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ON THE NORMAL DISCONNECTION OF A TREE

IN MEMORY OF LUCJAN SZAMKOŁOWICZ

Abstract. In the paper a certain generalization of trees, called *dendroids*, is described. A dendroid is expressed as a pair $\langle X, f\rangle$, where X is a set of arcs and f is a mapping of X into the set of all subsets of X. Finite dendroids coincide with finite directed trees; infinite ones include directed trees, but also other objects. This paper may be a good hint to a non-traditional access to the study of trees.

1. Definitions. By a *directed multigraph* [4] we mean a triple $G = \langle V, X, \varphi\rangle$, where V is a non-empty finite set of vertices, X is a finite set of arcs, and φ is a function mapping X into V^2. If φ is 1-1, then G is called a *directed graph.* In this case we can consider G as a pair $\langle V, R\rangle$, where R is a binary relation on V, and arcs are denoted by $[u, v]$. If for any $u, v \in V$, $uRv \Leftrightarrow \sim vRu$, then G is called *strongly directed.*

Let $G = \langle V, X\rangle$ be a directed graph. A sequence

$$[v_1, \ldots, v_k, x_1, \ldots, x_{k-1}]$$

where $v_i \in V$, $x_j \in X$ and $x_j = [v_j, v_{j+1}]$ or $x_j = [v_{j+1}, v_j]$ for $i = 1, \ldots, k$ and $j = 1, \ldots, k-1$ is a *chain* connecting v_1 and v_k. The chain $[v_1, \ldots, v_k, x_1, \ldots, x_{k-1}]$ is *simple* if $i \neq j$ implies $v_i \neq v_j$ for $i, j = 1, \ldots, k$. A chain is called a *cycle* if $v_1 = v_k$. A cycle is called *simple* if

$$k > 3 \quad \text{and} \quad v_i = v_j \Leftrightarrow \{i, j\} = \{1, k\}.$$

A graph is *connected* if for any pair of vertices there exists a chain connecting them. A connected graph without cycles is called a *tree*. Let us observe that the removal of an arc x from the set of arcs of a tree $G = \langle V, X\rangle$ disconnects G into exactly two trees

$$G_1 = \langle V_1, X_1\rangle \quad \text{and} \quad G_2 = \langle V_2, X_2\rangle$$

such that $V_1 \cup V_2 = V$ and $X_1 \cup X_2 = X \setminus \{x\}$. Such a disconnection is called *normal.* Using this property we shall generalize the notion of a tree into an infinite case. We shall resign from the definition of a chain as a sequence of vertices and arcs, next we shall show that, in the finite case, the new definition of a chain and the old one are equivalent.

2. Basic properties of normal disconnections of a tree. Let $\langle X, f\rangle$ be an ordered pair, where X is an arbitrary non-empty set of elements called *arcs* and f is a mapping $f\colon X \to 2^X$. For $a \in X$ write

$$g(a) = (X \setminus f(a) \setminus \{a\}).$$

The pair $\langle X, f\rangle$ is called a *dendroid* if the following conditions are satisfied:

(A1) $a \notin f(a)$,

(A2) $b \in f(a) \Rightarrow g(a) \cup \{a\} \subset f(b)$ or $g(a) \cup \{a\} \subset g(b)$,

(A3) $b \in g(a) \Rightarrow f(a) \cup \{a\} \subset f(b)$ or $f(a) \cup \{a\} \subset g(b)$.

Let $\langle X, f\rangle$ be a dendroid, $a, b \in X$, and $a \neq b$. The set of all $x \in X$ such that $x \neq a$, $x \neq b$ for which

$$a \in f(x) \wedge b \in g(x) \quad \text{or} \quad a \in g(x) \wedge b \in f(x)$$

is called an *open chain* and denoted by (a, b). We define the neighbourhood relation S in X by the formula

$$aSb \Leftrightarrow a \neq b \wedge (a, b) = \emptyset.$$

Let us put

$$f_S(a) = \{x \in f(a)\colon xSa\}, \qquad g_S(a) = \{x \in g(a)\colon xSa\}.$$

A subset Y of X is called a *node* of a dendroid $\langle X, f\rangle$ if Y satisfies one of the following two conditions:

(a) Y is maximal with respect to inclusion and such that

$$a, b \in Y \Rightarrow aSb;$$

(b) Y is a one-element set $\{a\}$ if $f_S(a) = \emptyset$ or $g_S(a) = \emptyset$.

