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# SOLUTION OF SOME PROBLEMS OF MINIMAX CONTROL FOR A MULTIVARIATE LINEAR STOCHASTIC SYSTEM

In the paper, problems of optimal control are considered for a multivariate linear stochastic system defined by (1) when the risk function is given by (4). The situation is investigated in which the disturbances in the system depend on an unknown parameter $\lambda$ but may have different other (known) parameters and belong to the exponential family with quadratic variance function. A special attention is devoted to the binomial distribution. Several results are obtained concerning the different considered situations connected with the available information. In the problems solved in the paper, Bayes, minimax and $\Gamma$-minimax control policies are obtained in an analytic form.

**1. Preliminary remarks and definitions.** The system considered in the paper is defined by the equation

$$x_{n+1} = A_n x_n + B_n u_n + C_n v_n \quad (n = 0, 1, \ldots, N-1), \tag{1}$$

where $x_n$ is the state variable, $u_n$ is the control, $v_n$ is the disturbance of the system at time $n$, $N$ $(1 \leqslant N \leqslant M)$ is a random variable independent of the process $v_n$ $(n = 0, 1, \ldots, M-1)$, $A_n$, $B_n$ and $C_n$ are $(s \times s)$-, $(s \times l)$- and $(s \times s)$-matrices, respectively.

Obviously, $x_n$, $u_n$, $v_n$ are $s$-, $l$-, $s$-dimensional column vectors, respectively, $v_n$ being a random variable.

It is assumed that the data available at time $n$ are $X_n = (x_0, x_1, \ldots, x_n)$, and $U_{n-1} = (u_0, u_1, \ldots, u_{n-1})$ and that $C_n$ $(n = 0, 1, \ldots, M-1)$ are nonsingular matrices. It follows then from (1) that at time $n$ the values of the random variables $v_j$ $(j = 0, 1, \ldots, n-1)$ are known. It follows also that at the beginning of the control the state $x_0$ is given.

The control $u_n$ is assumed to be a Borel (vector-valued) function of $(X_n, U_{n-1})$.

Put

$$v_n = \begin{bmatrix} v_{n1} \\ v_{n2} \\ \cdots \\ v_{ns} \end{bmatrix}.$$

It is supposed that the random variables $v_{nj}$ $(n = 0, 1, \ldots, M-1;\ j = 1, \ldots, s)$ are independent and have the distributions $P_\lambda^{(n,j)}$ belonging to the exponential family, i.e. their densities with respect to the $\sigma$-finite measure $\mu$ on $R^1 = (-\infty, \infty)$ are

(2) $$p_{nj}(v, \lambda) = S(v, q_{nj})\exp[q_{nj}A(\lambda)+vB(\lambda)],$$

where $\lambda \in \Lambda$ is a parameter.

From formula (2) it follows that the random variables

$$\hat{v}_n = \sum_{j=1} v_{nj} \quad \text{and} \quad z_n = \sum_{i=0}^{n-1} \sum_{j=1}^{s} v_{ij}$$

have distributions belonging to the same family as $v_{ij}$ and that $\hat{v}_n$ and $z_n$ are sufficient statistics for $\lambda$.

It is assumed that natural parametrization is chosen for which the conditions of the paper [6] hold and that the variance $D_\lambda^2(v_{nj})$ of the random variable $v_{nj}$ is quadratic in $\lambda$. Then we can assume that

$$E_\lambda(v_{nj}) = q_{nj}\lambda, \quad E_\lambda(v_{nj}^2) = q_{nj}^{(1)}\lambda^2 + q_{nj}^{(2)}\lambda + q_{nj}^{(3)}$$

for some constants $q_{nj} > 0$, $q_{nj}^{(1)}$, $q_{nj}^{(2)}$, $q_{nj}^{(3)}$. Let us notice that the constants

(3) $$\varepsilon_1 = \frac{q_{nj}^{(1)} - q_{nj}^2}{q_{nj}}, \quad \varepsilon_2 = \frac{q_{nj}^{(2)}}{q_{nj}}, \quad \varepsilon_3 = \frac{q_{nj}^{(3)}}{q_{nj}}$$

are independent of $n$ and $j$.

It is assumed that $p_{nj}(v, \lambda)$ $(n = 0, 1, \ldots, M-1;\ j = 1, \ldots, s)$ are known with the only exception of the parameter $\lambda$.

Random variables belonging to the exponential class are considered in [1], [2], and [4].

Let $u_0, u_1, \ldots, u_M$ be controls. The $(M+1)$-tuplet $U = (u_0, u_1, \ldots, u_M)$ is called a *control policy*.

It is supposed that the horizon $N$ of the control has a given distribution

$$P(N = n) = p_n, \quad p_M > 0, \quad \sum_{n=1}^{M} p_n = 1.$$

In the paper, the matrix transposed to a matrix $A$ is denoted by $A'$.

Let us define the *risk function* for the control policy $U$ by

(4) $$R(\lambda, U) = E_p\{E_\lambda[\sum_{i=0}^{N} (x_i', \lambda)S_i(x_i', \lambda)' + u_i' K_i u_i]\},$$

where

$E_p(\cdot)$ denotes the expectation with respect to the distribution of the random variable $N$:

$E_\lambda(\cdot)$ denotes the expectation with respect to the distribution of the random variables $v_{nj}$ ($n = 0, 1, \ldots, M-1$; $j = 1, \ldots, s$) for fixed $\lambda$;

$(x_i', \lambda)$ is the vector $x_i'$ with the added coordinate $\lambda$;

$S_i$, $K_i$ are $(s+1)\times(s+1)$, $l \times l$ symmetric nonnegative definite matrices, respectively.

We consider only these control policies $U$ for which the risk $R(\lambda, U)$ is finite for each $\lambda \in \Lambda$. The set of all these policies is denoted by $\Delta$.

A policy $U^{(0)} \in \Delta$ such that

(5) $$\sup_{\lambda\in\Lambda} R(\lambda, U^{(0)}) = \inf_{U\in\Delta} \sup_{\lambda\in\Lambda} R(\lambda, U)$$

is called a *minimax control policy.*

Sometimes the parameter $\lambda$ is a random variable with the a priori distribution $\pi$. Then, assuming the integral to exist, the functional

(6) $$r(\pi, U) = \int_{\Lambda} R(\lambda, U)\pi(d\lambda)$$

is called the *Bayes risk.*

A policy $U_\pi \in \Delta$ such that

$$r(\pi, U_\pi) = \inf_{U} r(\pi, U)$$

is called a *Bayes control policy with respect to* $\pi$.

A control policy which is Bayes with respect to some $\pi$ is called a *Bayes policy.*

We sometimes have the information that $\pi \in \Gamma$, where $\Gamma$ is known. Let $\Delta_\Gamma$ be the set of all control policies $U \in \Delta$ for which the Bayes risk $r(\pi, U)$ exists for all $\pi \in \Gamma$. A policy $U^{(0)} \in \Delta_\Gamma$ such that

$$\sup_{\pi\in\Gamma} r(\pi, U^{(0)}) = \inf_{U\in\Delta_\Gamma} \sup_{\pi\in\Gamma} r(\pi, U)$$

is called a *$\Gamma$-minimax control policy.*

The aim of this paper is to determine Bayes, minimax and $\Gamma$-minimax control policies for the system (1).

**2. Natural exponential families with quadratic variance function.** Suppose that the random variables $v_{nj}$ ($n = 0, 1, \ldots, M-1$; $j = 1, \ldots, s$) are distributed according to the densities (2). Denote by $\pi_{\beta,r}$ the a priori distribution of the parameter $\lambda$ with density

(7) $$g(\lambda, \beta, r) = C(\beta, r)\frac{dB(\lambda)}{d\lambda}\exp[\beta A(\lambda) + rB(\lambda)].$$

Write

$$\beta_n = \beta + \sum_{i=0}^{n-1}\sum_{j=1}^{s} q_{ij}, \qquad r_n = r + z_n = r + \sum_{i=0}^{n-1}\sum_{j=1}^{s} v_{ij}.$$

If the distribution $\pi_{\beta,r}$ is assigned to $\lambda$, then the density of the a posteriori distribution of $\lambda$ given $z_n$ is $g(\lambda, \beta_n, r_n)$. Then it is of the same form as the a priori density and only new parameters $\beta_n$ and $r_n$ must be computed.

