ resp. $\overline{B_{p,2}}$. To this rather delicate question we return in connection with combinatorial manifolds. We now briefly consider the question of the invariance of L_2-homology and cohomology. Let K be an n-dimensional u.l.f. simplicial complex. A simplicial subdivision K' of K is said to be of bounded degree of subdivision of there exists a number N such that every simplex of K is subdivided into at most N simplexes.

THEOREM 1.13. *Let* K' *be a subdivision of* K *of bounded degree. Then the subdivision mapping* θ: $C_{*2}(K) \to C_{*2}(K')$ *induces isomorphisms* θ_*: $H_{*2}(K) \to H_{*2}(K')$ *and topological isomorphisms* θ_*: $\bar{H}_{*2}(K) \to \bar{H}_{*2}(K')$. *The same holds for cohomology.* ∎

A complete proof for $\bar{H}_{*2}$ and $\bar{H}_2^*$ is contained in [7]. This proof also yields the invariance for the nonreduced theory.

Let K, K' be u.l.f. triangulations of $|K| = |K'|$. We say that $K' < K$ if the following two conditions are satisfied. 1) For each vertex $v' \in K'$ there exists a vertex $w \in K$ with $|\operatorname{st} v'| \subset |\operatorname{st} w|$.

2) There exists a number M such that every $\operatorname{st} w \subset K$ contains at most

M simplexes of K'. Under these assumptions $v' \to w$ induces a simplicial mapping η: $K' \to K$.

LEMMA 1.14. *If $K' < K$, then η induces an isomorphism η_*: $H_{*2}(K') \to H_{*2}(K)$ and a topological isomorphism $\bar{\eta}_*$: $\bar{H}_{*2}(K') \to \bar{H}_{*2}(K)$. The same holds for cohomology.*

Proof. All $K_i < K$ form a directed family and the barycentric subdivisions form a cofinal subfamily. From the transitivity of η_* and the isomorphism property of η_* for barycentric subdivisions we obtain the lemma. ∎

The isomorphism property of η_* for barycentric subdivisions follows from [7], the cofinality of barycentric subdivisions resulting from the proof of theorem 32.41 of [18], pp. 332–338.

THEOREM 1.15. *Let K_1 and K_2 be u.l.f. triangulations of $|K|$ such that for each simplex $\sigma \in K_i$ there exist at most M simplexes $\tau \in K_j$ with $\sigma \cap \tau = \emptyset$, $i \neq j$, $i, j = 1, 2$. Then $H_{*2}(K_1) \cong H_{*2}(K_2)$, $H_2^*(K_1) \cong H_2^*(K_2)$. The same holds in the topological sense for the reduced theory.*

Proof. The assumptions on K_1, K_2 imply the existence of a u.l.f. triangulation $K' < K_1$, K_2. From Lemma 1.14 we get the conclusion. ∎

We call two u.l.f. triangulations K_1, K_2 of $|K_1| = |K_2| = |K|$ equivalent, if they satisfy the assumptions of 1.15. The equivalence class of a u.l.f. triangulation K will be denoted by $[K]$. Then we can summarize our results into

THEOREM 1.16. *For an equivalence class $[K]$ of u.l.f. triangulations, H_{*2}, H_2^* are well defined up to isomorphism, $\bar{H}_{*2}$, $\bar{H}_2^*$ are well defined up to topological isomorphism.* ∎

Remark. For $\bar{H}_2^*$ this result is contained already in [4].

For further investigations concerning the general L_2 and Sobolev theory of infinite complexes we refer to [7].

As the most interesting case we now consider open oriented triangulated manifolds, t: $|K| \xrightarrow{\cong} M^n$. The first question which arises is the existence of a u.l.f. triangulation. The answer is yes. We state this as

THEOREM 1.17. *Every open PL-manifold admits a u.l.f. triangulation.* ∎

For the complete proof, which is a joint work of H. G. Bothe and the author, we refer to [8]. The proof follows from the next two lemmas.

LEMMA 1.18. *Every n-dimensional combinatorial manifold is homeomorphic to a cubical polyhedron in R^{2n+1}.* ∎

LEMMA 1.19. *Every cubical polyhedron admits a u.l.f. triangulation.* ∎

The last lemma is a consequence of the following

PROPOSITION. *There exists a number $\mu(m)$ such that every point $x \in R^m$ belongs to at most $\mu(m)$ cubes of a cubical polyhedron R^m.* ∎

Therefore it is no actual restriction to confine attention to u.l.f. triangulations of PL-manifolds. As a generalization of PL-manifolds we consider combinatorial homology manifolds. For a q-simplex $\sigma^q \in K$ we define the complex $\operatorname{lk} \sigma^q$ by $\operatorname{st} \sigma^q = \{\sigma^q \operatorname{lk} \sigma^q\}$. An n-dimensional combinatorial homology manifold is a u.l.f. connected complex $K = \{\sigma_\alpha^q\}_{\alpha,q}$ such that all $\operatorname{lk} \sigma^q$ are homology $(n-q-1)$-spheres. Every u.l.f. triangulated n-dimensional PL-manifold defines a combinatorial homology n-manifold, where, as usual, we identify K and $t(K)$ for combinatorial procedures by means of $t\colon |K| \to M^n$. If $K' = \{\tau_\beta^q\}$ denotes the first barycentric subdivision of K, then

$$\dot{\sigma}_\alpha^0 := \bigcup_{\tau_\beta^n > \sigma_\alpha^0} \tau_\beta^n$$

is an n-cell. For a q-simplex $\sigma_\alpha^q \in K$ we define

$$*\sigma_\alpha^q := \bigcap_{\sigma_\beta^0 < \sigma_\alpha^q} *\sigma_\beta^0.$$

Then $K^* = \{*\sigma_\alpha^q\}_{\alpha,q} = \{\mathscr{E}_\alpha^{n-q}\}_{\alpha,q}$ gives a cell decomposition of $|K|$, the cell complex dual to K. σ_α^q and $*\sigma_\alpha^q = \mathscr{E}^{n-q}$ intersect transversally and $\sigma_\alpha^q \cap +\sigma_\alpha^q = \hat{\sigma}_\alpha^q =$ barycenter of σ_α^q. We assume that K is oriented and we orient $*\sigma_\alpha^q = \mathscr{E}_\alpha^{n-q}$ in such a way that σ_α^q, $\mathscr{E}^{n-q}$ fit the orientation of K.