Let V denote the set of all nodes of a dendroid $\langle X, f\rangle$. In the set V we define the neighbourhood relation R putting for $u, v \in V$

$$uRv \Leftrightarrow u \neq v \wedge u \cap v \neq \emptyset \quad \text{or} \quad u = v = \{x\} \wedge f_S(x) = g_S(x) = \emptyset.$$

Now we present some properties of the notions defined above.

LEMMA 1. *If $a, b, c \in X$ and $b \in f(a)$, $c \in g(a)$, then $\sim bSc$.*

LEMMA 2. *For any node u and any $a \in u$, exactly one the following conditions holds:*

(a) $u \setminus \{a\} \subset f(a)$;

(b) $u \setminus \{a\} \subset g(a)$.

Proof. If $u = \{a, b\}$, the lemma is obvious. Let $b, c \in u \setminus \{a\}$, $b \neq c$. If $b \in f(a)$ and $c \in g(a)$, then $\sim bSc$, a contradiction with the assumption that u is a node. We obtain an analogous contradiction assuming that $b \in g(a)$ and $c \in f(a)$.

LEMMA 3. (a) *If* aSb, aSc *and* $b, c \in f(a)$, $b \neq c$, *then* bSc.

(b) *If* aSb, aSc *and* $b, c \in g(a)$, $b \neq c$, *then* bSc.

Proof. (a) Assume to the contrary that aSb, aSc, and $b, c \in f(a)$, $b \neq c$, and $\sim bSc$. Then there exists an arc d such that $d \in (b, c)$. Hence

$$b \in f(d) \text{ and } c \in g(d) \quad \text{or} \quad b \in g(d) \text{ and } c \in f(d).$$

Assume that $b \in f(d)$ and $c \in g(d)$. Observe that $d \neq a$. In fact, if $d = a$, then $c \in g(a)$ and $c \in f(a)$. This however is impossible in view of the definition of $g(a)$. According to Lemma 2, $a \in f(d)$ or $a \in g(d)$. Let $a \in f(d)$. Since $c \in g(d)$, so $d \in (a, c)$, whence $\sim aSc$. If $a \in g(d)$, then since $b \in f(d)$, we have $\sim aSb$. In both cases we get a contradiction with the assumption.

We argue similarly in the other case.

The proof of (b) is analogous.

COROLLARY 1. *If* u_1 *and* u_2 *are two non 1-element nodes,* $a \in u_1$, $a \in u_2$ *and* $u_1 \setminus \{a\} \subset f(a)$, $u_2 \setminus \{a\} \subset f(a)$, *then* $u_1 = u_2$. *Similarly, if* $u_1 \setminus \{a\} \subset g(a)$ *and* $u_2 \setminus \{a\} \subset g(a)$, *then* $u_1 = u_2$.

Proof. Assume that the elements of both nodes different from a belong to $f(a)$. Let $b \in u_1$, $c \in u_2$, $b \neq c$. By Lemma 3 we have bSc, so $u_1 = u_2$.

If an arc a belongs to the node u, then we say that a is *incident* with u.

COROLLARY 2. *If an arc* a *is incident with two different nodes* u_1, u_2 *such that none of* u_1, u_2 *is a 1-element node and* $u_1 \setminus \{a\} \subset f(a)$, *then*

$$u_2 \setminus \{a\} \subset g(a).$$

LEMMA 4. *Every arc is incident with at most two nodes.*

Proof. Let an arc a be incident with nodes u_1, u_2 and u_3. If none of u_1, u_2, u_3 is a 1-element node, then by Lemma 2 and Corollary 1 at least two of them are identical. If one of them, say u_1, is a 1-element node, then $u_1 = \{a\}$ and $f_S(a) = \emptyset$ or $g_S(a) = \emptyset$ and $u_2 = u_3$.