In [6] it is proved that there are only six distributions (and linear transformations of them) belonging to the exponential class for which the variance is a quadratic function of the mean. The distributions are the following (the measure $\mu$ with respect to which the densities $p(v, \lambda)$ are given is the Lebesgue or the counting measure):

(a) the Poisson distribution

$$p(v, \lambda) = \frac{(q\lambda)^v}{v!} e^{-q\lambda} \quad (\lambda > 0);$$

(b) the gamma distribution

$$p(v, \lambda) = \frac{1}{\Gamma(q)\lambda^q} v^{q-1} e^{-v/\lambda} I_{(0,\infty)}(v) \quad (\lambda > 0),$$

where $I_A$ is the characteristic function of the set $A$;

(c) the negative binomial distribution

$$p(v, \lambda) = \frac{\Gamma(q+v)}{\Gamma(q)\, v!} \frac{\lambda^v}{(1+\lambda)^{q+v}} \quad (\lambda > 0);$$

(d) the binomial distribution

$$p(v, \lambda) = \binom{q}{v} \lambda^v (1-\lambda)^{q-v} \quad (0 < \lambda < 1);$$

(e) the normal distribution (with variance $q$)

$$p(v, \lambda) = \frac{1}{\sqrt{2\pi q}} \exp\left[-\frac{(v-q\lambda)^2}{2q}\right] \quad (-\infty < \lambda < \infty);$$

(f) the generalized hyperbolic secant distribution (GEHS) ([3], [6])

$$p(v, \lambda) = \frac{1}{(1+\lambda^2)^{q/2}} \exp[v \operatorname{arc\,tg} \lambda]\, S(v, q),$$

where

$$S(v, q) = \frac{2^{q-2}}{\pi} B\left(\frac{q}{2} - i\frac{v}{2}, \frac{q}{2} + i\frac{v}{2}\right)$$

$$= \frac{2^{q-2}\Gamma^2(q/2)}{\pi\Gamma(q)} \prod_{k=0}^{\infty} \left(1 + \frac{v^2}{(q+2k)^2}\right)^{-1}.$$

For the random variable $v$ having the GEHS distribution we get

$$\mathrm{E}_\lambda(v) = q\lambda, \qquad \mathrm{E}_\lambda(v^2) = q(q+1)\lambda^2 + q.$$

From (a)–(f) it follows that $p_{nj}(v, \lambda) = p(v, \lambda)$ for $q = q_{nj}$.

Denote by $S$ the set of all $(\beta, r) \in R^2$ for which $\mathrm{E}_{\pi_{\beta,r}}(\lambda^2) < \infty$. We can verify that, for $(\beta, r) \in S$,

$$\mathrm{E}_{\pi_{\beta,r}}(\lambda) = \frac{r}{\beta},$$

(8)
$$\mathrm{E}_{\pi_{\beta,r}}(\lambda^2) = \frac{(r+q^{(2)}/q)r}{(\beta-(q^{(1)}-q^2)/q)\beta} + \frac{q^{(3)}/q}{\beta-(q^{(1)}-q^2)/q} \stackrel{\mathrm{df}}{=} k(\beta, r)$$
$$\stackrel{\mathrm{df}}{=} T_1^{(0)} r^2 + T_2^{(0)} r + T_3^{(0)}$$

(compare with (3)), where $\mathrm{E}_\pi(\cdot)$ is the expectation with respect to the a priori distribution $\pi$ of the parameter $\lambda$.

For the distributions listed in (a)–(f), respectively, the sets $S$ are

(a) $\{\beta > 0,\ r > 0\}$; (b) $\{\beta > 1,\ r > 0\}$; (c) $\{\beta > 1,\ r > 0\}$;

(d) $\{\beta - r > 0,\ r > 0\}$; (e) $\{\beta > 0,\ -\infty < r < \infty\}$; (f) $\{\beta > 1,\ -\infty < r < \infty\}$.

**3. Filtration problems.** Let the a priori density $g(\lambda, \beta, r)$ of the parameter $\lambda$ be given by (7). Then the a posteriori density of this parameter given $z_n$ is $g(\lambda, \beta_n, r_n)$. Moreover, by the Bayes formula we have

(9)
$$g(\lambda, \beta_{n+1}, r_{n+1}) = \frac{p_{n1}(v_{n1}, \lambda) \ldots p_{ns}(v_{ns}, \lambda) g(\lambda, \beta_n, r_n)}{\int_\Lambda p_{n1}(v_{n1}, \lambda) \ldots p_{ns}(v_{ns}, \lambda) g(\lambda, \beta_n, r_n) d\lambda}.$$

From equation (9) it follows that the distribution $p(v_{n1}, \ldots, v_{ns} \mid z_n)$ of the random variable $(v_{n1}, \ldots, v_{ns})$ given $z_n$ is

(10)
$$p(v_{n1}, \ldots, v_{ns} \mid z_n) = \int_\Lambda p_{n1}(v_{n1}, \lambda) \ldots p_{ns}(v_{ns}, \lambda) g(\lambda, \beta_n, r_n) d\lambda$$
$$= \frac{S(v_{n1}, q_{n1}) \ldots S(v_{ns}, q_{ns}) C(\beta_n, r_n)}{C(\beta_n + \sum_{j=0}^{s} g_{nj},\ r_n + \sum_{j=0}^{s} v_{nj})}.$$

In a similar way one obtains the distributions of the random variables $(v_{nj}, v_{nk})$ and $v_{nj}$ given $z_n$, respectively, in the forms

(11)
$$p(v_{nj}, v_{nk} \mid z_n) = \frac{S(v_{nj}, g_{nj}) S(v_{nk}, q_{nk}) C(\beta_n, r_n)}{C(\beta_n + q_{nj} + q_{nk},\ r_n + v_{nj} + v_{nk})},$$

(12)
$$p(v_{nj} \mid z_n) = \frac{S(v_{nj}, q_{nj}) C(\beta_n, r_n)}{C(\beta_n + q_{nj},\ r_n + v_{nj})}.$$

Assume that the random variable $Y$ is a function of the random variables $v_{01}, \ldots, v_{M-1,s}, \lambda$. We have

(13)
$$\mathrm{E}(Y \mid X_n, U_{n-1}) = \mathrm{E}(Y \mid z_n),$$

where $E(Y|z_n)$ is the conditional expectation of $Y$ given $z_n$, and $E(Y|X_n, U_{n-1})$ is the conditional expectation of $Y$ given $(X_n, U_{n-1})$ for unconditional measures of $(Y, z_n)$ and $(Y, v_{01}, \ldots, v_{n-1,s})$ determined by the distributions of $v_{01}, \ldots, v_{M-1,s}$ given $\lambda$ and the distribution of $\lambda$.

The equality in (13) follows from the remark that the pair $(X_n, U_{n-1})$ determines $v_{01}, \ldots, v_{n-1,s}$ (by (1)), that $v_{01}, \ldots, v_{n-1,s}$ determine $X_n$ for $U_{n-1}$ known, and that $z_n$ is a sufficient statistic for $v_{01}, \ldots, v_{n-1,s}$.

Taking into account (11)–(13) we obtain for $\pi = \pi_{\beta,r}$, $(\beta, r) \in S$, the equalities

$$E(v_{nj}|X_n, U_{n-1}) = q_{nj}\frac{r_n}{\beta_n} \stackrel{\mathrm{df}}{=} Q^{(n,j)} r_n, \tag{14}$$

$$E(v_{nj}^2|X_n, U_{n-1}) = Q_1^{(n,j)} r_n^2 + Q_2^{(n,j)} r_n + Q_3^{(n,j)}, \tag{15}$$

$$\begin{aligned} E(v_{nj} v_{nk}|X_n, U_{n-1}) &= q_{nj} q_{nk} k(\beta_n, r_n) \\ &= q_{nj} q_{nk} (T_1^{(n)} r_n^2 + T_2^{(n)} r_n + T_3^{(n)}) \qquad (j \neq k) \end{aligned} \tag{16}$$

and the coefficients $Q_1^{(n,j)}$, $Q_2^{(n,j)}$, $Q_3^{(n,j)}$ are

| | (a) | (b) | (c) | (d) | (e) | (f) |
|---|---|---|---|---|---|---|
| $Q_1^{(n,j)}$ | $\frac{q_{nj}}{\beta_n^2}$ | $\frac{q_{nj}(q_{nj}+1)}{\beta_n(\beta_n-1)}$ | $\frac{q_{nj}(q_{nj}+1)}{\beta_n(\beta_n-1)}$ | $\frac{q_{nj}(q_{nj}-1)}{\beta_n(\beta_n+1)}$ | $\frac{q_{nj}^2}{\beta_n^2}$ | $\frac{q_{nj}(q_{nj}+1)}{\beta_n(\beta_n-1)}$ |
| $Q_2^{(n,j)}$ | $\frac{q_{nj}(q_{nj}+\beta_n)}{\beta_n^2}$ | 0 | $\frac{q_{nj}(q_{nj}+\beta_n)}{\beta_n(\beta_n-1)}$ | $\frac{q_{nj}(q_{nj}+\beta_n)}{\beta_n(\beta_n+1)}$ | 0 | 0 |
| $Q_3^{(n,j)}$ | 0 | 0 | 0 | 0 | $\frac{q_{nj}(q_{nj}+\beta_n)}{\beta_n}$ | $\frac{q_{nj}(q_{nj}+\beta_n)}{\beta_n-1}$ |

for the distributions listed in (a)–(f), respectively. Moreover,

$$E(\lambda|X_n, U_{n-1}) = \frac{r_n}{\beta_n}, \tag{17}$$

$$E(\lambda^2|X_n, U_{n-1}) = T_1^{(n)} r_n^2 + T_2^{(n)} r_n + T_3^{(n)}. \tag{18}$$