LEMMA 1.20. *For the dual cell $\mathscr{E}^{n-q}$ to σ_α^q we have*

$$d(\mathscr{E}_\alpha^{n-q}) = (-1)^{n-q+1} *(d^* \sigma_\alpha^q).$$

Proof. [13], pp. 68–69. ∎

COROLLARY 1.21. *The mapping $\sigma_\alpha^q \to *\sigma_\alpha^q = \mathscr{E}_\alpha^{n-q}$ defines a topological chain isomorphism $D\colon C_{q,2}(K) \to C_2^{n-q}(K^*)$ and therefore induces an isomorphism*

$$D_*\colon H_{q,2}(K) \xrightarrow{\cong} H_2^{n-q}(K^*).$$

COROLLARY 1.22. *For an oriented combinatorial homology n-manifold K we have*

$$H_{q,2}([K]) = H_2^{n-q}([K]).$$

Proof. K' is a common subdivision of K and K^* of bounded degree. Then it remains to apply 1.14, 1.16 and 1.21. ∎

Another approach to duality is given by intersection numbers, as presented for example in [13] and for the L_2 case in [4]. For a q-cycle $f \in Z_{q,2}(K)$ and an $(n-q)$-cycle $g \in Z_{n-q}(K)$ we define their intersection number by $\#(f \cdot g) = \langle Df, \bar{g} \rangle$, where $\bar{g}$ is a representative of the homology class of g in $H_{n-q,2}(K^*)$ according to the isomorphism $H_{n-q,2}(K) = H_{n-q,2}(K^*)$. From 1.20 and 1.21 we get that this number is well defined and even defines a pairing between the homology classes of $H_{q,2}(K)$ and $H_{n-q,2}(K)$. The definition of intersection numbers immediately extends to the reduced theory. They are defined as above and one has to show the independence on representatives. Let $f \in \overline{B_{q,2}(K)}$, $g \in Z_{n-q,2}(K)$, $f = \lim f^{(\nu)}$, $f^{(\nu)} \in B_{q,2}(K)$. Then

$$\#(f \cdot g) = \langle Df, \bar{g} \rangle = \langle D \lim_\nu f^{(\nu)}, \bar{g} \rangle = \lim_\nu \langle Dd^* h^{(\nu)}, \bar{g} \rangle$$
$$= \lim_\nu (-1)^{n-q+1} \langle dDh^{(\nu)}, \bar{g} \rangle = (-1)^{n-q+1} \lim_\nu \langle Dh^{(\nu)}, d^* g \rangle = 0.$$

Analogously we get $\#(f \cdot g) = 0$ for $f \in Z_{q,2}(K)$, $g \in \overline{B_{n-q,2}(K)}$.

COROLLARY 1.23. *If $f \in Z_{q,2}(K)$, $g \in Z_{n-q,2}(K)$ and $\#(f \cdot g) = 0$, then f, g are nontrivial cycles in L_2-homology.* ∎

Corollary 1.23 provides a method of a proof of the nontriviality of L_2-cycles. For this we refer to the forthcoming paper [8].

According to 1.20 we have

$$dD = (-1)^{n-q+1} Dd^*$$

and according to [11], p. 139,

$$d^* D = (-1)^{q+1} Dd.$$

Hence we get $\Delta f = 0$ if and only if $\Delta Df = 0$, which implies together with 1.3 the following statement

THEOREM 1.24. *For a combinatorial homology n-manifold K, $\bar{H}_{q,2}([K])$ and $\bar{H}_2^{n-q}([K])$ are topologically isomorphic. Here as before we assume K to be oriented.* ∎

Finally we discuss the question of conditions under which $H_2^q(K)$ and $\bar{H}_2^q(K)$ coincide. From 1.16 it is clear that this is a property of the equivalence class $[K]$ of u.l.f. triangulations, in particular it is invariant under subdivisions of bounded degree. One has to find out conditions which provide $\operatorname{im} d_q = \overline{\operatorname{im} d_q}$.

THEOREM 1.25. *The following conditions are equivalent.*

(a) $\operatorname{im} d_q^*$ *and* $\operatorname{im} d_{q-1}^*$ *are closed.*

(b) $\operatorname{im} d_q$ *and* $\operatorname{im} d_{q-1}$ *are closed.*

(c) $\operatorname{im} \Delta_q$ *is closed.*

(d) $0 \notin \sigma_e(\Delta_q|_{(\ker \Delta_q)^\perp})$, *where σ_e denotes the essential spectrum.*

Proof. The equivalence of (a) and (b) is a standard fact of functional analysis (closed range theorem, [19], p. 205). According to the decomposition (1.1), $\operatorname{im} \Delta_q$ is closed if and only if $d_q^* d_q(\overline{\operatorname{im} d_q^*})$ and $d_{q-1} d_{q-1}^*(\overline{\operatorname{im} d_{q-1}})$ are closed. Again, according to the decomposition (1.1) and the continuity of all operators, this is equivalent to $\operatorname{im} d_q$, $\operatorname{im} d_{q-1}$ (i.e. $\operatorname{im} d_q^*$, $\operatorname{im} d_{q-1}^*$) being closed. It remains to show the equivalence between (c) and (d). Let $\tilde{\Delta} = \Delta|_{(\ker \Delta)^\perp}$. Then $\operatorname{im} \Delta = \operatorname{im} \tilde{\Delta}$. We have to consider only $\tilde{\Delta}$. Now assume

(c) im $\tilde{\Delta}$ is closed. According to a corollary to the closed range theorem, $\tilde{\Delta}^* = \tilde{\Delta}$ has a continuous inverse ([19], p. 208). From the spectral theorem we get that $0 \notin \sigma_e(\tilde{\Delta})$. Conversely, suppose $0 \notin \sigma_e(\tilde{\Delta})$. This implies $\|\tilde{\Delta}f\| \geq \delta \|f\|$ for some $\delta > 0$. From $\tilde{\Delta}f^{(\nu)} \to g$ we obtain that $(f^{(\nu)})_\nu$ has to be a Cauchy sequence and $(f^{(\nu)})_\nu$ converges in $D_{\tilde{\Delta}}$, $f^{(\nu)} \to f$, which implies $\tilde{\Delta}f = g$. Thus we obtain (c) from (d). ∎

1.25 includes a nice spectral-theoretic characterization of $H_{i,2} = \bar{H}_{i,2}$, $H^i_2 = \bar{H}^i_2$, $i = q, q-1$, which in fact admits a translation into geometrical language ([8]). Already the example of $K = S^1 \times R$ with a translation invariant u.l.f. triangulation, $\bar{H}_{1,2}(K) = (0)$, $\dim H_{1,2}(K) = \infty$, on the one hand and the fact that totally branched L_2 1-cycles of a surface define homology classes $\neq 0$ in $\bar{H}_{1,2}$ show, that $0 \notin \sigma_e(\Delta|_{(\ker\Delta)^\perp}$ has to do with the totally branching of cycles, resp. the possibility of totally branching, i.e. with the structure of K at infinity.