LEMMA 5. *Two different nodes have at most one common arc.*

Proof. If nodes u_1 and u_2 are different and one of them is a 1-element node, then the proof is obvious. Assume that none of u_1, u_2 is a 1-element node. Let u_1 and u_2 be different and have a common arc a. By Corollary 2, if $u_1 \setminus \{a\} \subset f(a)$, then $u_2 \setminus \{a\} \subset g(a)$. So, by Lemma 1, any arc $b \neq a$ cannot be incident with both u_1 and u_2.

By the definition of a node we have

LEMMA 6. *Every node is incident with at least one arc and every arc is incident with at least one node.*

By Lemmas 1–6 we get

THEOREM 1. *The pair* $\langle V, R \rangle$ *is a symmetric graph (finite or infinite) whose edge set can be identified with the set* X.

Note that if X is infinite, the graph $\langle V, R\rangle$ can have loops, i.e., elements $u\in V$ such that uRu, and need not be connected in the usual sense. Analogously, the notion of an open chain (a, b) does not coincide with that of the chain in the theory of finite graphs, since (a, b) cannot be empty, and does not contain any element adjacent to a.

EXAMPLE 1. Let X be the set of all real numbers. Assume that

$$f(x) = \{y\in X:\ y > x\}.$$

The pair $\langle X, f\rangle$ is a dendroid. An open chain is of the form

$$(a, b) = \{x\in X:\ a < x < b\}.$$

The set of nodes V coincides with X. Every arc x is a loop of $\langle V, R\rangle$.

EXAMPLE 2. Let X be the set of ordinals less than ω. For $\alpha\in X$ put

$$f(\alpha) = \{x\in X:\ x > \alpha\}.$$

Then $g(\omega) \neq \emptyset$ but $g_s(\omega) = \emptyset$, and we have a 1-element node ω.

We introduce a relation $\vec{R}$ defined in the set V as follows:

$$u\vec{R}v \Leftrightarrow uRv,\ u\cap v = \{x\} \text{ for some } x\in X,$$
$$v\setminus\{x\} \subset f(x) \text{ and } u\setminus\{x\}\subset g(x).$$

It is easy to see that

THEOREM 2. *The pair $\langle V, \vec{R}\rangle$ is a strongly directed graph, whose set of arcs is X.*

3. Properties of an open chain. Let $\langle X, f\rangle$ be a dendroid, V be the set of its nodes, and let $u\in V$. We define two sets of arcs:

$$u^+ = \{x\in X:\ x\in u \wedge u\setminus\{x\}\subset f(x)\},$$
$$u^- = \{x\in X:\ x\in u \wedge u\setminus\{x\}\subset g(x)\}.$$

The set u^+ consists of *entrance arcs* of u, and the set u^- consists of *exit arcs* of u. From Lemma 2 it follows that

$$u^+\cup u^- = u.$$

LEMMA 7. *For a node u and an open chain (a, b) we have*

$$|u\cap(a, b)| \leqslant 2.$$

Proof. Assume to the contrary that $u\cap(a, b) = \{x, y, z\}$. Let $x\in u^+$, $y\in u^-$, $z\in u^+$ and let $b\in f(y)$, $b\in f(z)$. Hence $a\in g(y)$ and $a\in g(z)$. Then $z, y\in f(x)$, $x\in f(z)$, and $x\in g(y)$. By (A3) it follows that $f(y)\subset f(x)$. In fact, the relation $f(y)\cup\{y\}\subset g(y)$ is impossible. Hence $b\in f(x)$, but $z\in f(x)$ and $x\in f(z)$, so by (A2) we have $g(z)\subset g(x)$. Consequently, $a\in f(x)$. Since $a\in f(x)$ and $b\in f(x)$, we get a contradiction with the assumption that $x\in(a, b)$. The proof in the remaining case is analogous.

COROLLARY 3. *If u is a node and $u \in (a, b)$, then $|u| \in \{1, 2\}$.*

Now we define the preceding relation in the set of arcs of an open chain. To this end we introduce the following notation:

$$x \vec{\in} (a, b) \Leftrightarrow x \in (a, b) \text{ and } b \in f(x),$$

$$x \overleftarrow{\in} (a, b) \Leftrightarrow x \in (a, b) \text{ and } a \in f(x).$$

The notation $x \vec{\in} (a, b)$ is read: the arc x is directed compatibly with the open chain (a, b).