**4. Bayes control policies.** Suppose that the a priori distribution $\pi = \pi_{\beta,r}$ of the parameter $\lambda$ is defined by (7) and that $(\beta, r) \in S$. Consider the problem of determining the Bayes control for the system (1) in the case where the risk function is given by (4). Write

$$\begin{aligned} r_n(\pi, U^{(n)}) &= E_p\left\{E\left[\sum_{i=n}^{N} ((x_i', \lambda) S_i (x_i', \lambda)' + u_i' K_i u_i) \mid X_n, U_{n-1}\right] \mid N \geqslant n\right\} \\ &= E\left\{\sum_{i=n}^{M} \frac{\pi_i}{\pi_n} ((x_i', \lambda) S_i (x_i', \lambda)' + u_i' K_i u_i) \mid X_n, U_{n-1}\right\}, \end{aligned}$$

where, for given $X_n$ and $U_{n-1}$, we have the policy $U^{(n)}=(u_n, \ldots, u_M)$ and

$$\pi_n = \sum_{i=n}^{M} p_i .$$

Obviously, $r(\pi, U) = r_0(\pi, U^{(0)})$.

Let for $\pi = \pi_{\beta, r}$, $(\beta, r) \in S$,

$$W_n = \min_{U^{(n)}} r_n(\pi, U^{(n)}).$$

Applying the optimality principle of dynamic programming we obtain

$$\begin{aligned}
(19)\quad W_n &= \min_{u_n} \Big\{ \mathrm{E}\,[(x_n', \lambda) S_n (x_n', \lambda)' + u_n' K_n u_n \mid X_n, U_{n-1}] + \\
&\quad + \min_{U^{(n+1)}} \mathrm{E}\Big[ \sum_{i=n+1}^{M} \frac{\pi_i}{\pi_n} [(x_i', \lambda) S_i (x_i', \lambda)' + u_i' K_i u_i] \mid X_n, U_{n-1} \Big] \Big\} \\
&= \min_{u_n} \Big\{ \mathrm{E}\,[(x_n', \lambda) S_n (x_n', \lambda)' \mid X_n, U_{n-1}] + u_n' K_n u_n + \\
&\quad + \frac{\pi_{n+1}}{\pi_n} \mathrm{E}\Big[ \mathrm{E}\Big[ \sum_{i=n+1}^{M} \frac{\pi_i}{\pi_{n+1}} [(x_i', \lambda) S_i (x_i', \lambda) + \\
&\qquad + u_i' K_i u_i] \mid X_{n+1}, U_n \Big] \mid X_n, U_{n-1} \Big] \Big\} \\
&= \min_{u_n} \Big\{ \mathrm{E}\,[(x_n', \lambda) S_n (x_n', \lambda)' \mid X_n, U_{n-1}] + u_n' K_n u_n + \\
&\qquad + \frac{\pi_{n+1}}{\pi_n} \mathrm{E}(W_{n+1} \mid X_n, U_{n-1}) \Big\}.
\end{aligned}$$

We show that $W_n$ can be presented in the form

$$(20)\qquad W_n = x_n' D_n x_n + 2F_n x_n \frac{r_n}{\beta_n} + G_n r_n^2 + H_n r_n + I_n,$$

where $D_n$ is an $(s \times s)$ symmetric matrix nonnegative definite, $F_n$ is an $s$-dimensional vector, and $G_n$, $H_n$, $I_n$ are scalars.

Divide the matrix $S_n$ into submatrices

$$S_n = \begin{bmatrix} S_n^{(1)} & s_n' \\ s_n & s_n^{s+1} \end{bmatrix},$$

where $s_n^{s+1}$ is a scalar. For $n = M$ equation (20) holds with

$$(21)\qquad \begin{aligned} D_M &= S_M^{(1)}, \quad F_M = s_M, \quad G_M = s_M^{s+1} T_1^{(M)}, \\ H_M &= s_M^{s+1} T_2^{(M)}, \quad I_M = s_M^{s+1} T_3^{(M)} \end{aligned}$$

and the optimal control $u_M^* = 0$.

Assume that equation (20) is true for $n+1$. Since $x_{n+1} = A_n x_n + B_n u_n + C_n v_n$, $W_n$ exists in (19), and to determine the Bayes control it is sufficient to solve the equation

$$2K_n u_n + \frac{\pi_{n+1}}{\pi_n} \operatorname{grad}_{u_n} \mathrm{E}(W_{n+1} \mid X_n, U_{n-1}) = 0, \tag{22}$$

where $\operatorname{grad}_{u_n} \mathrm{E}(W_{n+1} \mid X_n, U_{n-1})$ is the column vector defined as usual.

But $r_{n+1} = r_n + \hat{v}_n$ and we have

$$\begin{aligned} \mathrm{E}(W_{n+1} \mid X_n, U_{n-1}) &= \mathrm{E}\Big[(A_n x_n + B_n u_n + C_n v_n)' D_{n+1} (A_n x_n + B_n u_n + C_n v_n) + \\ &+ \frac{2}{\beta_{n+1}} (A_n x_n + B_n u_n + C_n v_n)(r_n + \hat{v}_n) + G_{n+1} (r_n + \hat{v}_n)^2 + \\ &+ H_{n+1} (r_n + \hat{v}_n) + I_{n+1} \mid X_n, U_{n-1}\Big]. \end{aligned} \tag{23}$$

Then from (22) using (14), we obtain, for the Bayes control $u_n^*$,

$$\begin{aligned} &\left[K_n + \frac{\pi_{n+1}}{\pi_n} B_n' D_{n+1} B_n\right] u_n^* \\ &\qquad = -\frac{\pi_{n+1}}{\pi_n}\left[B_n' D_{n+1} A_n x_n + B_n' (D_{n+1} C_n q_n + F_{n+1}') \frac{r_n}{\beta_n}\right], \end{aligned} \tag{24}$$

where

$$q_n = \begin{bmatrix} q_{n1} \\ q_{n2} \\ \cdots \\ q_{ns} \end{bmatrix}.$$

Assume that equation (24) has the solution $u_n^*$. Then the Bayes control is

$$u_n^* = -P_n x_n - Q_n \frac{r_n}{\beta_n}, \tag{25}$$

where

$$\begin{aligned} P_n &= \frac{\pi_{n+1}}{\pi_n}\left(K_n + \frac{\pi_{n+1}}{\pi_n} B_n' D_{n+1} B_n\right)^+ B_n' D_{n+1} A_n, \\ Q_n &= \frac{\pi_{n+1}}{\pi_n}\left(K_n + \frac{\pi_{n+1}}{\pi_n} B_n' D_{n+1} B_n\right)^+ B_n' (D_{n+1} C_n q_n + F_{n+1}'), \end{aligned} \tag{26}$$

and $A^+$ is the Moore–Penrose pseudoinverse matrix to the matrix $A$ (see [5], p. 407).