§ 2. Analytical L_2-cohomology of open manifolds

We consider open oriented Riemannian manifolds (M^n, g). $\Lambda^p = \Lambda^p(M^n) = C^\infty(\Lambda^p T^* M)$, resp. $\Lambda^p_0 = \Lambda^p_0(M) = C^\infty_0(\Lambda^p T^* M)$, will denote the vector space of all smooth p-forms, resp. p-forms, with compact support. Λ^p_0 becomes a pre-Hilbert space by the scalar product $\langle\ ,\ \rangle$, $\langle \omega, \omega' \rangle = \int \omega \wedge *\omega' = \int (\omega, \omega')_x \,\mathrm{dvol}$. $\Lambda^{p,0}$ denotes the vector space of all measurable p-forms with $\int \omega \wedge *\omega < \infty$. Then Λ^p_0 is a dense subspace of $\Lambda^{p,0}$ and $\Lambda^{p,0}$ is the completion of Λ^p_0 with respect to $\|\ \|^2_0$, $\|\omega\|^2_0 = \langle \omega, \omega \rangle$. $\Delta = \Delta_p = d\delta + \delta d = d_{p-1}\delta_p + \delta_{p+1} d_p$: $\Lambda^p \to \Lambda^p$ denotes the analytical Laplace operator acting on p-forms. Let S be a set of polynomials in d and δ, for example $S = \{d\}$, $S = \{\delta\}$, $S = \{d\delta + \delta d\} = \{\Delta\}$ or $S = \{\Delta^0, \Delta^1, \dots, \Delta^k\}$. Then we define

$$\Lambda^p_S = \{\omega \in \Lambda^{p,0} \cap \Lambda^p \mid \|D\omega\|^2_0 = \int D\omega \wedge *D\omega < \infty \text{ for all } D \in S\},$$

$$\Lambda^{p,S} = \text{closure of } \Lambda^p_S \text{ with respect to the norm}$$

$$\|\omega\|^2_S = \|\omega\|^2_0 + \sum_{D \in S} \|D\omega\|^2_0,$$

$\mathring{\Lambda}^{p,S}$ = closure of Λ^p_0 in $\Lambda^{p,S}$ with respect to the norm $\|\ \|_S$. Further, we set

$$\Lambda^{p,k} = \Lambda^{p,\{\Delta^0, \Delta^1, \dots, \Delta^k\}}.$$

If $S = \emptyset$, then $\|\ \|_\emptyset = \|\ \|_0$, $\Lambda^{p,\emptyset} = \Lambda^{p,0} = \mathring{\Lambda}^{p,0}$. We have the inclusions $\Lambda^{p,0} \supset \Lambda^p_S \supset \Lambda^p_0$, $\Lambda^{p,0} \supset \Lambda^{p,S} \supseteq \mathring{\Lambda}^{p,S}$. For complete (M^n, g), Δ is essentially self-adjoint on Δ^p_0, i.e. $\bar{\Delta}$, $D_{\bar{\Delta}} = \Lambda^{p,1}$, is self adjoint. Furthermore,

$$\mathscr{H}^p = \{\omega \mid d\omega = \delta\omega = 0 \text{ in distributional sense}\}$$
$$= \{\omega \in \Lambda^p \cap \Lambda^{p,0} \mid \Delta\omega = 0\}.$$

The analytical L_2-cohomology $H_2^p(M, d)$ of (M, g) is defined as the cohomology of the complex

$$\ldots \to \Lambda^p_{\{d\}} \to \Lambda^{p+1}_{\{d\}} \to \ldots,$$

$H_2^p(M, d) := \ker(d\colon \Lambda^p_{\{d\}} \to \Lambda^{p+1}_{\{d\}})/\mathrm{im}(d\colon \Lambda^{p-1}_{\{d\}} \to \Lambda^p_{\{d\}}) = \ker d_p/\mathrm{im}\, d_{p-1}$. An apparently other version is obtained by considering $H_2^p(M, \bar{d}) := \ker \bar{d}_p/\mathrm{im}\, \bar{d}_{p-1}$.

THEOREM 2.1. *The inclusion* $\Lambda^*_{\{d\}} \to \Lambda^{*,\{d\}}$ *induces an isomorphism* i_*: $H_2^*(M, d) \to H_2^*(M, \bar{d})$ ([3]). ∎

We define the reduced analytical L_2-cohomology by

$$\bar{H}_2^p(M, \bar{d}) := \ker \bar{d}_p/\overline{\mathrm{im}\, \bar{d}_{p-1}},$$

and more generally,

$$\bar{H}_2^{p,k}(M, d) := Z^{p,k}/\overline{B^{p,k}}^{\Lambda^{p,k}},$$

where $Z^{p,k} = \{\omega \in \Lambda^{p,k} \mid d\omega = 0\}$, $B^{p,k} = d\Lambda^{p-1,k+1}$.

THEOREM 2.2. *Let* (M^n, g) *be complete. Then the spaces* $\bar{H}_2^{p,k}$ *are independent of k. For every* $k \geqslant 1$, $0 \leqslant p \leqslant n$, $Z^{p,k}$ *admits an orthogonal decomposition*

$$Z^{p,k} = \mathscr{H}^p \oplus \overline{B^{p,k}}.$$

For $k = 0$ *we have*

$$Z^{p,0} = \mathscr{H}^p \oplus \overline{B^p},$$

where $B^p = \{\omega \in \Lambda^{p,0} \mid$ *There exists an* $\eta \in \Lambda^{p-1,0}$ *with* $\omega = d\eta\}$. ∎

For the proof we refer to [5], [6].

We have a natural surjection

$$j\colon H_2^p(M, d) \to \bar{H}_2^p(M, d), \qquad j(\omega + \mathrm{im}\, d_{p-1}) = \omega + \overline{\mathrm{im}\, d_{p-1}}$$

and a morphism

$$h\colon \mathscr{H}^p(M) \to H_2^p(M, \bar{d}), \qquad h(\omega) = \omega + \mathrm{im}\, \bar{d}_{p-1}.$$

We say that in (M^n, g) holds the strong Hodge theorem if h is an isomorphism.