The notation $x \overleftarrow{\in} (a, b)$ is read: the arc x is directed contrary to the open chain (a, b).

Let $x, y \in (a, b)$. We say that x *precedes* y and write $x \prec y$ iff

$$x \vec{\in} (a, b) \text{ and } y \vec{\in} (a, b) \Rightarrow f(y) \subset f(x), \tag{1}$$

$$x \overleftarrow{\in} (a, b) \text{ and } y \overleftarrow{\in} (a, b) \Rightarrow g(y) \subset g(x), \tag{2}$$

$$x \vec{\in} (a, b) \text{ and } y \overleftarrow{\in} (a, b) \Rightarrow g(x) \subset f(y), \tag{3}$$

$$x \overleftarrow{\in} (a, b) \text{ and } y \vec{\in} (a, b) \Rightarrow f(x) \subset g(y). \tag{4}$$

LEMMA 8. *For any two arcs x and y such that none of the sets $f(x)$, $g(x)$, $f(y)$, $g(y)$ is empty we have*

$$f(y) \subset f(x) \Leftrightarrow g(x) \subset g(y), \tag{5}$$

$$f(x) \subset g(y) \Leftrightarrow f(y) \subset g(x). \tag{6}$$

Proof. If $x = y$, then conditions (5) and (6) are obvious. So we can assume that $x \neq y$.

We prove the implication $\Rightarrow$ in (5). First we show that the inclusion

$$f(y) \subset f(x) \tag{7}$$

implies

$$y \in f(x). \tag{8}$$

Assume to the contrary that $y \in g(x)$; then by (A3) we have

$$f(x) \cup \{x\} \subset f(y) \tag{9}$$

or

$$f(x) \cup \{x\} \subset g(y). \tag{10}$$

Then by (9) and (7) we get $f(x) \cup \{x\} \subset f(x)$, contrary to (A1). From (10) and (7) it follows that $f(y) \subset g(y)$, which contradicts the definition of $g(y)$. Thus (8) holds. From (8) and (A2) it follows that one of the two cases holds:

$$g(x) \cup \{x\} \subset f(y) \tag{11}$$

or

$$g(x) \cup \{x\} \subset g(y). \tag{12}$$

Conditions (7) and (11) give a contradiction, since then

$$g(x) \subset g(x) \cup \{x\} \subset f(y) \subset f(x).$$

Thus (12) holds, which implies $g(x) \subset g(y)$, and this completes the proof of the implication $\Rightarrow$ in (5). The proofs of the remaining implications are analogous.

THEOREM 3. *The preceding relation $\prec$ is an order in an open chain (a, b).*

Proof. The proof of reflexivity is obvious.

Proof of transitivity. Let $x \prec y$ and $y \prec z$. We can assume that $x \neq y$ and $y \neq z$. To prove the transitivity we have to consider the following cases:

$$x \vec{\in} (a, b), \quad y \vec{\in} (a, b), \quad z \vec{\in} (a, b), \tag{13}$$

$$x \vec{\in} (a, b), \quad y \vec{\in} (a, b), \quad z \overleftarrow{\in} (a, b), \tag{14}$$

$$x \vec{\in} (a, b), \quad y \overleftarrow{\in} (a, b), \quad z \vec{\in} (a, b), \tag{15}$$

$$x \vec{\in} (a, b), \quad y \overleftarrow{\in} (a, b), \quad z \overleftarrow{\in} (a, b), \tag{16}$$

$$x \overleftarrow{\in} (a, b), \quad y \overleftarrow{\in} (a, b), \quad z \overleftarrow{\in} (a, b), \tag{17}$$

$$x \overleftarrow{\in} (a, b), \quad y \overleftarrow{\in} (a, b), \quad z \vec{\in} (a, b), \tag{18}$$

$$x \overleftarrow{\in} (a, b), \quad y \vec{\in} (a, b), \quad z \vec{\in} (a, b), \tag{19}$$

$$x \overleftarrow{\in} (a, b), \quad y \vec{\in} (a, b), \quad z \overleftarrow{\in} (a, b). \tag{20}$$

Assume that (13) holds. By (1) we have $f(y) \subset f(x)$ and $f(z) \subset f(y)$, so $f(z) \subset f(x)$. Since $x \vec{\in} (a, b)$ and $z \vec{\in} (a, b)$, so $x \prec z$. Assume (14) holds. By (1)' we have $f(y) \subset f(x)$, whence, by (5), $g(x) \subset g(y)$. From (14) and (3) it follows that $x \prec z$. The proofs in the remaining cases are similar.