Using (14)–(18), (23) and (25), we obtain

(27) $$\mathrm{E}[(x_n', \lambda) S_n (x_n', \lambda)' \mid X_n, U_{n-1}]$$
$$= x_n' S_n^{(1)} x_n + 2 s_n x_n \frac{r_n}{\beta_n} + s_n^{s+1} (T_1^{(n)} r_n^2 + T_2^{(n)} r_n + T_3^{(n)}),$$

(28) $$\mathrm{E}(W_{n+1} \mid X_n, U_{n-1})$$
$$= x_n' (A_n - B_n P_n)' D_{n+1} (A_n - B_n P_n) x_n -$$
$$-2Q_n' B_n' D_{n+1} (A_n - B_n P_n) x_n \frac{r_n}{\beta_n} + Q_n' B_n' D_{n+1} B_n Q_n \frac{r_n^2}{\beta_n^2} +$$
$$+2q_n' C_n' D_{n+1} (A_n - B_n P_n) x_n \frac{r_n}{\beta_n} - 2q_n' C_n' D_{n+1} B_n Q_n \frac{r_n^2}{\beta_n^2} +$$
$$+[q_n' C_n' D_{n+1} C_n q_n - q_n' \operatorname{diag}(C_n' D_{n+1} C_n) q_n] (T_1^{(n)} r_n^2 + T_2^{(n)} r_n + T_3^{(n)}) +$$
$$+Q_{1,1/2}^{(n)\prime} \operatorname{diag}(C_n' D_{n+1} C_n) Q_{1,1/2}^{(n)} r_n^2 +$$
$$+Q_{2,1/2}^{(n)\prime} \operatorname{diag}(C_n' D_{n+1} C_n) Q_{2,1/2}^{(n)} r_n + Q_{3,1/2}^{(n)\prime} \operatorname{diag}(C_n' D_{n+1} C_n) Q_{3,1/2}^{(n)} +$$
$$+2F_{n+1} (A_n - B_n P_n) x_n \frac{r_n}{\beta_n} - 2F_{n+1} B_n Q_n \frac{r_n^2}{\beta_n^2} +$$
$$+\frac{2}{\beta_{n+1}} F_{n+1} C_n \left[ q_n \frac{r_n^2}{\beta_n} + (q_n \hat{q}_n - \overline{q_n^2})(T_1^{(n)} r_n^2 + T_2^{(n)} r_n + T_3^{(n)}) + \right.$$
$$\left. + Q_1^{(n)} r_n^2 + Q_2^{(n)} r_n + Q_3^{(n)} \right] + G_{n+1} \left[ r_n^2 + 2\hat{q}_n \frac{r_n^2}{\beta_n} + \right.$$
$$+(\hat{q}_n^2 - q_n' q_n)(T_1^{(n)} r_n^2 + T_2^{(n)} r_n + T_3^{(n)}) +$$
$$\left. + \hat{Q}_1^{(n)} r_n^2 + \hat{Q}_2^{(n)} r_n + \hat{Q}_3^{(n)} \right] + H_{n+1} \left(1 + \frac{\hat{q}_n}{\beta_n}\right) + I_{n+1},$$

where

$$Q_k^{(n)} = \begin{bmatrix} Q_k^{(n,1)} \\ Q_k^{(n,2)} \\ \cdots \\ Q_k^{(n,s)} \end{bmatrix}, \qquad Q_{k,1/2}^{(n)} = \begin{bmatrix} \sqrt{Q_k^{(n,1)}} \\ \sqrt{Q_k^{(n,2)}} \\ \cdots \\ \sqrt{Q_k^{(n,s)}} \end{bmatrix},$$

$$\hat{q}_n = \sum_{j=1}^{s} q_{nj}, \qquad \overline{q_n^2} = \begin{bmatrix} q_{n1}^2 \\ q_{n2}^2 \\ \cdots \\ q_{ns}^2 \end{bmatrix}, \qquad \hat{Q}_k^{(n)} = \sum_{j=1}^{s} Q_k^{(n,j)} \quad (k = 1, 2, 3).$$

Using (19) with the help of (25)–(28), we prove (20) with $D_n$, $F_n$, $G_n$, $H_n$, $I_n$ satisfying the equations

$$D_n = S_n^{(1)} + \frac{\pi_{n+1}}{\pi_n} A_n' D_{n+1} (A_n - B_n P_n),$$

$$F_n = s_n + \frac{\pi_{n+1}}{\pi_n} (q_n' C_n' D_{n+1} + F_{n+1})(A_n - B_n P_n),$$

(29)
$$\begin{aligned}
G_n = {} & s_n^{s+1} T_1^{(n)} + \\
& + \frac{\pi_{n+1}}{\pi_n} \Big\{ [q_n' C_n' D_{n+1} C_n q_n - q_n' \operatorname{diag}(C_n' D_{n+1} C_n) q_n] T_1^{(n)} + \\
& + Q_{1,1/2}^{(n)\prime} \operatorname{diag}(C_n' D_{n+1} C_n) Q_{1,1/2}^{(n)} - (q_n' C_n' D_{n+1} + F_{n+1}) B_n Q_n \frac{1}{\beta_n^2} + \\
& + \frac{2}{\beta_{n+1}} F_{n+1} C_n \left[ \frac{q_n}{\beta_n} + (q_n \hat{q}_n - \overline{q_n^2}) T_1^{(n)} + Q_1^{(n)} \right] + \\
& + G_{n+1} \left[ 1 + 2 \frac{\hat{q}_n}{\beta_n} + (\hat{q}_n^2 - q_n' q_n) T_1^{(n)} + \hat{Q}_1^{(n)} \right] \Big\}, \\
H_n = {} & s_n^{s+1} T_2^{(n)} + \frac{\pi_{n+1}}{\pi_n} \Big\{ q_n' C_n' D_{n+1} C_n q_n T_2^{(n)} - \\
& - q_n' \operatorname{diag}(C_n' D_{n+1} C_n) q_n T_2^{(n)} + Q_{2,1/2}^{(n)\prime} \operatorname{diag}(C_n' D_{n+1} C_n) Q_{2,1/2}^{(n)} + \\
& + \frac{2}{\beta_{n+1}} F_{n+1} C_n [(q_n \hat{q}_n - \overline{q_n^2}) T_2^{(n)} + Q_2^{(n)}] + \\
& + G_{n+1} [(\hat{q}_n^2 - q_n' q_n) T_2^{(n)} + \hat{Q}_2^{(n)}] + H_{n+1} \left(1 + \frac{\hat{q}_n}{\beta_n}\right) \Big\}, \\
I_n = {} & s_n^{s+1} T_3^{(n)} + \frac{\pi_{n+1}}{\pi_n} \Big\{ q_n' C_n' D_{n+1} C_n q_n T_3^{(n)} - \\
& - q_n' \operatorname{diag}(C_n' D_{n+1} C_n) q_n T_3^{(n)} + Q_{3,1/2}^{(n)\prime} \operatorname{diag}(C_n' D_{n+1} C_n) Q_{3,1/2} + \\
& + \frac{2}{\beta_{n+1}} F_{n+1} C_n [(q_n \hat{q}_n - \overline{q_n^2}) T_3^{(n)} + Q_3^{(n)}] + \\
& + G_{n+1} [(\hat{q}_n^2 - q_n' q_n) T_3^{(n)} + \hat{Q}_3^{(n)}] + I_{n+1} \Big\}
\end{aligned}$$

and the boundary conditions (21).

From (26) and (29) we obtain also the other form of the equation for $D_n$:

(30)
$$D_n = S_n^{(1)} + P_n' K_n P_n + \frac{\pi_{n+1}}{\pi_n} (A_n - B_n P_n)' D_{n+1} (A_n - B_n P_n).$$

From (21) and (30) it follows that $D_n$ is a symmetric nonnegative definite matrix.

In the paper we assume that

(A) For each $n = 0, 1, \ldots, M-1$ and for each $y' \in R^s$, $z \in R^1$ the equation

$$\left[K_n + \frac{\pi_{n+1}}{\pi_n} B_n' D_{n+1} B_n\right] u_n = -\frac{\pi_{n+1}}{\pi_n}[B_n' D_{n+1} A_n y + B_n'(D_{n+1} C_n q_n + F_{n+1}') z]$$

has a solution $u_n$, where $D_n$ and $F_n$ satisfy the first two equations in (29) and the boundary conditions (21).