LEMMA 2.3. (a) *h is injective if* (M^n, g) *satisfies the Stokes theorem in the* L_2*-sense, i.e.* $\langle \bar{d}\varrho, \omega\rangle = \langle \varrho, \bar{\delta}\omega\rangle$ *for all* $\varrho \in \Lambda^{p-1,\{d\}} = D_{\bar{d}}$, $\omega \in \Lambda^{p,\{\delta\}} = D_{\bar{\delta}}$.

(b) *The condition* (a) *is satisfied if* $\bar{d}_{p-1} = \delta_p^*$. *This is equivalent to* $\bar{d}^*_{p-1} = \bar{\delta}_p$ *or to* $\bar{d}_{p-1} = \overline{d_{p-1}|_{\Lambda_0^{p-1}}}$ *or to* $\bar{\delta}_p = \overline{\delta_p|_{\Lambda_0^p}}$. *All these conditions are satisfied if* (M^n, g) *is complete.*

(c) *h is surjective if and only if* $\mathrm{im}\, \bar{d}_{p-1} \supseteq \overline{\mathrm{im}\, \bar{d}_{p-1}|_{\Lambda_0^p}}$.

Proof. All these propositions follow from the classical Hodge theorem and the work of Gaffney ([12]). ∎

We summarize our considerations into

THEOREM 2.4. (a) *The strong Hodge theorem holds if and only if* $\operatorname{im}\bar{d}_{p-1} = \overline{d_{p-1}\Lambda_0^{p-1}}$. *In particular, this holds if* $\dim H_2^p(M, \bar{d}) < \infty$ *and Stokes theorem holds in the* L_2*-sense.*

(b) *If* (M^n, g) *is complete, then* $\mathcal{H}^p(M)$, $\bar{H}_2^p(M, \bar{d})$ *and* $\bar{H}^{p,k}(M, d)$ *are all topologically canonically isomorphic and* h^p: $\mathcal{H}^p(M) \to H_2^p(M, \bar{d})$ *is an injection.* ∎

The natural questions which now arise concern the connections between combinatorical and analytical L_2-cohomology and applications. Subsequent sections are devoted to these question.

§ 3. The de Rham–Hodge isomorphism in the L_2-category

To build up connections between analytical and combinatorial L_2-theory of (M^n, g), one has to triangulate M^n. Let σ^n be a curved n-simplex in M^n. We define the fullness $\theta(\sigma)$ by $\theta(\sigma) = \operatorname{vol}(\sigma)/(\operatorname{diam}(\sigma))^n$. We consider smooth triangulations t: $|K| \xrightarrow{\cong} M^n$ which satisfy the following conditions ([6]).

(a) There exists a $\theta_0 > 0$ such that for every curved simplex σ^n the fullness satisfies the inequality $\theta(\sigma) \geqslant \theta_0$.

(b) There exist constants $c_1 > c_2 > 0$ such that for every σ^n we have

$$c_2 \leqslant \operatorname{vol}(\sigma) \leqslant c_1 .$$

(c) There exists a constant $c > 0$ such that for every vertex $v \in K$ the barycentric coordinate function φ_v: $M \to R$ satisfies the condition $|\nabla\varphi_v| \leqslant c$.

If one assumes (a), then (b) is equivalent to the existence of bounds $d_1 > d_2 > 0$ with $d_2 \leqslant \operatorname{diam}(\sigma) \leqslant d_1$ for all $\sigma \in K$. (a) and (b) are equivalent to the boundedness of the volumes from below and the diameters from above.

We call triangulations which satisfy conditions (a)–(c) uniform.

Now the existence problem arises for such triangulations. For this we consider two conditions.

CONDITION (I). The injectivity radius is positively bounded from below on (M^n, g), i.e. $\inf_{x \in M} r_{\text{inj}}(x) > 0$. ($r_{\text{inj}}(x) =$ distance between x and the cut locus, i.e. between x and the first cut point of geodesics, starting at x).

CONDITION (B_k): $\nabla^i R$ is bounded at M, $0 \leqslant i \leqslant k$, where R denotes the curvature tensor.

(M^n, g) is of bounded geometry up to order k if (M^n, g) satisfies the conditions (I) and (B_k).

The following theorem of Calabi gives a sufficient condition for the existence of uniform triangulations.

THEOREM 3.1. *If* (M^n, g) *has bounded geometry up to order* 0, *then* (M^n, g) *admits a uniform triangulation* t: $|K| \xrightarrow{\cong} M^n$. ∎

We will give some examples for bounded geometry.

THEOREM 3.2. (a) *Every homogeneous Riemannian manifold has bounded geometry of arbitrary high order.*

(b) *The class of open manifolds which admit a metric of bounded geometry is closed with respect to finite connected sums and to coverings.*

(c) *Every open manifold which arises by infinite pasting of a finite number of bordisms admits a metric of bounded geometry of arbitrary order.* ∎

Uniform triangulations have nice combinatorial properties.

THEOREM 3.3. (a) *Every uniform triangulation is u.l.f.*

(b) *The Whitney standard subdivision of a uniform triangulation is again uniform.*

(c) *Two uniform triangulations K_1, K_2 under the same Riemannian metric g satisfy the assumptions of Theorem* 1.15. *In particular, $H_2^*(K_1) \cong H_2^*(K_2)$, $\bar{H}_2^*(K_1) \cong \bar{H}_2^*(K_2)$.* ∎

Now we can ask for connections between analytical and combinatorical L_2-theory. According to 2.1 and 2.4, we identify $\bar{H}_2^p(M, d)$, $\bar{H}^{p,k}(M, d)$ with $\mathscr{H}^p(M)$ for complete (M^n, g). A certain answer is given by the following

THEOREM 3.4. *Let (M^n, g) be open, complete, of bounded geometry up to order $k > n/2 - 1$ and t: $|K| \xrightarrow{\cong} M$ a smooth uniform triangulation. Then the integration of forms over simplexes of K defines a topological isomorphism $\int$: $\mathscr{H}^*(M) \xrightarrow{\cong} \bar{H}_2^*(K)$.* ∎

Remark. In 1977 Dodziuk proved this theorem for normal coverings of closed Riemannian manifolds ([5]). A version of this proof was applied to the class occurring in Theorem 3.2. (c) ([10]). From the proofs it was more or less clear that the only fact needed was bounded geometry. This was firstly recognized by Dodziuk ([6]). As an interesting corollary of 1.9 and 3.4 we obtain