Proof of antisymmetry. We have to show that if $x, y \in (a, b)$ and $x \neq y$, then

$$\sim (x \prec y) \quad \text{or} \quad \sim (y \prec x).$$

We have the following cases:

$$x \vec{\in} (a, b), \quad y \vec{\in} (a, b) \wedge y \in f(x), \tag{21}$$

$$x \vec{\in} (a, b), \quad y \vec{\in} (a, b) \wedge y \in g(x), \tag{22}$$

$$x \vec{\in} (a, b), \quad y \overleftarrow{\in} (a, b) \wedge y \in f(x), \tag{23}$$

$$x \vec{\in} (a, b), \quad y \overleftarrow{\in} (a, b) \wedge y \in g(x), \tag{24}$$

$$x \overleftarrow{\in} (a, b), \quad y \vec{\in} (a, b) \wedge y \in f(x), \tag{25}$$

$$x\overleftarrow{\in}(a, b), \quad y\vec{\in}(a, b) \wedge y\in g(x), \tag{26}$$

$$x\overleftarrow{\in}(a, b), \quad y\overleftarrow{\in}(a, b) \wedge y\in f(x), \tag{27}$$

$$x\overleftarrow{\in}(a, b), \quad y\overleftarrow{\in}(a, b) \wedge y\in g(x). \tag{28}$$

Consider case (21). We shall prove that $y \prec x$. Otherwise, we have $f(x) \subset f(y)$ and $y\in f(x)$, so $y\in f(y)$, which is impossible. The proof in the other cases is similar.

Proof of connectivity. Let $x, y\in(a, b)$. If $x = y$, then $x \prec y$. Assume that $x \neq y$; then we have to consider cases (21)–(28).

Assume that (21) holds. We shall prove that $x \prec y$. Let $x\in(a, b)$, $y\in(a, b)$, $y\in f(x)$. Then $b\in f(x)$ and $b\in f(y)$. Hence $a\in g(x)$ and $a\in g(y)$. Since $y\in f(x)$, so by (A2) we have

$$g(x)\cup\{x\} \subset f(y) \tag{29}$$

or

$$g(x)\cup\{x\} \subset g(y). \tag{30}$$

Case (29) is impossible since (29) implies

$$a\in g(x) \subset g(x)\cup\{x\} \subset f(y).$$

Hence (30) holds, which means that $g(x) \subset g(y)$ and by (5) we get $f(y) \subset f(x)$. Thus $x \prec y$. The proofs of the other cases are analogous.

LEMMA 9. *If $y\in(a, b)$ and z is an element such that $y\in f(z)$ and $a\in g(z)$ or $y\in g(z)$ and $a\in f(z)$, then $z\in(a, b)$ and $z \prec y$.*

Proof. Assume that $y\in f(z)$ and $a\in g(z)$; the proof of the other case is analogous. We shall prove that $b\in f(z)$. Assume to the contrary that $b\in g(z)$. Since $y\in f(z)$, so by (A2) we have

$$g(z)\cup\{z\} \subset f(y) \quad \text{or} \quad g(z)\cup\{z\} \subset g(y).$$

In the first case we have $a, b\in f(y)$, and in the second case $a, b\in g(y)$. This however contradicts the assumption that $y\in(a, b)$. Thus $a\in g(z)$ and $b\in f(z)$, and since $z \neq a$ and $z \neq b$, so $z\in(a, b)$. We show that $z \prec y$. Assume that $y\vec{\in}(a, b)$; the proof in the other case is analogous. Since $y\in f(z)$, so by (A2) we get

$$g(z)\cup\{z\} \subset g(y) \quad \text{or} \quad g(z)\cup\{z\} \subset f(y).$$

The second case cannot hold since then we would have $a\in f(y)$, contrary to the assumption that $y\vec{\in}(a, b)$. So $b\in f(y)$. Hence $g(z) \subset g(y)$ and, by Lemma 8, we have $f(y) \subset f(z)$. Thus we obtain $z \prec y$.