**5. Determining the risk.** Let $u_n^*$ be the control defined by (25), (26) and (29), and let $U_{\beta,r}^* = (u_0^*, u_1^*, \ldots, u_M^*)$. For $(\beta, r) \in S$ we put

$$R_n(\lambda, U_{\beta,r}^*) = \mathrm{E}_\lambda\left[\sum_{i=n}^{M} \frac{\pi_i}{\pi_n}[(x_i', \lambda) S_i (x_i', \lambda)' + u_i^{*\prime} K_i u_i^*] \mid X_n, U_{n-1}^*\right], \tag{31}$$

where $U_{n-1}^* = (u_0^*, \ldots, u_{n-1}^*)$. Obviously,

$$R(\lambda, U_{\beta,r}^*) = R_0(\lambda, U_{\beta,r}^*). \tag{32}$$

Moreover,

$$\begin{aligned} R_n(\lambda, U_{\beta,r}^*) &= (x_n', \lambda) S_n (x_n', \lambda)' + u_n^{*\prime} K_n u_n^* + \\ &\quad + \frac{\pi_{n+1}}{\pi_n} \mathrm{E}_\lambda\left[\mathrm{E}_\lambda\left[\sum_{i=n+1}^{M} \frac{\pi_i}{\pi_{n+1}}[(x_i', \lambda) S_i (x_i', \lambda)' + u_i^{*\prime} K_i u_i^*] \right.\right. \\ &\qquad\qquad \left.\left. \mid X_{n+1}, U_n^* \right] \mid X_n, U_{n-1}^*\right] \\ &= (x_n', \lambda) S_n (x_n', \lambda)' + u_n^{*\prime} K_n u_n^* + \\ &\quad + \frac{\pi_{n+1}}{\pi_n} \mathrm{E}_\lambda [R_{n+1}(\lambda, U_{\beta,r}^*) \mid X_n, U_{n-1}^*]. \end{aligned} \tag{33}$$

From equations (32) and (33) the risk $R(\lambda, U_{\beta,r}^*)$ can be determined. We prove that $R_n(\lambda, U_{\beta,r}^*)$ can be expressed in the form

$$R_n(\lambda, U_{\beta,r}^*) = x_n' a_n x_n + 2b_n x_n \lambda + c_n r_n^2 + e_n \lambda^2 + 2f_n r_n \lambda + i_n \lambda + j_n. \tag{34}$$

For $n = M$ it is satisfied with

$$a_M = S_M^{(1)}, \quad b_M = s_M, \quad e_M = s_M^{s+1}, \quad c_M = f_M = i_M = j_M = 0 \tag{35}$$

because $u_M^* = 0$. But, for example,

$$\mathrm{E}_\lambda(x'_{n+1}a_{n+1}x_{n+1} \mid X_n, U^*_{n-1})$$

$$= \mathrm{E}_\lambda\,[(A_n x_n + B_n u_n^* + C_n v_n)' a_{n+1}(A_n x_n + B_n u_n^* + C_n v_n)]$$

$$= x'_n(A_n - B_n P_n)' a_{n+1}(A_n - B_n P_n) x_n -$$

$$-2Q'_n B'_n a_{n+1}(A_n - B_n P_n) x_n \frac{r_n}{\beta_n} + Q'_n B'_n a_{n+1} B_n Q_n \frac{r_n^2}{\beta_n^2} +$$

$$+2q'_n C'_n a_{n+1}(A_n - B_n P_n) x_n \lambda - 2q'_n C'_n a_{n+1} B_n Q_n \frac{r_n}{\beta_n}\lambda +$$

$$+q'_n C'_n a_{n+1} C_n q_n \lambda^2 - q'_n \operatorname{diag}(C'_n a_{n+1} C_n) q_n \lambda^2 +$$

$$+q^{(1)\prime}_{n,1/2} \operatorname{diag}(C'_n a_{n+1} C_n) q^{(1)}_{n,1/2} \lambda^2 + q^{(2)\prime}_{n,1/2} \operatorname{diag}(C'_n a_{n+1} C_n) q^{(2)}_{n,1/2} \lambda +$$

$$+q^{(3)\prime}_{n,1/2} \operatorname{diag}(C'_n a_{n+1}) q^{(3)}_{n,1/2},$$

where

$$q^{(k)}_{n,1/2} = \begin{bmatrix} \sqrt{q^{(k)}_{n1}} \\ \sqrt{q^{(k)}_{n2}} \\ \cdots\cdots \\ \sqrt{q^{(k)}_{ns}} \end{bmatrix} \qquad (k = 1, 2, 3).$$

Assuming the equation (34) to hold for $n+1$ and using this in (33) we prove (34) for $n$ with $a_n, \ldots, j_n$ satisfying the equations

$$a_n = S_n^{(1)} + P'_n K_n P_n + \frac{\pi_{n+1}}{\pi_n}(A_n - B_n P_n)' a_{n+1}(A_n - B_n P_n),$$

$$b_n = s_n + \frac{\pi_{n+1}}{\pi_n}(q'_n C'_n a_{n+1} + b_{n+1})(A_n - B_n P_n),$$

$$c_n = Q'_n K_n Q_n \frac{1}{\beta_n^2} + \frac{\pi_{n+1}}{\pi_n}\left(Q'_n B'_n a_{n+1} B_n Q_n \frac{1}{\beta_n^2} + c_{n+1}\right),$$

$$e_n = s_n^{s+1} + \frac{\pi_{n+1}}{\pi_n}[q'_n C'_n a_{n+1} C_n q_n - q'_n \operatorname{diag}(C'_n a_{n+1} C_n) q_n +$$

$$+q^{(1)\prime}_{n,1/2} \operatorname{diag}(C'_n a_{n+1} C_n) q^{(1)}_{n,1/2} + 2b_{n+1} C_n q_n +$$

$$+c_{n+1}(\hat{q}_n^2 - q'_n q_n + \hat{q}_n^{(1)}) + e_{n+1} + 2f_{n+1}\hat{q}_n],$$

$$f_n = \frac{\pi_{n+1}}{\pi_n}\left[-(q'_n C'_n a_{n+1} + b_{n+1}) B_n Q_n \frac{1}{\beta_n} + c_{n+1}\hat{q}_n + f_{n+1}\right],$$

$$i_n = \frac{\pi_{n+1}}{\pi_n}[q^{(2)\prime}_{n,1/2} \operatorname{diag}(C'_n a_{n+1} C_n) q^{(2)}_{n,1/2} + c_{n+1}\hat{q}_n^{(2)} + i_{n+1}],$$

$$j_n = \frac{\pi_{n+1}}{\pi_n}[q^{(3)\prime}_{n,1/2} \operatorname{diag}(C'_n a_{n+1} C_n) q^{(3)}_{n,1/2} + c_{n+1}\hat{q}_n^{(3)} + j_{n+1}],$$

where

$$\hat{q}_n^{(k)} = \sum_{j=1}^{s} q_{nj}^{(k)} \quad (k = 1, 2, 3).$$

Solving this system of equations with the boundary conditions (35) we obtain

$$a_n = D_n, \quad b_n = F_n, \quad c_n = \frac{1}{\pi_n} \sum_{i=n}^{M-1} \frac{\varkappa_i}{\beta_i^2},$$

$$e_n = \frac{1}{\pi_n}\left[ \varkappa_n^{(1)} - 2 \sum_{i=n+1}^{M-1} (h_i - h_n)\frac{\varkappa_i}{\beta_i} + \right.$$

(36)
$$\left. + \sum_{i=n+1}^{M-1} (h_i^{(0)} + h_i^{(1)} - h_n^{(0)} - h_n^{(1)})\frac{\varkappa_i}{\beta_i^2} + 2 \sum_{i=n}^{M-1} \hat{q}_i \sum_{j=i+1}^{M-1} (h_j - h_{i+1})\frac{\varkappa_j}{\beta_j^2}\right],$$

$$f_n = \frac{1}{\pi_n}\left[ - \sum_{i=n}^{M-1} \frac{\varkappa_i}{\beta_i} + \sum_{i=n+1}^{M-1} (h_i - h_n)\frac{\varkappa_i}{\beta_i^2}\right],$$

$$i_n = \frac{1}{\pi_n}\left[ \varkappa_n^{(2)} + \sum_{i=n+1}^{M-1} (h_i^{(2)} - h_n^{(2)})\frac{\varkappa_i}{\beta_i^2}\right],$$

$$j_n = \frac{1}{\pi_n}\left[ \varkappa_n^{(3)} + \sum_{i=n+1}^{M-1} (h_i^{(3)} - h_n^{(3)})\frac{\varkappa_i}{\beta_i^2}\right],$$

where

$$\varkappa_n = \pi_n Q_n' K_n Q_n + \pi_{n+1} Q_n' B_n' D_{n+1} B_n Q_n,$$

$$h_n = \sum_{i=0}^{n-1} \hat{q}_i = \sum_{i=0}^{n-1} \sum_{j=1}^{s} q_{ij}, \quad h_n^{(0)} = \sum_{i=0}^{n-1} (\hat{q}_i^2 - q_i' q_i),$$

$$h_n^{(k)} = \sum_{i=0}^{n-1} \hat{q}_i^{(k)} \quad (k = 1, 2, 3),$$

$$\varkappa_n^{(1)} = \pi_n s_n^{s+1} + \sum_{i=n}^{M-1} \pi_{i+1} [q_i' C_i' D_{i+1} C_i q_i -$$

$$- q_i' \operatorname{diag}(C_i' D_{i+1} C_i) q_i + q_{i,1/2}^{(1)\prime} \operatorname{diag}(C_i' D_{i+1} C_i) q_{i,1/2}^{(1)} +$$

$$+ 2F_{i+1} C_i q_i + s_{i+1}^{s+1}],$$

$$\varkappa_n^{(2)} = \sum_{i=n}^{M-1} \pi_{i+1} q_{i,1/2}^{(2)\prime} \operatorname{diag}(C_i' D_{i+1} C_i) q_{i,1/2}^{(2)},$$

$$\varkappa_n^{(3)} = \sum_{i=n}^{M-1} \pi_{i+1} q_{i,1/2}^{(3)\prime} \operatorname{diag}(C_i' D_{i+1} C_i) q_{i,1/2}^{(3)}.$$