THEOREM 3.5. *Let (M^n, g) be open, connected, complete and of bounded geometry up to order $k > n/2 - 1$. Then there exist no square integrable harmonic p-forms on M^n, $p = 0$, n.* ∎

There arises the natural question of similar results for the nonreduced theory; i.e., what are the relations between $H^{p,k}(M, d) = Z^{p,k}/B^{p,k}$ and $H_2^p(K)$? We are far from giving here a full discussion and full proofs, but we will present some results in this direction. For the proofs we refer to ([9]). One gets immediately

THEOREM 3.6. *Let (M^n, g) be open, of bounded geometry up to order $k > n/2 - 1$ and K a uniform triangulation. Then the integration of forms over the simplexes of K induces a surjection*

$$\int: H^{*,k}(M, d) \to H_2^*(K). \ \blacksquare$$

For the next theorems we assume that the reader is well acquainted with the Whitney mapping W, which maps cochains into forms. If K is a uniform triangulation, $K^{(r)}$ the rth standard subdivision, then we denote the associated Whitney mapping with W_r.

THEOREM 3.7. *Let (M^n, g) be open, of bounded geometry up to order $k > n/2-1$, K a uniform triangulation and $\overline{\bigcup_{r=0}^{\infty}(\mathrm{im}(dW_r))} \subset d_{p-1}\Lambda^{p-1,k+1}$. Then $\int: H^{p,k}(M,d) \to H_2^p(K)$ is injective.* ∎

To distinguish between d on a combinatorial and an analytical level, we write d^c for the combinatorical coboundary operator.

THEOREM 3.9. *Let (M^n, g) be open, complete and of bounded geometry up to order $k > n/2-1$. Under each of the following conditions $\int: H^{p,k}(M^n, d) \to H_2^p(K)$ is an isomorphism.*

(a) $\mathrm{im}\,\bar{d}_{p-1}$ *is closed*;

(b) $\mathrm{im}\,d^c_{p-1}$ *is closed and* $\overline{\bigcup_{r=0}^{\infty}(\mathrm{im}(dW_r))} \subset \mathrm{im}\,d_{p-1}$;

(c) $\mathrm{im}\,\Delta_p$ *or* $\mathrm{im}\,\Delta_{p-1}$ *is closed*;

(d) $0 \notin \sigma_e(\Delta_p|_{(\ker\Delta_p)^\perp})$ *or* $0 \notin \sigma_e(\Delta_{p-1}|_{(\ker\Delta_{p-1})^\perp})$. ∎

§ 4. Applications of L_2-cohomology to algebraic geometry

We conclude our presentation with two applications of L_2-cohomology to algebraic geometry, namely, the proof of Hirzebruchs conjecture and intersection homology. According to the foregoing sections, L_2-cohomology includes the following objects: $H_2^*(K)$, $\bar{H}_2^*(K)$, $\mathscr{H}^*(K)$ at the combinatorial level and $H_2^*(M)$, $\bar{H}_2^*(M)$, $H^{*,k}(M)$, $\mathscr{H}^*(M)$ at the analytical level. For complete manifolds, $\bar{H}_2^*(M)$, $\bar{H}^{*,k}(M)$ and $\mathscr{H}^*(M)$ are L_2-isomorphic in a canonical way. If additionally (M^n, g) is of bounded geometry and K is a uniform triangulation, then also $\bar{H}_2^*(M) \cong \bar{H}^{*,k}(M) \cong \mathscr{H}^*(M) \cong \bar{H}_2^*(K) = \mathscr{H}^*(K)$. In this case the computation of any one of these spaces yields all the other spaces and it depends in fact on the actual situation, which computation is most convenient, most handy. Unfortunately, the manifolds which occur in Hirzebruch's conjecture are complete but do not have bounded geometry. Consider a $4k$-dimensional complete oriented manifold. The $*$-operator defines an involution $\tau_p = (-1)^{p(p-1)/2}*_p$, and this induces an orthogonal decomposition $\Lambda^{2k,0} = \Lambda_+^{2k,0} \oplus \Lambda_-^{2k,0}$. If D denotes the operator acting on certain L_2-forms and $\dim\mathscr{H}^*(M) < \infty$, then $\mathrm{ind}_{L_2} D = \dim\mathscr{H}_+^{2k}(M) - \dim\mathscr{H}_-^{2k}(M)$. Thus we have a simple relation between the L_2-index of D and L_2-harmonic forms, i.e. L_2-cohomology. The computation of $\dim\mathscr{H}_+^{2k} - \dim\mathscr{H}_-^{2k}$ is the heart of Müller's proof of Hirzebruch's conjecture as shall see.

The other class of examples is furnished by projective varieties with certain types of singularities Σ. After removing the singularities, the resulting Kähler manifolds are neither complete nor of bounded geometry, i.e. $H_2^*(M\backslash\Sigma)$, $H_2^*(K)$ are in general independent objects. We indicate an isomorphism between $H_2^*(M\backslash\Sigma)$ and the dual of the Goresky–McPherson homology.

At first we introduce Hirzebruch's conjecture.

Let N^{4k-1} be a closed smooth oriented $(4k-1)$-dimensional manifold with stable trivial tangent bundle. Then all its Stiefel-Whitney and Pontriagin numbers vanish and N bounds a $4k$-dimensional compact smooth oriented manifold X^{4k}. If α is a trivialization of the stable tangent bundle of X, then the stable tangent bundle of X can be pushed down to an SO-bundle over X/N. Let $\tilde{p}_j \in H^{4j}(X/N; Z) = H^{4j}(X, \partial X; Z)$ be its Pontriagin classes and L_k the Hirzebruch polynomial. According to Novikov's signature additivity theorem, the number

$$\delta(N) = L_k(\tilde{p}_1, \ldots, \tilde{p}_k)[X, \partial X] - \operatorname{sign} X \tag{4.1}$$

is independent of the choice of X and is well defined. The above situation arises in a natural way in the investigation of cusps of Hilbert modular varieties (compactification of each single cusp by adding a point, resolution of the corresponding singularity and taking for N a connected boundary of the cusp). The computation of $\delta(N)$ by L_2-methods for uncompactified cusp is one of the most beautiful applications of L_2-cohomology. To make this clear, we recall briefly some well known facts on Hilbert modular varieties. Here we follow [14], [15], [16]. Let F be a totally real algebraic number field of degree n over Q, i.e. such that there are n different embeddings $F \to R$ into the real numbers R, $x \to x^{(j)} \in R$, $x \in F$. If O_F denotes the ring of algebraic integers, then the Hilbert modular group $\Gamma = Sl_2(O_F)/\{-1, 1\}$ acts on the n-fold product $H^n = H \times \ldots \times H$ of the upper half-plane H by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z_1, \ldots, z_n) = \left(\frac{a^{(1)} z_1 + b^{(1)}}{c^{(1)} z_1 + d^{(1)}}, \ldots, \frac{a^{(n)} z_n + b^{(n)}}{c^{(n)} z_n + d^{(n)}}\right). \tag{4.2}$$