LEMMA 10. *If $y\in(a, b)$ and z is an element such that $y\in f(z)$ and $b\in g(z)$ or $y\in g(z)$ and $b\in f(z)$, then $z\in(a, b)$ and $y \prec z$.*

The proof is analogous to that of Lemma 9.

LEMMA 11. *If* $y, z \in (a, b)$, $y \prec z$, $y \neq z$ *and* $y \in g(t)$, $z \in f(t)$ *or* $y \in f(t)$, $z \in g(t)$, *then* $t \in (a, b)$ *and* $y \prec t \prec z$.

Proof. Assume that $y \in g(t)$ and $z \in f(t)$; the proof in the second case is analogous. Let, for example, $y, z \vec{\in} (a, b)$, the other cases are similar. First we shall prove that

$$z \in f(y).$$

In fact, if $z \in g(y)$, then by (A3) we have

$$f(y) \cup \{y\} \subset f(z) \quad \text{or} \quad f(y) \cup \{y\} \subset g(z).$$

The first case is impossible, since then we have $y \in f(z)$ but $f(z) \subset f(y)$, whence $y \in f(y)$. The second case is also impossible, since $y \vec{\in} (a, b)$ implies $b \in f(y)$, so $b \in g(z)$. But $z \vec{\in} (a, b)$ implies $b \in f(z)$. So we have $z \in f(y)$. We shall prove that

$$t \in f(y).$$

In fact, if $t \in g(y)$, then by (A3) we have

$$f(y) \cup \{y\} \subset f(t) \quad \text{or} \quad f(y) \cup \{y\} \subset g(t).$$

Thus $z \in g(t)$, contrary to the assumption. Thus $t \in f(y)$, and hence and by (A2) we have

$$g(y) \cup \{y\} \subset f(t) \quad \text{or} \quad g(y) \cup \{y\} \subset g(t).$$

The first case is impossible since then we have $y \in f(t)$, and the second case implies $a \in g(t)$. From the assumption $z \in f(t)$ it follows by Lemma 9 that $t \in (a, b)$ and $t \prec z$. The proof that $y \prec t$ is dual by using Lemma 10.

THEOREM 4. *If* $a \subset (a, b)$, $u = \{y\}$, $a \in g(y)$, $b \in f(y)$, *then*

1° *If* $g_S(y) = \emptyset$, *then there is no maximal element in the set*

$$P_1 = \{x\colon\ x \in (a, b),\ x \prec y,\ x \neq y\}.$$

2° *If* $f_S(y) = \emptyset$, *then there is no minimal element in the set*

$$P_2 = \{x\colon\ x \in (a, b),\ y \prec x,\ x \neq y\}.$$

Proof. We shall prove 1°. The set P_1 is non-empty since otherwise it would follow from Lemma 9 that aSy, contrary to the assumption. If there exists a maximal element z_0 in P_1, then by Lemma 11 we have $z_0 Sy$, contrary to the assumption that $g_S(y) = \emptyset$. The proof of 2° is analogous by using Lemma 10.

COROLLARY 4. *If* $u = \{x\}$, $u \subset (a, b)$ *and there is a loop at* u, *then there is no maximal element in the set* P_1 *and there is no minimal element in the set* P_2.

4. Locally finite dendroids. A dendroid $\langle X, f\rangle$ is called *locally finite* if the following condition holds:

(A4) For any $a, b \in X$ we have $|(a, b)| < \aleph_0$.

Let $\langle X, f\rangle$ be a locally finite dendroid. Consider two of its arcs a and b such that $(a, b) \neq \emptyset$. According to Theorem 3 we can order all arcs of the open chain (a, b) into the sequence $[x_1, x_2, \ldots, x_k]$ such that if $i < j$, then $x_i \prec x_j$. Using this we shall prove Lemmas 12–15.