But

$$2\sum_{i=0}^{M-1}\hat{q}_i\sum_{j=i+1}^{M-1}(h_j-h_{i+1})\frac{\varkappa_j}{\beta_j^2}=2\sum_{j=1}^{M-1}\Big(h_j^2-\sum_{i=0}^{j-1}\hat{q}_i h_{i+1}\Big)\frac{\varkappa_j}{\beta_j^2}$$

$$=\sum_{j=1}^{M-1}\Big[2h_j^2+\sum_{i=0}^{j-1}(h_i^2-h_{i+1}^2-\hat{q}_i^2)\Big]\frac{\varkappa_j}{\beta_j^2}=\sum_{j=1}^{M-1}\Big(h_j^2-\sum_{i=0}^{j-1}\hat{q}_i^2\Big)\frac{\varkappa_j}{\beta_j^2}$$

and the constant $e_0$ can be written in the simple form

$$e_0=\varkappa_0^{(1)}-2\sum_{i=1}^{M-1}h_i\frac{\varkappa_i}{\beta_i}+\sum_{i=1}^{M-1}\Big(h_i^{(1)}+h_i^2-\sum_{j=0}^{i-1}q_j'q_j\Big)\frac{\varkappa_i}{\beta_i^2}. \tag{37}$$

Taking into account equations (32), (34), (36) and (37) we obtain the risk for the control policy $U^*_{\beta,r}$:

$$\begin{aligned}R(\lambda, U^*_{\beta,r})=&\Big[\varkappa_0^{(1)}-2\sum_{i=1}^{M-1}h_i\frac{\varkappa_i}{\beta_i}+\sum_{i=1}^{M-1}\Big(h_i^{(1)}+h_i^2-\sum_{j=0}^{i-1}q_j'q_j\Big)\frac{\varkappa_i}{\beta_i^2}\Big]\lambda^2+\\&+\Big[2F_0x_0+\varkappa_0^{(2)}-2r\beta\sum_{i=0}^{M-1}\frac{\varkappa_i}{\beta_i^2}+\sum_{i=1}^{M-1}h_i^{(2)}\frac{\varkappa_i}{\beta_i^2}\Big]\lambda+\\&+x_0'D_0x_0+\varkappa_0^{(3)}+r^2\sum_{i=0}^{M-1}\frac{\varkappa_i}{\beta_i^2}+\sum_{i=1}^{M-1}h_i^{(3)}\frac{\varkappa_i}{\beta_i^2}\\&\overset{\mathrm{df}}{=}Z_1(\beta)\lambda^2+Z_2(\beta,r)\lambda+Z_3(\beta,r).\end{aligned} \tag{38}$$

Let $\pi$ be an a priori distribution of the parameter $\lambda$ for which $\mathrm{E}(\lambda^2)<\infty$. From (6) and (38) it follows that

$$r(\pi, U^*_{\beta,r})=Z_1(\beta)\,\mathrm{E}_\pi(\lambda^2)+Z_2(\beta,r)\,\mathrm{E}_\pi(\lambda)+Z_3(\beta,r). \tag{39}$$

In particular, for $(\beta, r)\in S$ we have

$$r(\pi_{\beta,r}, U^*_{\beta,r})=Z_1(\beta)k(\beta,r)+Z_2(\beta,r)\frac{r}{\beta}+Z_3(\beta,r), \tag{40}$$

where $k(\beta, r)$ is defined in (8).

**6. Limit Bayes control policies.** Denote by $U_m^+=(u_0^+, u_1^+, \ldots, u_M^+)$ the control policy for which

$$u_M^+=0,\quad u_n^+=-P_nx_n-Q_nm \ (n=0, 1, \ldots, M-1).$$

Denote by $U^-_{\beta,m}=(u_0^-, u_1^-, \ldots, u_M^-)$ the control policy for which

$$u_M^-=0,$$

$$u_0^-=-P_0x_0-Q_0\bar{m},\quad u_n^-=-P_nx_n-Q_n\frac{r_n^{(\bar{m})}}{\beta_n}\ (n=1, \ldots, M-1),$$

where

$$r_n^{(m)} = \sum_{i=1}^{n-1} \sum_{j=1}^{s} v_{ij} + m\beta.$$

The control policy $U_{\beta,m}^-$ is defined also for $\beta = 0$.

Obviously, $U_{\beta,\bar{m}}^- = U_{\beta,\bar{m}\beta}^*$ for $(\beta, \bar{m}) = S$ since in this case $\beta > 0$.

We say that $m \in \hat{S}_1$ if there is a sequence $\{(\gamma_k, s_k)\}_1^\infty$, $(\gamma_k, s_k) \in S$, such that $s_k/\gamma_k \to m$ for $k \to \infty$.

Similarly, we say that $(\beta, \bar{m}) \in \hat{S}_2$ if there is a sequence $\{(\gamma_k, s_k)\}_1^\infty$, $(\gamma_k, s_k) \in S$, such that $\gamma_k \to \beta$, $s_k/\gamma_k \to \bar{m}$ for $k \to \infty$.

As we have noticed, from the specification of the sets $S$ for the particular distributions it follows that $\beta > 0$ if $(\beta, r) \in S$. Then from (25), (38) and (39) for $m \in \hat{S}_1$, $(\beta, \bar{m}) \in \hat{S}_2$, $(\gamma, s) \in S$ we obtain

$$U_m^+ = \lim_{\substack{s \to \infty \\ s/\gamma \to m}} U_{\gamma,s}^*, \qquad R(\lambda, U_m^+) = \lim_{\substack{s \to \infty \\ s/\gamma \to m}} R(\lambda, U_{\gamma,s}^*),$$

$$r(\pi, U_m^+) = \lim_{\substack{s \to \infty \\ s/\gamma \to m}} r(\pi, U_{\gamma,s}^*) \quad \text{if } \mathrm{E}_\pi(\lambda^2) < \infty$$

and

$$U_{\beta,\bar{m}}^- = \lim_{\substack{\gamma \to \beta \\ s/\gamma \to \bar{m}}} U_{\gamma,s}, \qquad R(\lambda, U_{\beta,\bar{m}}^-) = \lim_{\substack{\gamma \to \beta \\ s/\gamma \to \bar{m}}} R(\lambda, U_{\gamma,s}^*),$$

$$r(\pi, U_{\beta,\bar{m}}^-) = \lim_{\substack{\gamma \to \beta \\ s/\gamma \to \bar{m}}} r(\pi, U_{\gamma,s}^*) \quad \text{if} \quad \mathrm{E}_\pi(\lambda^2) < \infty,$$

respectively.

A control policy $U^{(0)}$ is called a *limit Bayes policy* if there is a sequence $\{U_k\}_1^\infty$ of Bayes control policies such that $U^{(0)} = \lim_{k \to \infty} U_k$ with probability 1. Then $U_m^+$ and $U_{\beta,\bar{m}}^-$ are limit Bayes policies for $m \in \hat{S}_1$ and $(\beta, \bar{m}) \in \hat{S}_2$, respectively.