More generally, we consider discrete irreducible subgroups $\Gamma \subset (Sl_2(R))^n$. $Sl_2(R)$ acts on $P^1(C)$ by

$$z \to \frac{az+b}{cz+d} \tag{4.3}$$

An element of $Sl_2(R)$ is called parabolic if it has under this action exactly one fixed point in $P^1(C)$. Clearly, this point belongs to $P^1(R) = R \cup \{\infty\}$, since fixed points with imaginary part $\neq 0$ do not exist. An element $\gamma = (\gamma^{(1)}, \ldots, \gamma^{(n)}) \in (Sl_2(R))^n$ is called parabolic if all $\gamma^{(i)}$ are parabolic. A parabolic element of $(Sl_2(R))^n$ has exactly one fixed point in $(P^1(C))^n$ which belongs to $(P^1(R))^n$. The parabolic points of Γ are defined as fixed points of parabolic elements of Γ. The orbits of parabolic points under the action of Γ on $(P^1(R))^n$ are called parabolic orbits.

LEMMA 4.1. *If Γ is irreducible, then there are only finitely many parabolic orbits. In particular, this holds for the action of the Hilbert modular group.* ∎

H^n has the invariant metric

$$ds^2 = \sum_{j=1}^{n} \frac{(dx_j)^2 + (dy_j)^2}{y_j^2}. \tag{4.4}$$

LEMMA 4.2. *For $\Gamma = Sl_2(O_F)/\{-1, 1\}$ and the volume form*

$$\omega = \frac{dx_1 \wedge dy_1}{y_1^2} \wedge \ldots \wedge \frac{dx_n \wedge dy_n}{y_n^2}$$

we have

$$\int_{H^n/\Gamma} \omega = 2\pi^{-n}(D_{F/Q})^{3/2}\zeta_F(2),$$

where $D_{F/Q}$ is the discriminant and $\zeta_F(s)$ is the zeta function of F. ∎

The parabolic orbits correspond to cusps of H^n/Γ as follows. Γ acts by (4.3) on $P^1(F) = F \cup \{\infty\}$ and by (4.2), (4.3) on $H^n \cup P^1(F)$. $\overline{H^n/\Gamma} := (H^n \cup P^1(F))/\Gamma$ is a compact algebraic variety with a finite number of isolated singularities. H^n/Γ is noncompact with finitely many isolated ends $=$ (open) cusps and each end is compactified by a point of $P^1(F)/\Gamma$. The number of ends $=$ the number of parabolic orbits coincides with the class number of F, i.e. the order of the class group. Let $x = \frac{p}{q}$, $p, q \in O_F$, be a representative for the point of $P^1(F)$ which compactifies a cusp. We label the cusps by these points. Write $\Gamma_x = \{\gamma \in \Gamma \mid \gamma x = x\}$. Then there exists a $\varrho \in Sl_2(F)$ with $\varrho x = \infty$. We define ${}^{\varrho}\Gamma_x = \varrho \Gamma_x \varrho^{-1}$ and $W(d) = \{z \in H^n \mid \prod_{j=1}^{n} \mathrm{Im}(z_j) \leqslant d\}$, $d > 0$. By T^k we denote the k-torus.

LEMMA 4.3. *$W(d)/{}^{\varrho}\Gamma_x$ is a smooth neighbourhood of the cusp and $\partial(W(d)/{}^{\varrho}\Gamma_x)$ is a T^n-bundle over T^{n-1}.* ∎

The stable tangent bundle of $\partial(W(d)/{}^{\varrho}\Gamma_x)$ is trivial and therefore, for a cusp represented by the point $x \in P^1(F)/\Gamma$, the signature defect $\delta(x) = \delta(\partial(W(d)/{}^{\varrho}\Gamma_x))$ of (4.1) is well defined. We define the norm $N(\mu)$ of an element $\mu \in F$ by $\mu^{(1)} \ldots \mu^{(n)}$. An element $\varepsilon \in F$ is called totally positive ($\varepsilon \gg 0$) if $\varepsilon^{(1)}, \ldots, \varepsilon^{(n)} > 0$. By O_F^* we denote as usual the units of O_F. Let M be an additive subgroup of F which is free abelian of rank n i.e. is a lattice of rank n. Define $U_M^+ = \{\varepsilon \in O_F^* \mid \varepsilon M = M, \varepsilon \gg 0\}$. Two lattices M_1, M_2 are called strictly equivalent if there exists a $\lambda \gg 0$ with $\lambda M_1 = M_2$. This implies $U^+_{M_1} = U^+_{M_2}$.

LEMMA 4.4. *For each cusp x there exist a lattice $M_x \subset F$ and a subgroup $V_x \subset U^+_{M_x}$ of finite index such that*

$$^e\Gamma_x = \left\{ \begin{pmatrix} \varepsilon & \mu \\ 0 & 1 \end{pmatrix} \middle| \; \varepsilon \in V_x, \; \mu \in F \text{ and } \mu \in M_x \text{ if } \varepsilon = 1 \right\},$$

i.e. $^e\Gamma_x$ is an extension of V_x by M_x:

$$0 \to M_x \to {}^e\Gamma_x \to V_x \to 1. \;\blacksquare$$

For each pair (M, V), $M \subset F$ a lattice and $V \subset U^+_M$ a subgroup of finite index, we define the L-series

$$L(M, V, s) = \sum_{\mu \in (M \setminus 0)/V} \frac{\operatorname{sign} N(\mu)}{|N(\mu)|^s}. \tag{4.5}$$

This Dirichlet series converges for $\operatorname{Re}(s) > 1$ and admits an analytic continuation to C, which is regular at $s = 1$. Further, we can embed M as a lattice in R^n, $\mu \to (\mu^{(1)}, \dots, \mu^{(n)})$. To each cusp x we now associate an L-series $L(M_x, V_x, s)$ by (4.5). Let $z \in H^n$ be a fixed point of Γ, Γ_z the isotropy group. Γ_z is finite cyclic of order $q > 1$. After a choice of a geodesic ball D_z around z and normal coordinates $z_1, \dots, z_n$, Γ_z acts on D_z by

$$\gamma \cdot (z_1, \dots, z_n) = (\zeta^{q_1} z_1, \dots, \zeta^{q_n} z_n),$$

where γ is a generator of Γ_z, $\zeta^q = 1$ and $(q_i, q) = 1$. Then

$$\delta(z) = \frac{(-1)^k}{q} \left(\sum_{j=1}^{q-1} \cot\left(\frac{\pi q_1^j}{q}\right) \dots \cot\left(\frac{\pi q_n^j}{q}\right) \right)$$

is called the cotangent sum associated with the quotient singularity $z \in H^n/\Gamma$. Now we recall the following fundamental result of [14].