LEMMA 12. *For any x_i we have*

(i) *if $j < i$, then $x_j \in (a, x_i)$;*

(ii) *if $i < j$, then $x_j \in (x_i, b)$.*

Proof. We prove (i); the proof of (ii) is analogous. Assume first that

$$x_i \vec{\in} (a, b) \quad \text{and} \quad x_j \vec{\in} (a, b).$$

Since $x_j \prec x_i$, according to (1) we have

$$f(x_i) \subset f(x_j).$$

We also have $b \in f(x_j)$, and $a \in g(x_j)$. So it remains to prove that $x_i \in f(x_j)$. Assume to the contrary that $x_i \in g(x_j)$. By (A3) we have

$$f(x_j) \cup \{x_j\} \subset f(x_i) \quad \text{or} \quad f(x_j) \cup \{x_j\} \subset g(x_i).$$

In the former case we get $\{x_j\} \subset f(x_j)$, a contradiction. In the latter case we have $b \in g(x_i)$, contrary to the assumption that $x_i \vec{\in} (a, b)$. Thus $x_i \in f(x_j)$. The other cases can be proved similarly.

LEMMA 13. *For every $i = 1, 2, \ldots, k-1$ we have $x_i S x_{i+1}$.*

Proof. If for some i there exists $t \in (x_i, x_{i-1})$, then by Lemma 11 we have $t \in (a, b)$ and $x_i \prec t \prec x_{i+1}$. This however is impossible, since x_1 and x_k are the only elements of the chain (a, b) and are different from t by Lemma 12.

Hence by Lemma 7 we have

LEMMA 14. *For every $i = 1, \ldots, k-1$ the node u_i with arcs x_i, x_{i+1} has the property that $u_i \cap (a, b) = \{x_i, x_{i+1}\}$.*

LEMMA 15. *aSx_1 and $x_k Sb$.*

Proof. The relation aSx_1 follows from Lemma 9. In fact, if there exists z such that $z \in (a, x_1)$, then $z \in (a, b)$ and $z \prec x_1$, contrary to the assumption that the elements $x_1, \ldots, x_k$ are the only elements of the chain (a, b). The fact that $x_k Sb$ follows from Lemma 10.

THEOREM 5. *If a dendroid $\langle X, f\rangle$ is locally finite, then every two its nodes can be connected with a chain in the classical sense.*

Proof. If $u = v$, then $[u]$ is the required chain. Let v and v' be two different nodes. If $v \subset v'$, then v must be a 1-element node. If $v = \{a\}$, then the required chain is $[v, v', a]$. We argue analogously if $v' \subset v$. Let $v \setminus v' \neq \emptyset$,

$v'\setminus v \neq \emptyset$ and $a \in v\setminus v'$, $b \in v'\setminus v$. Let us consider an open chain (a, b). The elements of (a, b) can be ordered into a sequence

$$[a_1, a_2, \ldots, a_n], \quad \text{where } a_1 \prec a_2 \prec \ldots \prec a_n.$$

By Lemmas 13 and 15 we have

$$aSa_1, a_1 Sa_2, a_2 Sa_3, \ldots, a_{n-1} Sa_n, a_n Sb.$$

Let us note that if $a \in u$ and $u \cap (a, b) \neq \emptyset$, then $u \cap (a, b) = \{a_1\}$ by Lemmas 7 and 12. Similarly, if $b \in u$ and $u \cap (a, b) \neq \emptyset$, then $u \cap (a, b) = \{a_n\}$. The pairs $\{a, a_1\}$, $\{a_n, b\}$, $\{a_i, a_{i+1}\}$ $(i = 1, \ldots, n-1)$ can be extended to the nodes v_0, v_n, v_i, respectively, where $v_0 = v$ if $a_1 \in v$ or $v_n = v'$ if $a_n \in v$. Then:

(a) if $a_1 \in v$, then $[v, v_1, \ldots, v_{n-1}, v_n, v', a_1, \ldots, a_n, b]$ is the required chain;

(b) if $a_n \in v_n$, then $[v, v_0, \ldots, v_{n-1}, v', a, a_1, \ldots, a_n]$ is the chain from v to v'.