**7. A lemma from decision theory.** Limit Bayes control policies are frequently minimax or $\Gamma$-minimax as it is seen from the following lemma:

LEMMA. *Let $\{\pi_k\}_1^\infty$, $\pi_k \in \Gamma$, be a sequence of a priori distributions on $\Lambda$ and let $\{U_k\}_1^\infty$ and $\{r(\pi_k, U_k)\}_1^\infty$ be the corresponding sequences of Bayes control policies and Bayes risks. If $U^{(0)}$ is a control policy for which the Bayes risk $r(\pi, U^{(0)})$ satisfies the condition*

$$\sup_{\pi \in \Gamma} r(\pi, U^{(0)}) \leqslant \limsup_{k \to \infty} r(\pi_k, U_k)$$

*then $U^{(0)}$ is a $\Gamma$-minimax policy.*

A proof similar to that of Theorem 6.5.2 in [8] is omitted. The lemma is a generalization of that theorem.

The control policy $U$ is a *constant Bayes risk policy* if $r(\pi, U) = c$ for all $\pi \in \Gamma$.

COROLLARY 1. *A constant Bayes risk policy which is Bayes with respect to the distribution $\pi \in \Gamma$ on $\Lambda$ is a $\Gamma$-minimax policy.*

For minimax policies we have

COROLLARY 2. *Let $\{\pi_k\}_1^\infty$ be a sequence of a priori distributions on $\Lambda$ and let $U_k$ be Bayes with respect to $\pi_k$. If $U^{(0)}$ is a control policy for which the risk function $R(\lambda, U^{(0)})$ satisfies the condition*

$$\sup_{\lambda \in \Lambda} R(\lambda, U^{(0)}) \leqslant \limsup_{k \to \infty} r(\pi_k, U_k),$$

*then $U^{(0)}$ is a minimax policy.*

The control policy $U$ is a *constant risk policy* if $R(\lambda, U) = c$ for all $\lambda \in \Lambda$.

COROLLARY 3. *A Bayes constant risk policy is minimax.*

**8. $\Gamma$-minimax control policies for disturbances belonging to an exponential family.** Suppose that the disturbances $v_{ij}$ have the distributions belonging to an exponential family with the natural parameter $\lambda$ being also a random variable. Then, by (10) and (16), $v_{ij}$ become dependent and the conditional distribution of $v_{nj}$ given $(X_n, U_{n-1})$ in general does not belong to the exponential class.

Let the a priori distribution of the parameter $\lambda$ belong to the set $\Gamma_1$ of all distributions $\pi$ on $\Lambda$ for which $\mathrm{E}_\pi(\lambda^2) = m_2$, where $m_2 \in (\Lambda)^2 - \{0\}$ is given. Let $T$ be the set of all $(\beta, r)$ for which $k(\beta, r) = m_2$. We have

THEOREM 1. I. *If there is a point $(\beta, r) \in S$ such that*

$$k(\beta, r) = m_2 \quad \text{and} \quad Z_2(\beta, r) = 0,$$

*then the $\Gamma_1$-minimax control policy is $U^*_{\beta,r}$.*

II. *If $Z_2(\beta, r) > 0$ for each $(\beta, r) \in S \cap T$, then the $\Gamma_1$-minimax control policy is $U^+_{m_2^{1/2}}$.*

III. *If $Z_2(\beta, r) < 0$ for each $(\beta, r) \in S \cap T$, then the $\Gamma_1$-minimax control policy is*

(i) *for the normal and GEHS distributions, the policy $U^+_{-m_2^{1/2}}$;*

(ii) *for the Poisson distribution, the policy $U^-_{0,0}$;*

(iii) *for the gamma and the negative binomial distributions, the policy $U^-_{1,0}$;*

(iv) *for the binomial distribution, the policy $U^-_{0,m_2}$.*

Proof. Since the full proof of the theorem in the case $s = 1$ was presented by the author in [7] and the proof for any $s$ differs from it in points of minor importance, we prove only point II to present the idea of the proof. Put

$$\mu_1 = \inf_{(\beta,r) \in S \cap T} r/\beta \quad \text{and} \quad \mu_2 = \sup_{(\beta,r) \in S \cap T} r/\beta.$$

The intervals $(\mu_1, \mu_2)$ for the particular distributions listed in (a)–(f) are

$$(41)\qquad \begin{matrix} \text{(a) } (0, \sqrt{m_2}), & \text{(b) } (0, \sqrt{m_2}), & \text{(c) } (0, \sqrt{m_2}), \\ \text{(d) } (m_2, \sqrt{m_2}), & \text{(e) } (-\sqrt{m_2}, \sqrt{m_2}), & \text{(f) } (-\sqrt{m_2}, \sqrt{m_2}), \end{matrix}$$

respectively, and it can be proved that $r/\beta \to \sqrt{m_2}$ if $\beta \to \infty$.

Suppose that $Z_2(\beta, r) > 0$ for all $(\beta, r) \in S \cap T$. Then for $\pi \in \Gamma_1$ and $(\gamma, s) \in S$ we have

$$r(\pi, U^+_{m_2^{1/2}}) = \lim_{\substack{s\to\infty \\ s/\gamma \to m_2^{1/2}}} r(\pi, U^*_{\gamma,s})$$

$$\overset{(39)}{=} \lim_{\substack{s\to\infty \\ s/\gamma \to m_2^{1/2}}} [Z_1(\gamma) m_2 + Z_2(\gamma, s) \mathrm{E}_\pi(\lambda) + Z_3(\gamma, s)]$$

$$\leqslant \lim_{\substack{s\to\infty \\ s/\gamma \to m_2^{1/2}}} [Z_1(\gamma) m_2 + Z_2(\gamma, s)\sqrt{m_2} + Z_3(\gamma, s)] \overset{(40)}{=} \lim_{\substack{s\to\infty \\ s/\gamma \to m_2^{1/2}}} r(\pi_{\gamma,s}, U^*_{\gamma,s})$$

From (41) and the remark after (41) it follows that there is a sequence $(\gamma, s) \in S \cap T$ for which $s \to \infty$, $s/\gamma \to \sqrt{m_2}$. Then from the Lemma we infer that the control policy $U^+_{m_2^{1/2}}$ is $\Gamma_1$-minimax.

Let now the set $\Gamma = \Gamma_2$ of the a priori distributions of the parameter $\lambda$ be defined by the conditions $\mathrm{E}_\pi(\lambda) = m$ and $\mathrm{E}_\pi(\lambda^2) = m_2$, where $m \in \Lambda$, $m_2 \in (\Lambda)^2$, $m^2 < m_2$.

Since for the binomial distribution we have also $\lambda^2 < \lambda$ if $\lambda \in \Lambda$, to determine the class $\Gamma_2$ we assume in addition for this distribution that $m_2 < m$.

THEOREM 2. *The control policy $U^*_{\beta,r}$ for which $r/\beta = m$ and $k(\beta, r) = m_2$ $((\beta, r) \in S)$ is a $\Gamma_2$-minimax policy. This policy always exists assuming the condition* (A) *to be satisfied.*

For the proof it is sufficient to use the definition of the class $\Gamma_2$, the specification of the set $S$ for the particular distributions, Corollary 1, and table (41).

**9. Minimax and $\Gamma$-minimax control policies for binomial distribution.** Suppose that the random variables $v_{ij}$ have the binomial distribution for fixed $\lambda$. In this case the risk $R(\lambda, U^*_{\beta,r})$ is

$$R(\lambda, U^*_{\beta,r}) = \left[\varkappa_0^{(1)} - 2\sum_{i=1}^{M-1} h_i \frac{\varkappa_i}{\beta_i} + \sum_{i=1}^{M-1} h_i(h_i - 1)\frac{\varkappa_i}{\beta_i^2}\right]\lambda^2 +$$

$$+\left[2F_0 x_0 + \varkappa_0^{(2)} - 2r\beta \sum_{i=0}^{M-1} \frac{\varkappa_i}{\beta_i^2} + \sum_{i=1}^{M-1} h_i \frac{\varkappa_i}{\beta_i^2}\right]\lambda +$$

$$+ x_0' D_0 x_0 + r^2 \sum_{i=0}^{M-1} \frac{\varkappa_i}{\beta_i^2}.$$

The constants $h_i \geqslant 0$ by definition. It can be shown that for particular parameters of the matrices $S_n$ and $K_n$ all of the following three cases can occur:

I. there is $\beta$ such that $Z_1(\beta) = 0$;

II. $Z_1(\beta) < 0$ for each $\beta > 0$;

III. $Z_1(\beta) > 0$ for each $\beta > 0$.

The remaining five distributions do not have this property.