THEOREM 4.5. *Let $\Gamma \subset (Sl_2(R))^{2k}$ be a discrete irreducible subgroup such that $\operatorname{vol}(H^{2k}/\Gamma) < \infty$. If $z_1, \dots, z_s$ represent the quotient singularities of H^{2k}/Γ and $x_1, \dots, x_t$ is a complete system of inequivalent parabolic points, then*

$$\operatorname{sign}(H^{2k}/\Gamma) = \sum_{i=1}^{s} \delta(z_i) + \sum_{j=1}^{t} \delta(x_j). \;\blacksquare \tag{4.6}$$

Thus the computation of $\operatorname{sign}(H^{2k}/\Gamma)$ includes as a main step the computation of the $\delta(x_j)$. In the case $k = 1$ Hirzebruch was able to compute $\delta(x)$ for a cusp by compactification of the cusp, explicit resolution of the singularity and computation the terms in (4.1). Computation of $L(M_x, V_x, 1)$ for a real quadratic field and comparison gives

$$\delta(x) = -\frac{d(M)}{\pi^2} L(M_x, V_x, 1), \tag{4.7}$$

where $d(M_x) = \mathrm{vol}(R^n/M_x)$. Now, Hirzebruch conjectured that for every $n = 2k$ and every cusp x of H^{2k}/Γ, denoting the Hilbert modular group associated with a totally real number field F of field degree $2k$, the following equality holds:

$$\delta(x) = \frac{(-1)^k}{\pi^{2k}} d(M_x) L(M_x, V_x, 1). \tag{4.8}$$

This conjecture – actually a more general version of it – was proved in meantime by Atiyah, Donnelly and Singer ([1], [2]). A second proof is announced and partially published by W. Müller. Reduced L_2-cohomology, i.e. a suitable space of square-integrable harmonic forms, is an essential tool in that proof. We briefly sketch W. Müller's idea, following [15], and [16]. For the sake of simplicity we restrict ourself to the case of $k = 1$, $n = 2$, $F = Q(\sqrt{D})$ a real quadratic number field of class number 1. Thus we have one end = cusp. The involution $\tau = i^{p(p-1)}*$ on $\Lambda^*(H^2/\Gamma)$ induces a decomposition $\Lambda^* = \Lambda^*_+ \oplus \Lambda^*_-$ into ± 1-eigenspaces of τ. Let $D = d+\delta|_{\Lambda^*_{+,0}}: \Lambda^*_{+,0} \to \Lambda^*_{-,0}$ be the signature operator, $\bar{D}$ its closure, D^* the adjoint operator, $\Delta_+ = D^* D$, $\Delta_- = DD^+$ and $\bar{\Delta}_\pm$ the self-adjoint closure of $\Delta_\pm$. H^2/Γ is complete; thus $\ker \Delta \cap \Lambda^{*,0}$ coincides with the square-integrable closed and coclosed forms. τ anticommutes with D and induces a decomposition $\mathscr{H}^*(H^2/\Gamma) = \ker \Delta \cap \Lambda^{*,0} = \mathscr{H}^*_+ \oplus \mathscr{H}^*_-$. Then $\ker \bar{\Delta}_\pm = \mathscr{H}^*_\pm$.

LEMMA 4.6. *$\mathscr{H}^*_\pm$ are finite-dimensional.* ∎

COROLLARY 4.7.

$$\mathrm{ind}_{L_2} D = \dim\ker \bar{D} - \dim\mathrm{coker}\, \bar{D} = \dim\ker \Delta_+ - \dim\ker \Delta_-$$

is well defined. ∎

Let $L_2\Lambda^*(H^2/\Gamma) = \Lambda^{*,0}(H^2/\Gamma) = L_{2,d}(\Lambda^*(H^2/\Gamma)) \oplus L_{2,ac}(\Lambda^*(H^2/\Gamma))$ be the decomposition corresponding to the discrete, resp. absolutely continuous part of Δ, $L_{2,d}\Lambda^* = L_{2,\mathrm{cus}}\Lambda^* \oplus L_{2,\mathrm{res}}\Lambda^*$ the decomposition into cusp and residual forms. According to Selberg there exists a torsion free normal subgroup $\Gamma_1 \subset \Gamma$ of finite index.

THEOREM 4.8. $\mathrm{sign}(H^2/\Gamma_1) = \dim \mathscr{H}^2_{\mathrm{cus},+}(\Gamma_1) - \dim \mathscr{H}^2_{\mathrm{cus},-}(\Gamma_1)$. ∎

Further, $\dim \mathscr{H}^*_{\mathrm{res},+} = \dim \mathscr{H}^*_{\mathrm{res},-}$. From this and the behaviour of the signature under finite coverings one gets finally

$$\mathrm{ind}_{L_2} D = \dim \mathscr{H}_+(H^2/\Gamma) - \dim \mathscr{H}_-(H^2/\Gamma) = \mathrm{sign}(H^2/\Gamma).$$

Therefore it remains to compute $\mathrm{ind}_{L_2} D$. Set $A_\pm = \bar{\Delta}_\pm|_{L_{2,d}\Lambda^*_\pm}$.