Lemma 16. (a) *If $b \in f(a)$, $b \in u$, then $u\setminus\{a\} \subset f(a)$.*

(b) *If $b \in g(a)$, $b \in u$, then $u\setminus\{a\} \subset g(a)$.*

Proof. Assume that there exists $c \in u$ such that $c \in g(a)$; then $a \subset (b, c)$, so $\sim bSc$, contrary to the definition of a node.

Lemma 17. *In a locally finite dendroid for any $x \in X$ we have*

(i) *if $f_S(x) = \emptyset$, then $f(x) = \emptyset$;*

(ii) *if $g_S(x) = \emptyset$, then $g(x) = \emptyset$.*

Proof. We shall show (i). The proof of (ii) is analogous. Assume that $f(x) \neq \emptyset$, $y \in f(x)$, and $(x, y) \neq \emptyset$. Hence, by Lemmas 13 and 15, we have $xSa_1, \ldots, a_n Sy$. We prove that $a_n \in f(x)$. Otherwise we have $a_n \in g(x)$, $y \in f(x)$, which is impossible since ySa_n. Similarly we have $a_1 \in f(x)$. Since $a_1 Sx$, we have $f_S(x) \neq \emptyset$.

Theorem 6. *If $\langle V, \vec{R}\rangle$ is the graph of a locally finite dendroid $\langle X, f\rangle$ such that $|X| > 1$, then any two nodes of this graph are connected exactly by one chain in the classical sense.*

Proof. Assume to the contrary that there are two different chains connecting two nodes u and v; then we have one of the following situations:

(a) there exists a loop at u or at v;

(b) there exist different nodes u_1 and u_2 connected by means of two different arcs x_1 and x_2;

(c) there exists a simple cycle $[u_1, u_2, \ldots, u_n = u_1]$, where $n \geqslant 2$.

(a) If there exists a loop x at u, then according to the definition of the relation $\vec{R}$ we have:

$$u = \{x\} \quad \text{and} \quad f_S(x) = g_S(x) = \emptyset.$$

Then by Lemma 17 we obtain $f(x) = g(x) = \emptyset$, so $|X| = 1$, contrary to the assumption of the theorem.

(b) The existence of two different nodes connected with two different arcs contradicts Lemma 5.

(c) Let (c) hold. Denote by x_i the arc between u_i and u_{i+1} for $i = 1, \ldots, n-1$. Obviously, all the arcs $x_1, \ldots, x_{n-1}$ are different by Lemma 2 and by the definition of a 1-element node. The nodes u_1 and u_2 are different by the assumption. So by Corollary 2 we can assume, e.g., that

$$u_2 \setminus \{x_1\} \subset f(x_1)$$

and

$$u_1 \setminus \{x_1\} \subset g(x_1), \qquad x_2 \in u_3, \qquad x_2 \in u_2 \setminus \{x_1\} \subset f(x_2).$$

Then by Lemma 16 we have $u_3 \setminus \{x_1\} \subset f(x_1)$ and going on we conclude that

$$u_n = u_1 \setminus \{x_1\} \subset f(x_1).$$

This however gives a contradiction since $u_1 \setminus \{x_1\} \subset g(x_1)$.

From Theorem 6 it follows that the graph $\langle V, \vec{R} \rangle$ is connected and does not contain simple cycles, so it is a generalization of a tree to an infinite case. A dendroid $\langle X, f \rangle$ is called *finite* if it satisfies axioms (A1)–(A3) and additionally the condition

$$|X| < \infty.$$

Since a finite dendroid is locally finite, Theorem 6 implies the following

THEOREM 7. *The graph $\langle V, \vec{R} \rangle$ of a finite dendroid $\langle X, f \rangle$ in which $|X| > 1$ is a directed tree.*

References

[1] C. Berge, *Graphs and Hypergraphs*, North-Holland, Amsterdam 1973.
[2] A. Koślinski, *Decompositions of graphs into the sum of direct systems and decompositions of trees* (in Polish), Report No. 211, Institute of Material Science and Applied Mechanics, Technical University of Wrocław, 1978.
[3] – and L. Szamkołowicz, *On a definition of the directed tree* in: *Hypergraphs and Block Systems*, Zielona Góra 1976.
[4] L. Szamkołowicz, *Theory of Finite Graphs* (in Polish), Ossolineum, Wrocław 1971.

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