Let $\Gamma = \Gamma_3$ be the set of all a priori distributions $\pi$ of the parameter $\lambda$ for which

$$E_\pi(\lambda) = m, \quad 0 < m < 1. \tag{42}$$

We have

THEOREM 3. *Let the disturbances $v_{nj}$ in the system* (1) *have the binomial distributions with parameters $q_{nj}$ and $\lambda$, where $\lambda$ is a random variable satisfying condition* (42) *with given $m$ $(0 < m < 1)$.*

I. *If there is $\beta > 0$ such that $Z_1(\beta) = 0$, then the $\Gamma_3$-minimax policy is $U^*_{\beta,m\beta}$.*

II. *If $Z_1(\beta) < 0$ for each $\beta > 0$, then the $\Gamma_3$-minimax policy is $U^+_m$.*

III. *If $Z_1(\beta) > 0$ for each $\beta > 0$, then the $\Gamma_3$-minimax policy is $U^-_{0,m}$.*

Proof. We prove only proposition III. Suppose that $Z_1(\beta) > 0$ for each $\beta > 0$. Since $E_\pi(\lambda^2) \leqslant E_\pi(\lambda) = m$, we have

$$\begin{aligned} r(\pi, U^-_{0,m}) &= \lim_{\beta \to 0+} r(\pi, U^*_{\beta,m\beta}) \\ &= \lim_{\beta \to 0+} [Z_1(\beta) E_\pi(\lambda^2) + Z_2(\beta, m\beta)m + Z_3(\beta, m\beta)] \\ &\leqslant \lim_{\beta \to 0+} [(Z_1(\beta) + Z_2(\beta, m\beta))m + Z_3(\beta, m\beta)] \\ &= \lim_{\beta \to 0+} r(\pi_{\beta,m\beta}, U^*_{\beta,m\beta}) \end{aligned}$$

because

$$r(\pi_{\beta,m\beta}, U^*_{\beta,m\beta}) = \left(Z_1(\beta)\frac{m\beta+1}{\beta+1} + Z_2(\beta, m\beta)\right)m + Z_3(\beta, m\beta).$$

Since $\pi_{\beta,m\beta} \in \Gamma_3$, the policy $U^-_{0,m}$ is $\Gamma_3$-minimax.

Sometimes we have no information about the parameter $\lambda$. In this case we can use minimax control policies for the binomial distribution.

THEOREM 4. *Let the disturbances $v_{nj}$ in the system* (1) *be distributed according to the binomial law with the parameters $q_{nj}$ and $\lambda$, respectively.*

I. *If there is $\beta > 0$ such that $Z_1(\beta) = 0$ and for this $\beta$*

(a) *there is $r$ $(0 < r < \beta)$ such that $Z_2(\beta, r) = 0$, then the policy $U^*_{\beta,r}$ is minimax;*

(b) *if* $Z_2(\beta, r) < 0$ *for all* $r$ $(0 < r < \beta)$, *then the policy* $U^-_{\beta,0}$ *is minimax;*
(c) *if* $Z_2(\beta, r) > 0$ *for all* $r$ $(0 < r < \beta)$, *then the policy* $U^-_{\beta,1}$ *is minimax.*

II. *Let*

$$m_0 = 2F_0 x_0 + \varkappa_0^{(2)} \quad \text{and} \quad m = \frac{m_0}{2\left(\sum_{i=0}^{M-1} \varkappa_i - \varkappa_0^{(1)}\right)}.$$

*If* $Z_1(\beta) < 0$ *for all* $\beta > 0$ *and if* $\sum_{i=0}^{M-1} \varkappa_i - \varkappa_0^{(1)} > 0$, *then the minimax control policy is*

(a) $U_m^+$ *if* $0 < m < 1$,
(b) $U_0^+$ *if* $m \leqslant 0$,
(c) $U_1^+$ *if* $m \geqslant 1$.

*If* $Z_1(\beta) < 0$ *for all* $\beta > 0$ *and if* $\sum_{i=0}^{M-1} \varkappa_i - \varkappa_0^{(1)} = 0$, *then the minimax control policy is*

(d) $U_0^+$ *if* $m_0 \leqslant 0$,
(e) $U_1^+$ *if* $m_0 \geqslant 0$.

III. *Let*

$$m_0 = 2F_0 x_0 + \varkappa_0^{(1)} + \varkappa_0^{(2)} - \sum_{i=1}^{M-1} \varkappa_i \quad \text{and} \quad m = \frac{m_0}{2\varkappa_0}.$$

*If* $Z_1(\beta) > 0$ *for all* $\beta > 0$ *and if* $\varkappa_0 > 0$, *then the minimax control policy is*

(a) $U^-_{0,m}$ *if* $0 < m < 1$,
(b) $U^-_{0,0}$ *if* $m \leqslant 0$,
(c) $U^-_{0,1}$ *if* $m \geqslant 1$.

*If* $Z_1(\beta) > 0$ *for all* $\beta > 0$ *and if* $\varkappa_0 = 0$, *then the minimax control policy is*

(d) $U^-_{0,0}$ *if* $m_0 \leqslant 0$,
(e) $U^-_{0,1}$ *if* $m_0 \geqslant 0$.

*The condition* $\sum_{i=0}^{M-1} \varkappa_i - \varkappa_0^{(1)} \geqslant 0$ *always holds when* $Z_1(\beta) < 0$ *for all* $\beta > 0$ *and always* $\varkappa_i \geqslant 0$.

Proof. To present the dodges used we prove propositions I(b), II(c) and III(a). The remaining cases can be proved similarly.

Suppose that the case I(b) occurs. We have

$$R(\lambda, U^-_{\beta,0}) = \lim_{\substack{\gamma \to \beta \\ s/\gamma \to 0+}} R(\lambda, U^*_{\gamma,s}) = \lim_{s \to 0} \left(Z_1(\beta, s)\lambda + Z_3(\beta, s)\right)$$

$$\leqslant \lim_{s \to 0} Z_3(\beta, s) = x_0' D_0 x_0 = \lim_{r \to 0} r(\pi_{\beta,r}, U^*_{\beta,r}),$$

which proves, by Corollary 3, that $U^-_{\beta,0}$ is a minimax control policy.

Suppose that $Z_1(\beta) < 0$ for all $\beta > 0$. Since $Z_1(\beta)$ is a nondecreasing function of $\beta > 0$, we have $\varkappa_0^{(1)} \leqslant 0$. Since $\varkappa_i \geqslant 0$, we get $\sum_{i=0}^{M-1} \varkappa_i - \varkappa_0^{(1)} \geqslant 0$.

Assume that $m \geqslant 1$. We have

$$R(\lambda, U_1^+) = \varkappa_0^{(1)}(1-\lambda)^2 + \left(2F_0 x_0 + \varkappa_0^{(2)} + 2\varkappa_0^{(1)} - 2\sum_{i=0}^{M-1} \varkappa_i\right)\lambda + \\ + x_0' D_0 x_0 + \sum_{i=0}^{M-1} \varkappa_i - \varkappa_0^{(1)}$$

$$\leqslant 2F_0 x_0 + \varkappa_0^{(1)} + \varkappa_0^{(2)} - \sum_{i=0}^{M-1} \varkappa_i + x_0' D_0 x_0$$

$$= \lim_{\substack{r \to \infty \\ r/\beta \to 1-}} r(\pi_{\beta,r}, U_{\beta,r}^*).$$

Thus proposition II(c) is proved.

Let us suppose that $Z_1(\beta) > 0$ for all $\beta > 0$. Since $Z_1(\beta)$ is nondecreasing for $\beta > 0$, we obtain

$$\varkappa_0^{(1)} - \sum_{i=1}^{M-1} \frac{h_i+1}{h_i} \varkappa_i \geqslant 0.$$

Let $\varkappa_0 > 0$ and let $0 < m < 1$. We have

$$R(\lambda, U_{0,\bar{m}}^-) = \left(\varkappa_0^{(1)} - \sum_{i=1}^{M-1} \frac{h_i+1}{h_i} \varkappa_i\right)\lambda(\lambda-1) + x_0' D_0 x_0 + \varkappa_0 \bar{m}^2$$

$$\leqslant x_0' D_0 x_0 + \varkappa_0 m^2 = \lim_{\substack{\beta \to 0+ \\ r/\beta \to m}} r(\pi_{\beta,r}, U_{\beta,r}^*).$$

Thus proposition III(a) is proved.

The most important in applications is here the special case $q_{nj} = 1$, $n = 0, 1, \ldots, M-1$, $j = 1, \ldots, s$, since then the binomial distributions reduce to two-point distributions.

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