THEOREM 4.9. *For every $t > 0$, $e^{-tA_\pm}$ has a smooth kernel $k_0^\pm(z, z', t)$, is of trace class and thus*

$$\mathrm{ind}_{L_2} D = \int_{H^2/\Gamma} \mathrm{tr}\, k_0^+(z, z, t) - \int_{H^2/\Gamma} \mathrm{tr}\, k_0^-(z, z, t) \tag{4.9}$$

The remaining main problem is the computation of the right hand side of (4.9). Here we examine the contribution of various conjugacy classes in Γ to (4.9). Hyperbolic conjugacy classes give contribution zero. The contribution of the conjugacy classes with an elliptic fixed point z is the cotangent sum $\delta(z)$. The parabolic contribution is $-\pi^{-2}d(M)L(M, V, 1)$, where $M = O_F$, $V = O_F^{*2}$ and $d(M) = (D_{F/Q})^{1/2}$. Thus

$$\mathrm{ind}_{L_2} D = \sum_{j=1}^{k} \delta(z_j) - \pi^{-2} d(M) L(M, V, 1) = \mathrm{sign}(H^2/\Gamma), \quad \text{i.e.}$$

$$\delta(x) = -\frac{d(M)}{\pi^2} L(M, V, 1).$$

Concluding this class of examples we state one of the main theorems of [15].

THEOREM 4.10. *Let F/Q be a totally real number field of degree $2k$ and let $\Gamma \subset Sl_2(F)$ be an arithmetic subgroup. Let x_j, $j = 1, \ldots, p$, be a complete system of Γ-inequivalent parabolic fixed points of Γ and let $\delta(x_j)$ be the signature defect of x_j. If the lattice $M_j \subset F$ and the subgroup $V_j \subset U^+_{M_j}$ of finite index are associated with x_j, then*

$$\sum_{j=1}^{p} \delta(x_j) = \frac{(-1)^{2k}}{2k} \sum_{j=1}^{p} d(M_j) L(M_j, V_j, 1). \quad \blacksquare$$

As a second application of L_2-cohomology in algebraic geometry we present the isomorphism between the L_2-cohomology $H_2^*(X \setminus \Sigma)$ and the dual $(IH_*^{\bar{p}}(X))^*$ of the intersection homology for certain classes of stratified spaces with singularity Σ and a suitable metric. To formulate a precise theorem, the introduction of some concepts is unavoidable. Let X^n be a pseudomanifold, i.e. a polyhedron such that there exists a closed subspace Σ with $\dim \Sigma \leqslant n - 2$ and $X \setminus \Sigma$ being a dense oriented manifold in X. Let X be a pseudomanifold with triangulation T. A stratification of X is a filtration by closed subspaces

$$X^n = X_n \supset X_{n-1} = X_{n-2} \supset X_{n-3} \supset \ldots \supset X_1 \supset X_0$$

such that for each point $p \in X_i \setminus X_{i-1}$ there is a filtered space

$$V = V_n \supset V_{n-1} \supset \ldots \supset V_i = \text{a point},$$

and a mapping $V \times B^i \to X$ which maps $V_j \times B^i$, for each j, PL-homeomorphically onto a neighbourhood of p in X_j, where B^i denotes the PL i-ball. In particular, $X_{(i)} = X_i \setminus X_{i-1}$ is an i-manifold or is empty. Denote by $C_*^T(X; R) = C_*^T(X)$ the corresponding (to T) chain complex of all simplicial chains with real coefficients and $C_*(X) = \varinjlim_T C_*^T(X)$ the group of all PL geometric chains. Each $\xi \in C_i(X)$ has a well defined support $|\xi|$. For a perversity $\bar{p}$, i.e. a sequence of integers $\bar{p} = (p_2, p_3, \ldots, p_n)$ with $p_2 = 0$ and $p_k \leqslant p_{k+1} \leqslant p_k + 1$, we define

$$IC_i^{\bar{p}}(X) = \{\xi \in C_i(X) | \dim(|\xi| \cap X_{n-k}) \leqslant i - k + p_k, \dim(|\partial\xi| \cap X_{n-k}) \leqslant i - 1 - k + p_k \text{ for all } k\}.$$

The i-th intersection homology group $IH_i^{\bar{p}}(X)$ of X with perversity $\bar{p}$ and with a fixed stratification is defined to be the i-th homology group of the chain complex $IC_*^{\bar{p}}(X)$.

Remark. Cheeger considers in his paper [3] only the so-called middle perversity $\bar{m} = (0, 0, 1, 1, 2, 2, \ldots, n/2 - 1)$.

THEOREM 4.11. *Let X be a compact pseudomanifold without boundary. Then $IH_*^{\bar{p}}(X)$ is finitely generated and independent of stratification.* ∎

For a sequence $\bar{c} = (c_2, \ldots, c_n)$ of nonnegative real numbers, a metric g on $X_{(n)}$ is said to be associated with $\bar{c}$ if, at a local product representation for a tubular neighbourhood, for every stratum $X_{(j)}$ the metric g is locally of the kind

$$g_U + dr \otimes dr + r^{2c} n - j \cdot g_S,$$

where g_U is a metric in the base U, $dr \otimes dr$ is the metric in radial direction and g_S is the induced metric in some fixed sphere of radius r ([17], p. 345).

LEMMA 4.12. *For any $\bar{c}$ there exists a metric at $X_{(n)}$ which is associated with $\bar{c}$.* ∎

For any perversity $\bar{p} \leqslant \bar{m}$ (i.e. such that $p_i \leqslant m_i$), a metric g is said to be associated with $\bar{p}$, if g is associated with $\bar{c} = (c_2, \ldots, c_n)$ and

$$\frac{1}{k - 1 - 2p_k} \leqslant c_k < \frac{1}{k - 3 - 2p_k} \quad \text{if } 2p_k \leqslant k - 3,$$

$$1 \leqslant c_k < \infty \quad \text{if } 2p_k = k - 2.$$

THEOREM 4.13. *Let X^n be an n-dimensional compact stratified space with a fixed* PL *structure and a stratification $X = X_n \supset X_{n-1} \ldots \supset X_0$ such that $X_{n-1} = X_{n-2}$ and each stratum $X_{(j)}$ of dimension $j \leqslant n - 2$ is diffeomorphic to the disjoint union of $]0, 1[^j$, $X_{(j)} = \bigcup_\alpha (]0, 1[^j)_\alpha$. Let $\bar{p} \leqslant \bar{m}$ be a perversity and g a metric on $X_{(n)}$ associated with $\bar{p}$. Then*

$$H_2^q(X_{(n)}, d) \cong (IH_q^{\bar{p}}(X))^* \quad ([17]). \ \blacksquare$$

For examples one can take certain singular projective varieties. The above class includes and generalizes the conical singularities of Cheeger ([3]). We conclude with the final remark that L_2-cohomology methods become more and more an essential ingredient of topology, geometry, anaylsis and harmonic analysis.

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